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IMPLICIT VISCOSITY ITERATIVE ALGORITHM FOR NONEXPANSIVE MAPPING ON HADAMARD MANIFOLDS

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Abstract. In this paper, an implicit viscosity iterative algorithm for nonexpansive mappings is proposed and investigated on Hadamard manifolds. A convergence theorem of a fixed point of a nonexpansive mapping is established on Hadamard manifolds. It is proved that the fixed point also solves a variational inequality.
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1. Introduction

The fixed point problem of a nonlinear mapping T is to find x such that

$$x = Tx. \tag{1.1}$$

A number practical problems can be convertible to the fixed point problem, such as, optimization problems, variational inequality problems, equilibrium problems, and split feasibility problems, etc. In view of its real applications, such as, image recovery, signal processing and so on, the fixed point problems have been extensively studied by many researchers; see, e.g., [2, 8, 7, 17] and the references therein. However, most of these results were established on linear spaces; see, e.g., [4, 10, 14, 21, 22] and the references therein.

Recall that Halpern iterative algorithm (1.2) generates an iterative sequence $\{x_n\}$ as follows

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \forall n \geq 1, \tag{1.2}$$

where u is the anchorm and the sequence $\{\alpha_n\} \subset (0,1)$. It is known that the Halpern iterative algorithm is an efficient algorithm dealing with fixed points of nonexpansive mappings. The main advantage is that the strong convergence can be guaranteed in various linear spaces without any compact restrictions. Recently, many results based on the Halpern iterative algorithm were established; see, e.g, [13, 15, 24] and the references therein. In particular, in 2000, Moudafi [16] first considered a viscosity algorithm for nonexpansive mappings in Hilbert spaces. The fixed point in Moudafi's results was proved that it also uniquely solves some variational inequality associated to a contractive mapping. In 2004, Xu [25] further improved Moudafi's results from Hilbert spaces to Banach spaces. Subsequence, various viscosity algorithms were introduced and investigated; see, e.g., [6, 18, 19] and the references therein.

Let M be an Hadamard manifold, and let TM be the tangent bundle of M. Let K be a nonempty closed convex subset of M. Let exp be a exponential mapping. Recently, Li et al. [11] studied fixed points of nonexpansive mapping by using Halpern iterative algorithm, and obtained the strong convergence on Hadamard manifolds. This interesting result extends many results from classical linear spaces to the setting of manifolds.

On Hadamard manifolds, the Halpern iterative algorithm is presented as follows

$$x_{n+1} = \exp_u (1 - \alpha_n) \exp_u^{-1} Tx_n, \quad n \geq 0, \tag{1.3}$$

where $u, x_0 \in K$ and the sequence $\{\alpha_n\} \subset (0, 1)$. It is equivalent

$$x_{n+1} = \gamma_n (1 - \alpha_n), \quad n \geq 0,$$

where $\gamma_n : [0, 1] \to M$ is the geodesic joining u to Tx_n (i.e. $\gamma(0) = u$ and $\gamma(1) = T(x_n)$).

Limited by the nonlinearity of manifolds, the research progress on fixed point problem (1.1) is slow. As far as now, only few related researches are presented; see, e.g., [11, 5, 1, 9]. On the other hand, to the best of our knowledge, all of these results were restricted to explicit algorithms.

Motivated by the results of Li [11], Qin et al. [18], and Xu [25], our goal here is to present an implicit viscosity iterative algorithm for approximating fixed points of nonexpansive mappings on Hadamard manifolds

$$x_n = \exp_{f(x_n)} (1 - \alpha_n) \exp_{f(x_n)}^{-1} Tx_n, \quad n \geq 0,$$

and we proved the sequence $\{x_n\}$ generated by this implicit viscosity iterative algorithm strongly converges to a fixed point of nonexpansive mapping T. Our results extend the results of Xu [25] from the classical linear spaces to the setting of Hadamard manifolds and perfected fixed point theory of nonexpansive mapping on Hadamard manifolds.

2. Preliminaries

Let M be a connected $m-$dimensional manifold and $p \in M$, the T_pM denotes the tangent space of M at p. To become Riemannian manifold, we always assume M is endowed with the Riemannian metric $\langle , \rangle$ and the corresponding norm $\| \cdot \|$.

Given a piecewise smooth curve $c : [a, b] \to M$ joining p to q, we define the length of c by

$$L(c) = \int_a^b \|c'(t)\| dt.$$

Then, the Riemannian distance $d(p, q)$ is the minimal length over all such curves joining p to q.

Let ∇ be a Levi-Civita connection associated with the Riemannian manifold M. If ϕ is a smooth curve, a smooth vector field F along ϕ is called parallel if $\nabla_{\phi'} F = 0$. If ϕ' is parallel, then ϕ is a geodesic, and $\|\phi'\|$ is a constant. Based on the definition of Riemannian distance $d(p, q)$, it is easily seen that a geodesic joining p to q in M is called a minimizing geodesic if its length equals to $d(p, q)$.

A Riemannian manifold is complete if its geodesics are defined for any $t \in R$, Hopf-Rinow theorem asserts that if M is complete then any pair of points in M can be joined by a minimizing geodesic. A complete simply connected Riemannian manifold of non-positive sectional curvature is named a Hadamard manifold.

Let $\gamma(t) : [a, b] \to R$, the parallel transport $P_{\gamma,\gamma(a),\gamma(b)} : T_{\gamma(a)}M \to T_{\gamma(b)}M$ on the tangent bundle TM on the $\gamma(t)$ is defined by

$$P_{\gamma,\gamma(b),\gamma(a)}(\nu) = F(\gamma(b)), \forall a, b \in R, \nu \in T_{\gamma(a)}M,$$

where F is a unique vector field such that $F(\gamma(a)) = \nu$ and $\nabla_{\gamma'(t)} F = 0, \forall t \in [a, b]$.

If $\gamma(t) : [a, b] \to R$ is a minimizing geodesic joining a to b, $P_{\gamma,b,a}$ is denoted by $P_{b,a}$ and $P_{b,a}^{-1} = P_{a,b}$ generally. Recall that, for $a, b \in R$, for all $u, v \in T_{\gamma(a)}M$, we have

$$\langle P_{\gamma(b),\gamma(a)} u, P_{\gamma(b),\gamma(a)} v\rangle = \langle u, v\rangle.$$

Definition 2.1. The mapping $T : K \to K$ is called to be nonexpansive, if the following inequality holds

$$d(Tx, Ty) \le d(x, y).$$

Definition 2.2. The mapping $T : K \to K$ is called to be contraction, if there exists a constant $\alpha \in (0, 1)$ and the following inequality holds

$$d(Tx, Ty) \le \alpha d(x, y).$$

Lemma 2.1.[3] Let $\triangle(p, q, r)$ be a geodesic triangle in a Hadamard manifold M, then there exists $p', q', r' \in R^2$ such that

$$d(p, q) = \|p' - q'\|, d(q, r) = \|q' - r'\|, d(r, p) = \|r' - p'\|.$$

Remark 2.1. The triangle $\triangle(p', q', r')$ is called to be the comparison triangle of the geodesic triangle $\triangle(p, q, r)$, which is unique up to isometry of M.

Lemma 2.2. [11] Let $\triangle(p, q, r)$ be a geodesic triangle in a Hadamard manifold M, and $\triangle(p', q', r')$ is its comparison triangle.

(i) Let $\alpha, \beta, \gamma(\alpha', \beta', \gamma')$ be the angles of $\triangle(p, q, r)(\triangle(p', q', r'))$ at the vertices $p, q, r(p', q', r')$. Then the following inequalities hold:

$$\alpha \le \alpha',\ \beta \le \beta',\ \gamma \le \gamma'.$$

(ii) Let z be a point in the geodesic joining p to q, and z' is its comparison point in the interval $[p', q']$. Suppose that $d(z, p) = \|z' - p'\|$ and $d(z, q) = \|z' - q'\|$. Then the the following inequality holds:

$$d(z, r) \le \|z - r'\|.$$

Lemma 2.3. [12] Let $x^* \in M$ and $\{x_n\} \subset M$ with $x_n \to x^*$ as $n \to \infty$. Then the following conclusions hold:

(i) For any $y \in M$, then $\exp_{x_n}^{-1} y \to \exp_{x^*}^{-1} y$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x^*$ as $n \to \infty$.
(ii) If $v_n \in T_{x_n}M$ and $v_n \to v^*$ as $n \to \infty$, then $v^* \in T_{x^*}M$.
(iii) Let $\eta_n, \nu_n \in T_{x_n}M$ and $\eta_*, \nu^* \in T_{x^*}M$ if $\eta_n \to \eta^*$ and $\nu_n \to \nu^*$ as $n \to \infty$, then $\langle \eta_n, \nu_n \rangle \to \langle \eta^*, \nu^* \rangle$ as $n \to \infty$.

Lemma 2.4. [12] If $x, y \in M$ and $w \in T_yM$, then

$$\langle w, -\exp_y^{-1} x \rangle = \langle w, P_{y,x} \exp_x^{-1} y \rangle = \langle P_{y,x} w, \exp_y^{-1} x \rangle.$$

Lemma 2.5. [20] Let $d : M \times M \to R$ be the distance function. Then d is a convex function with respect to the product Riemannian metric, i.e., given any pair of geodesics $\gamma_1 : [0,1] \to M$ and $\gamma_2 : [0,1] \to M$, the following inequality holds for all $t \in [0,1]$:

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)).$$

Let P_K denote the projection onto K, and for a point $p \in M$,$P_K(p)$ is defined by

$$P_K(p) = \{p_0 \in K | d(p, p_0) \leq d(p, q) \forall q \in K\}$$

Lemma 2.6. [20] Let $\triangle(x_1, x_2, x_3)$ be a geodesic triangle in M. Then
(i) $d^2(x_1, x_2) + d^2(x_2, x_3) - 2\langle \exp_{x_2}^{-1} x_1, \exp_{x_2}^{-1} x_3 \rangle \leq d^2(x_3, x_1)$,
(ii) $d^2(x_1, x_2) \leq \langle \exp_{x_1}^{-1} x_3, \exp_{x_1}^{-1} x_2 \rangle + \langle \exp_{x_2}^{-1} x_3, \exp_{x_2}^{-1} x_1 \rangle$.
(iii) If γ is the angle at x_1, then we have

$$\langle \exp_{x_1}^{-1} x_2, \exp_{x_1}^{-1} x_3 \rangle = d(x_2, x_1) d(x_1, x_3) \cos\gamma.$$

Lemma 2.7. [23] For any point $p \in M$, $P_K(p)$ is a singleton and the following inequality holds

$$\langle \exp_{P_K(p)}^{-1} p, \exp_{P_K(p)}^{-1} q \rangle \leq 0, \forall q \in K.$$

3. Main results

Let $x_0 \in M$, $\{\alpha_n\} \subset (0,1)$, $f : M \to M$ a contraction with coefficient α, consider the iteration scheme

$$x_n = \exp_{f(x_n)}(1-\alpha_n) \exp_{f(x_n)}^{-1} Tx_n, \quad n \geq 0, \tag{3.1}$$

or equivalently

$$x_n = \gamma_n(1-\alpha_n), \quad n \geq 0, \tag{3.2}$$

where $\gamma_n : [0,1] \to M$ is the geodesic joining $f(x_n)$ to Tx_n (i.e. $\gamma(0) = f(x_n)$ and $\gamma(1) = T(x_n)$).

Theorem 3.1. Let K be a closed convex subset of M, $T : K \to K$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, and $f : K \to K$ a contraction with coefficient α. Let $\{x_n\}$ be generated by the algorithm (3.1), $\{\alpha_n\} \subset (0,1)$ satisfies $\lim\limits_{n\to\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ converges to $\tilde{x}$, where $\tilde{x}$ is the unique solution of the variation inequality

$$\langle \exp_{\tilde{x}}^{-1} f(\tilde{x}), \exp_{\tilde{x}}^{-1} z \rangle \leq 0, \forall z \in Fix(T). \tag{3.3}$$

Proof. The proof is divided into four steps.

Step 1. We show $\{x_n\}$ is bounded.

Take $x \in Fix(T)$ and fix n, then, by the convexity of the Riemannian distance in lemma 2.5 and the nonexpansive of T, we have that

$$\begin{aligned} d(x_n, x) &= d(\gamma_n(1-\alpha_n), x) \\ &\le \alpha_n d(\gamma_n(0), x) + (1-\alpha_n) d(\gamma_n(1), x) \\ &= \alpha_n d(f(x_n), x) + (1-\alpha_n) d(Tx_n, x) \\ &\le \alpha_n d(f(x_n), f(x)) + \alpha_n d(f(x), x) + (1-\alpha_n) d(x_n, x) \\ &\le \alpha_n \alpha d(x_n, x) + \alpha_n d(f(x), x) + (1-\alpha_n) d(x_n, x). \end{aligned}$$

By induction

$$d(x_n, x) \le \frac{1}{1-\alpha} d(f(x), x), \forall n \ge 0.$$

Then $\{x_n\}$ is bounded, so are $\{Tx_n\}$ and $\{f(x_n)\}$.
Step 2. We show $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.
By Step 1, there exists a constant C such that

$$d(f(x_n), Tx_n) \le C, \forall n \ge 0.$$

Using the convexity of the distance function in lemma 2.5, we have that

$$\begin{aligned} d(x_n, Tx_n) &= d(\gamma_n(1-\alpha_n), Tx_n) \\ &\le \alpha_n d(\gamma_n(0), Tx_n) + (1-\alpha_n) d(\gamma_n(1), Tx_n) \\ &\le \alpha_n d(f(x_n), Tx_n) + (1-\alpha_n) d(Tx_n, Tx_n) \\ &\le \alpha_n d(f(x_n), Tx_n) \\ &\le \alpha_n C. \end{aligned}$$

Together with the condition $\lim_{n\to\infty} \alpha_n = 0$, we get $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.
Step 3. We show that there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \tilde{x}$ as $k \to \infty$, and $\tilde{x} \in Fix(T)$ solves the variational inequality (3.3)

Since $\{x_n\}$ is bounded by step1, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \tilde{x}$ as $k \to \infty$.

From Step 2, we know $\lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = 0$, therefore,

$$\begin{aligned} d(\tilde{x}, T\tilde{x}) &\le d(\tilde{x}, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, T\tilde{x}) \\ &\le 2d(\tilde{x}, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) \end{aligned}$$

which implies that $d(\tilde{x}, T\tilde{x}) = 0$, i.e. $\tilde{x} \in Fix(T)$.

Fix $n, k \in \mathbb{N}$, let $z \in Fix(T)$ and $\triangle(x_{n_k}, f(x_{n_k}), z) \subseteq M$ be a geodesic triangle with vertices x_{n_k}, $f(x_{n_k})$ and z, and $\triangle(\overline{x_{n_k}}, \overline{f(x_{n_k})}, \overline{z}) \subseteq R^2$. Let β and $\overline{\beta}$ denote the angles at x_{n_k} and $\overline{x_{n_k}}$, respectively. And we know $\beta < \overline{\beta}$ by lemma 2.2.

Likewise, let $\triangle(Tx_{n_k}, f(x_{n_k}), z) \subseteq M$ be a geodesic triangle with vertices Tx_{n_k}, $f(x_{n_k})$ and z, and $\triangle(\overline{Tx_{n_k}}, \overline{f(x_{n_k})}, \overline{z}) \subseteq R^2$. Let θ and $\overline{\theta}$ denote the angles at Tx_{n_k} and $\overline{Tx_{n_k}}$, respectively. And we know $\theta < \overline{\theta}$ by lemma 2.2.

The comparison point $\overline{x_{n_k}} = \alpha_n \overline{f(x_{n_k})} + (1-\alpha_n)\overline{Tx_{n_k}}$, by lemma 2.1 then

$$\begin{aligned}
&\langle \overline{x_{n_k}} - \overline{Tx_{n_k}}, \overline{x_{n_k}} - \overline{z} \rangle \\
&= \langle \overline{x_{n_k}} - \overline{z} - (\overline{Tx_{n_k}} - \overline{z}), \overline{x_{n_k}} - \overline{z} \rangle \\
&= \|\overline{x_{n_k}} - \overline{z}\|^2 - \langle \overline{Tx_{n_k}} - \overline{z}, \overline{x_{n_k}} - \overline{z} \rangle \\
&\geq \|\overline{x_{n_k}} - \overline{z}\|^2 - \|\overline{Tx_{n_k}} - \overline{z}\| \cdot \|\overline{x_{n_k}} - \overline{z}\| \\
&= d^2(x_{n_k}, z) - d(Tx_{n_k}, z) d(x_{n_k}, z) \\
&\geq d^2(x_{n_k}, z) - d^2(x_{n_k}, z) = 0
\end{aligned}$$

It follows lemma 2.4, we have that

$$\begin{aligned}
&\langle \exp^{-1}_{x_{n_k}} f(x_{n_k}), \exp^{-1}_{x_{n_k}} z \rangle \\
&= \langle -P_{x_{n_k}, f(x_{n_k})} \exp^{-1}_{f(x_{n_k})} x_{n_k}, \exp^{-1}_{x_{n_k}} z \rangle \\
&= -d(x_{n_k}, f(x_{n_k})) d(z, x_{n_k}) \cos\beta \\
&\leq -d(x_{n_k}, f(x_{n_k})) d(z, x_{n_k}) \cos\overline{\beta} \\
&\leq -\|\overline{x_{n_k}} - \overline{f(x_{n_k})}\| \|\overline{z} - \overline{x_{n_k}}\| \cos\overline{\beta} \\
&\leq -\langle \overline{x_{n_k}} - \overline{f(x_{n_k})}, \overline{z} - \overline{x_{n_k}} \rangle \\
&\leq \frac{1-\alpha_{n_k}}{\alpha_{n_k}} \langle \overline{x_{n_k}} - \overline{Tx_{n_k}}, \overline{z} - \overline{x_{n_k}} \rangle \\
&= -\frac{1-\alpha_{n_k}}{\alpha_{n_k}} \langle \overline{x_{n_k}} - \overline{Tx_{n_k}}, \overline{x_{n_k}} - \overline{z} \rangle \\
&\leq 0
\end{aligned}$$

Taking the limit through $k \to \infty$ by lemma 2.3, we have

$$\langle \exp^{-1}_{\tilde{x}} f(\tilde{x}), \exp^{-1}_{\tilde{x}} z \rangle \leq 0, \forall z \in Fix(T). \tag{3.4}$$

Step 4. We Show $\lim_{n\to\infty} x_n = \tilde{x}$.

Assume there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to \hat{x}$ as $j \to \infty$. Similarly, we get $\hat{x} \in Fix(T)$ satisfying the variational inequality

$$\langle \exp^{-1}_{\hat{x}} f(\hat{x}), \exp^{-1}_{\hat{x}} z \rangle \leq 0, \forall z \in Fix(T). \tag{3.5}$$

Replacing $z \in Fix(T)$ with $\hat{x}$ in (3.4) and replacing $z \in Fix(T)$ with $\tilde{x}$ in (3.5), we obtain

$$\langle \exp^{-1}_{\tilde{x}} f(\tilde{x}), \exp^{-1}_{\tilde{x}} \hat{x} \rangle \leq 0 \tag{3.6}$$

and

$$\langle \exp^{-1}_{\hat{x}} f(\hat{x}), \exp^{-1}_{\hat{x}} \tilde{x} \rangle \leq 0. \tag{3.7}$$

Let $\triangle(\tilde{x}, f(\tilde{x}), \hat{x}) \subseteq M$ be a geodesic triangle with vertices $\tilde{x}$, $f(\tilde{x})$ and $\hat{x}$, and $\triangle(\overline{\tilde{x}}, \overline{f(\tilde{x})}, \overline{\hat{x}}) \subseteq R^2$. Let μ and $\overline{\mu}$ denote the angles at $\tilde{x}$ and $\overline{\tilde{x}}$, respectively. And we know $\mu < \overline{\mu}$ by lemma 2.2.

Likewise, let $\triangle(\hat{x}, f(\hat{x}), \tilde{x}) \subseteq M$ be a geodesic triangle with vertices $\hat{x}$, $f(\hat{x})$ and $\tilde{x}$, and $\triangle(\overline{\hat{x}}, \overline{f(\hat{x})}, \overline{\tilde{x}}) \subseteq R^2$. Let μ and $\overline{\mu}$ denote the angles at $\hat{x}$ and $\overline{\hat{x}}$, respectively. And we know $\nu < \overline{\nu}$ by lemma2.2.

By lemma 2.1 and lemma 2.6, we obtain

$$\begin{aligned}\langle \overline{\tilde{x}} - \overline{f(\tilde{x})}, \overline{\tilde{x}} - \overline{\hat{x}} \rangle &= \|\tilde{x} - f(\tilde{x})\| \|\tilde{x} - \hat{x}\| \cos \overline{\mu} \\ &= d(f(\tilde{x}), \tilde{x}) d(\tilde{x}, \hat{x}) \cos \overline{\mu} \\ &\le d(f(\tilde{x}), \tilde{x}) d(\tilde{x}, \hat{x}) \cos \mu \\ &= \langle \exp_{\tilde{x}}^{-1} f(\tilde{x}), \exp_{\tilde{x}}^{-1} \hat{x} \rangle \le 0\end{aligned} \tag{3.8}$$

Repeated the same as the above technique, it yields

$$\langle \overline{\hat{x}} - \overline{f(\hat{x})}, \overline{\hat{x}} - \overline{\tilde{x}} \rangle \le \langle \exp_{\hat{x}}^{-1} f(\hat{x}), \exp_{\hat{x}}^{-1} \tilde{x} \rangle \le 0. \tag{3.9}$$

Adding up (3.8) and (3.9), we have

$$\begin{aligned}&\langle \overline{\tilde{x}} - \overline{f(\tilde{x})} - (\overline{\hat{x}} - \overline{f(\hat{x})}), \overline{\tilde{x}} - \overline{\hat{x}} \rangle \\ &= \|\overline{\tilde{x}} - \overline{\hat{x}}\|^2 - \langle \overline{f(\tilde{x})} - \overline{f(\hat{x})}, \overline{\tilde{x}} - \overline{\hat{x}} \rangle \\ &= d^2(\tilde{x}, \hat{x}) - \langle \overline{f(\tilde{x})} - \overline{f(\hat{x})}, \overline{\tilde{x}} - \overline{\hat{x}} \rangle \le 0,\end{aligned} \tag{3.10}$$

since

$$\begin{aligned}\langle \overline{f(\tilde{x})} - \overline{f(\hat{x})}, \overline{\tilde{x}} - \overline{\hat{x}} \rangle &\le \|\overline{f(\tilde{x})} - \overline{f(\hat{x})}\| \|\overline{\tilde{x}} - \overline{\hat{x}}\| \\ &\le d(f(\tilde{x}), f(\hat{x})) d(\tilde{x}, \hat{x}) \\ &\le \alpha d^2(\tilde{x}, \hat{x}).\end{aligned} \tag{3.11}$$

Combine (3.10) with (3.11), we get $(1-\alpha)d^2(\tilde{x}, \hat{x}) \le 0$. Thus $\tilde{x} = \hat{x}$. So $\{x_n\}$ converges strongly to $\tilde{x}$.

Remark 3.1. Theorem 3.1 constructed firstly the implicit iteration approximation theory on Hadamard manifolds and extend the results of Xu [25] from the classical linear space to Hadamard manifolds.

4. Conclusions

In this paper, an implicit viscosity iterative algorithm for nonexpansive mapping on Hadamard manifolds has been proposed, and we have proved the sequence generated by the algorithm (3.1) strongly converges to the fixed point of the nonexpansive mapping $T : K \to K$ on Hadamard manifolds. The results present in this paper extended the results of Xu [25] from the classical linear space to Hadamard manifolds and the implicit iterative approximation theory is constructed firstly on Hadamard manifolds.

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