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STUDY OF IMPLICIT RELATION IN w-DISTANCE AND $(\eta, \theta, \mathcal{Z}, \phi)_\beta$-CONTRACTION IN wt-DISTANCE WITH AN APPLICATION

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To the memory of Professor Wataru Takahashi (January 22, 1944 - November 19, 2020)

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Abstract. In this paper, we study some new fixed point results in the context of w-distance as well as on wt-distance by using α-admissible mappings. Our results enhance and generalize several well known results available in the literature. We present examples to support our new findings. As an application, we utilize our results to obtain a solution of a nonlinear integral equation.
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1. Introduction and preliminaries

The notion of w-distance was first introduced by Kada *et al.*([33]). Many authors put their attentions on the said paper and proved several well known results in this set up (see [28], [42], [43], [45], [46]). Consequently, their work became very popular in fixed point theory. On the other site, the concept of w-distance was generalized in different way. One of the important generalization of w-distance is wt-distance over a b-metric space which can be found in [27]. In recent times, many interesting fixed point results have been established in the setting of b-metric spaces as well as in wt-distance (see e.g. [1], [6], [10], [11], [20], [26], [36], [37] and the references therein). In year 2012 the idea of α-admissible mappings was first initiated by Samet *et al.*([51]). This is another outstanding work in fixed point theory. Researchers utilized this concepts and generalized many results of fixed point theory (see [5], [7], [8], [9], [14],

[15], [24], [31], [35], [38], [40] and the references cited therein). In this paper, we establish some new kind of fixed point results in setting of w-distance as well as in wt-distance by using α-admissible mappings. Basically, the main aim of our work is to generalize and extend the main results of Liu *et al.* [47] into two directions. In the first direction, we have used the famous implicit functions (Theorem 2.1), introduced by Popa [49] by considering α-admissible mappings and altering distance functions in the setting of w-distance. α-admissible mappings are involved here as they combine many different structures such as structure of standard metric spaces, the structure of metric spaces endowed with a partial order, the structure of a metric spaces endowed with a graph, the structure of cyclic mappings via closed subsets of a metric space etc. Again, in the paper of Liu *et al.* [47], the authors have used integral type conditions. We know that any integral type function is always an altering distance function but not conversely. Motivated by this fact, we have utilized altering distance function inside the implicit function.

In the second direction, we introduce $(\eta, \theta, \mathcal{Z}, \phi)_\beta$-contraction in the context of wt-distance (Theorem 2.5) that we are going to discuss later.

Next, we move to define α-admissible mappings.

Definition 1.1. [51] Let X be a non-empty set. Suppose $\alpha : X \times X \to \mathbb{R}_+$ be a mapping. A mapping $J : X \to X$ is said to be α-admissible if the following satisfies:

$$\forall\ x, y\ \in X \text{ with } \alpha(x, y) \geq 1 \text{ implies } \alpha(Jx, Jy) \geq 1.$$

Very recently, the following definition was introduced by Shahi *et al.*([54]).

Definition 1.2. [54] Let X be a non-empty set. Let $I, J : X \to X$ and $\alpha : X \times X \to \mathbb{R}_+$ be three given mappings. Then J is said to be α-admissible w.r.t I if the following satisfies:

$$\forall\ x, y \in X \text{ with } \alpha(Ix, Iy) \geq 1 \text{ implies } \alpha(Jx, Jy) \geq 1.$$

From now, we write (I, J) to mean J is α-admissible w.r.t I and the collection of all such (I, J) is denoted by $\alpha_{\mathcal{A}}(X)$.

Definition 1.3. Let X be a non-empty set and δ be a positive real number with $\delta \in [1, \infty)$. Let $I, J : X \to X$ and $\alpha : X \times X \to \mathbb{R}_+$ be three given mappings. Then J is said to be α-admissible w.r.t I associated with δ if the following holds:

$$\forall\ x, y \in X \text{ with } \alpha(Ix, Iy) \geq \delta \text{ implies } \alpha(Jx, Jy) \geq \delta.$$

From now, we write $(I, J)_\delta$ to mean J is α-admissible w.r.t I associated with δ and the collection of all such $(I, J)_\delta$ is denoted by $\alpha^\delta_{\mathcal{A}}(X)$. First, we make an observation, that the class of $\alpha_{\mathcal{A}}(X)$ and $\alpha^\delta_{\mathcal{A}}(X)$ are totally independent. We have the following examples for verification where $X = \mathbb{R}_+$ and $\delta = 2$.

Example 1.1. Let $\alpha : X \times X \to \mathbb{R}_+$ and $I, J : X \to X$ be three given mappings defined by:

$$\alpha(x, y) = \begin{cases} \frac{\sqrt{x}+\sqrt{y}+5}{3} & \text{if } x, y \in [0, 1], \\ \frac{e^{x+y}}{2+e^{x+y}} & \text{otherwise.} \end{cases}$$

$$I(x) = \begin{cases} \frac{1+x}{2} & \text{if } x \in [0,1], \\ 3+x & \text{otherwise.} \end{cases}$$

$$J(x) = \begin{cases} \frac{x^6+1}{200} & \text{if } x \in [0,1], \\ 2+\sinh x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha(Ix, Iy) \geq 1 \Rightarrow \alpha(Jx, Jy) \geq 1$. Now, if we take $x = .4, y = .6$, then we have $\alpha(I(.4), I(.6)) > 2$ and $\alpha(J(.4), J(.6)) < 2$. Thus, the pair $(I, J) \in \alpha_{\mathcal{A}}(X)$ but $(I, J) \notin \alpha^2_{\mathcal{A}}(X)$. Hence $\alpha_{\mathcal{A}}(X) \nsubseteq \alpha^2_{\mathcal{A}}(X)$.

Now, we move to our second example.

Example 1.2. Let $\alpha : X \times X \to \mathbb{R}_+$ and $I, J : X \to X$ be three given mappings defined by:

$$\alpha(x,y) = \begin{cases} x+y & \text{if } x, y \in [1,2], \\ \min\left\{\frac{1}{4}, \frac{2|x-y|}{1+4|x-y|}\right\} & \text{otherwise.} \end{cases}$$

$$I(x) = \begin{cases} 1+\frac{x}{3} & \text{for } x \in [1,2], \\ \frac{3}{4}+\frac{x^2}{5} & \text{for } x \in [0,1), \\ \frac{1}{1+x} & \text{for } x \in (2,\infty). \end{cases}$$

$$J(x) = \begin{cases} 1+\ln(\frac{1}{2}+x) & \text{for } x \in [1,2], \\ \frac{x^2}{6}+2 & \text{for } x \in [0,1) \cup (2,\infty). \end{cases}$$

Clearly, $\alpha(Ix, Iy) \geq 2 \Rightarrow \alpha(Jx, Jy) \geq 2$. Now, if we take $x = .3, y = .2$, then we obtain $\alpha(I(.3), I(.2)) > 1$ and $\alpha(J(.3), J(.2)) < 1$. Consequently, the pair $(I, J)_2 \in \alpha^2_{\mathcal{A}}(X)$ but $(I, J)_2 \notin \alpha_{\mathcal{A}}(X)$. Hence, $\alpha^2_{\mathcal{A}}(X) \nsubseteq \alpha_{\mathcal{A}}(X)$, and our claim is justified.

Note. In general the mapping α may not be symmetric, i.e., $\alpha(x,y) \neq \alpha(y,x)$. To check this, consider the following example.

Example 1.3. Let $X = [0,3]$ with $\delta = 2$. Let $\alpha : X \times X \to \mathbb{R}_+$ and $I, J : X \to X$ be three given mappings defined by:

$$\alpha(x,y) = \begin{cases} x^2+y^3 & \text{if } x, y \in [2,3], \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

$$I(x) = \begin{cases} \sqrt{x} & \text{for } x \in [2,3], \\ x^2 & \text{otherwise.} \end{cases}$$

$$J(x) = \begin{cases} \sqrt[3]{x} & \text{for } x \in [2,3], \\ x^3 & \text{otherwise.} \end{cases}$$

Here J is α-admissible w.r.t I associated with 2. But α is not symmetric.

Next, we state the definition of wt-distance over a b-metric space and some related results.

Definition 1.4. ([22]) Let X be a non-empty set. Let $\rho_\delta : X \times X \mapsto [0, \infty)$ be a mapping which satisfies the following relations:

(1) $\rho_\delta(x, y) = 0 \Leftrightarrow x = y$;
(2) $\rho_\delta(x, y) = \rho_\delta(y, x), \forall x, y \in X$;
(3) $\rho_\delta(x, y) \leq \delta(\rho_\delta(x, z) + \rho_\delta(z, y))$, for any point $x, y, z \in X$ and for some $\delta \geq 1$.

Then the pair (X, ρ_δ) is called a $b-$metric space.

Definition 1.5. [27] Let (X, ρ_δ) be a $b-$metric space. Suppose $\{x_n\}$ is a sequence in X. Then,

(1) $x_n \to x (\in X),$ as $n \to \infty \Leftrightarrow \rho_\delta(x_n, x) \to 0$, as $n \to \infty$, and
(2) $\{x_n\}$ is Cauchy $\Leftrightarrow \rho_\delta(x_n, x_m) \to 0$, as $n, m \to \infty$.

(X, ρ_δ) is complete $\Leftrightarrow$ every Cauchy sequence in X is convergent. In the rest of the paper, we denote **cms** and **cbms** as **complete metric space** and **complete b-metric space**, respectively.

Definition 1.6. [18] Let (X, ρ_δ) and (X^*, ρ^*_δ) be two b-metric spaces, then:

(1) A function $J : X \mapsto X^*$ is said to be a $b-$continuous at a point $x \in X$, if for a sequence $x_n (\subseteq X)$ with $x_n \to x$, as $n \to \infty \Rightarrow J(x_n) \to J(x)$, as $n \to \infty$.

From now we write **lsc** to mean **lower semi continuous**.

Definition 1.7. [27] Let (X, ρ_δ) be a $b-$metric space with constant $\delta \geq 1$. Then a function $\omega_\delta : X \times X \mapsto \mathbb{R}^+$ is said to a $wt-$distance on X if the following holds:

(1) $\omega_\delta(x, y) \leq \delta(\omega_\delta(x, z) + \omega_\delta(z, y))$, for any point $x, y, z \in X$;
(2) for any $x \in X$, $\omega_\delta(x, \cdot) : X \mapsto \mathbb{R}^+$ is δ-lsc;
(3) for any $\epsilon > 0$, there exists $\nu > 0$ such that $\omega_\delta(x, y) \leq \nu$ and $\omega_\delta(x, z) \leq \nu \Rightarrow \rho_\delta(y, z) \leq \epsilon$.

Definition 1.8. [28] Let (X, ρ_δ) be a b-metric space. Then a function $J : X \mapsto \mathbb{R}$ is called $\delta-$lsc at a point $x_0 \in X$ if either $\underline{\lim}_{x_n \to x_0} J(x_n) = \infty$ or $J(x_0) \leq \underline{\lim}_{x_n \to x_0} \delta J(x_n)$, whenever $x_n \in X$ for every $n \in \mathbb{N}$ with $x_n \to x_0$.

Examples of wt-distance over b-metric space are given in [27].

Remark 1.1. Every $b-$metric space is wt-distance but the converse is not true.

Remark 1.2. If $\delta = 1$, then ω_δ becomes a w-distance. In that case, we denote "ω" instead of "ω_1" in w-distance. Hence, the concept of wt-distance is more general than the concept of w-distance.

Remark 1.3. Note that it had been shown in Example 3.4 [27], $\omega_\delta(x, y) = \omega_\delta(y, x)$ does not always satisfy for all $x, y \in X$, i.e., $wt-$distance is not symmetric.

The following important lemma will be used to prove our main results.

Lemma 1.1. [27] *Let (X, ρ_δ) be a $b-$metric space with constant $\delta \geq 1$ and ω_δ be a $wt-$distance on X. Consider two sequences $\{x_n\}$ and $\{t_n\}$ in X with $x, t, y \in X$. Let $\{\beta_n\}$ and $\{\gamma_n\}$ be two sequences in $[0, \infty)$ converging to 0. Then the following assertions hold.*

(l_1) *If* $\omega_\delta(x_n, t) \leq \beta_n$ *and* $\omega_\delta(x_n, y) \leq \gamma_n$ *for* $n \in \mathbb{N}$*, then* $t = y$*. In fact, if* $\omega_\delta(x, t) = 0$ *and* $\omega_\delta(x, y) = 0$*, then* $t = y$*.*
(l_2) *If* $\omega_\delta(x_n, t_n) \leq \beta_n$ *and* $\omega_\delta(x_n, y) \leq \gamma_n$ *hold for* $n \in \mathbb{N}$*, then* $t_n \to y (\in X)$*, as* $n \to \infty$*.*
(l_3) *If* $\omega_\delta(t, x_n) \leq \beta_n$ *holds for* $n \in \mathbb{N}$*, then* $\{x_n\}$ *is a Cauchy in* X*.*
(l_4) *If* $\omega_\delta(x_n, x_m) \leq \beta_n$ *holds for* $m > n$*, then* $\{x_n\}$ *is a Cauchy in* X*.*

Definition 1.9. [32] Suppose E, F are two self mappings defined on X. Then E, F are said to be weakly compatible if they commute at their coincidence point, i.e., if $Fx = Ex$ holds for some $x \in X$ then $FEx = EFx$.

Next, we state the definition of compatibility of two self mappings in a b-metric space.

Definition 1.10. ([3], [32]) Let (X, ρ_δ) be a b-metric space. Suppose C and D be two self mappings on X. We say that the pair (C, D) satisfies the compatibility condition in a b-metric space if and only if

$$\lim_{n\to\infty} \rho_\delta(CDx_n, DCx_n) = 0,$$

whenever the sequence $\{x_n\}(\subseteq X)$ satisfies $\lim_{n\to\infty} Cx_n = x = \lim_{n\to\infty} Dx_n$, for some $x \in X$.

Next, we move to the definition of altering distance functions and comparison functions.

Definition 1.11. [41] A function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an altering distance function if the following two conditions are satisfied:
(i) θ is strictly monotone non-decreasing and continuous;
(ii) $\theta(\tau) = 0$ if and only if $\tau = 0$.

From now we write Θ to denote the collection of all altering distance function.

Definition 1.12. [18] A function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if
(i) it is an increasing function;
(ii) $\mu^k(l) \to 0$ as $k \to \infty$, for all $l \in \mathbb{R}_+$.

Note. Every comparison function satisfies the following properties.
(i) $\mu(l) < l$ for $l > 0$, (ii) $\mu(0) = 0$, (iii) μ is continuous at 0.

Definition 1.13. [18] A function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a b-comparison function(with $\tau \geq 1$) if μ is a increasing function and there exists a $k_0 \in \mathbb{N}$, $\gamma \in (0, 1)$ and a convergent series of non-negative terms $\Sigma_{k=1}^{\infty} d_k$ such that

$$\tau^{k+1}\mu^{k+1}(l) \leq \gamma\tau^k\mu^k(l) + d_k \text{ for } k \geq k_0 \text{ and any } l \in \mathbb{R}_+.$$

From now, we write Δ to denote the class of all b-comparison functions.
Note. The series $\Sigma_{k=1}^{\infty} \tau^k\mu^k(l)$ converges for every $l \in \mathbb{R}_+$, whenever $\mu \in \Delta$. Furthermore, each b-comparison function is a comparison function (for details see [18]).

In this paper, our aim is to present some new kind of fixed point results which extend and generalize several well known results of fixed point theory including the very recent result of Liu *et al.*[47].

2. Main results

2.1. **Implicit function.** In this section, first we have investigated implicit function in the context of w-distance by using α-admissible mappings and altering distance functions. To do this, we consider the following implicit function (readers may also look into [16], [17], [49], [52], [53]).

Definition 2.1. Let Ω be the collection of all functions $G(\xi_1, \cdots, \xi_4) : \mathbb{R}_+^4 \to \mathbb{R}$ such that it satisfies the following properties:
(G_1) G is continuous in each variable and decreasing in ξ_3 variable;
(G_{2a}) for all $c, d \geq 0$ with $G(c, d, d, c) \leq 0$ there exists a $\mu \in \Delta$ such that $c \leq \mu(d)$;
(G_{2b}) for all $c, d \geq 0$ with $G(c, d, \tau, \tau) \leq 0$ implies $c \leq \mu(d)$ or $c \leq \mu(\tau)$ for all $\tau > 0$, where $\mu \in \Delta$;
(G_3) for all $\tau > 0$, we obtain $0 < G(\tau, \tau, 0, 0)$;
(G_4) G satisfies (G_γ) condition, i.e., if $G(\tau_1, \tau_2, 0, \tau_3) \leq 0$ for all $\tau_1, \tau_2, \tau_3 \geq 0$, there exists a $\gamma \in [0, 1)$ such that $\tau_1 \leq \gamma \max\{\tau_2, \tau_3\}$.

Now, we provide some examples of G.

Example 2.1. $G(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 - k\xi_2 - l\xi_3 - m\xi_4$, where $k, l, m \geq 0$ with $k+l+m < 1$.

Example 2.2. $G(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 - k\max\{\xi_2, \xi_3, \xi_4\}$, where $k \in [0, \frac{1}{2})$.

Example 2.3. $G(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1^2 - k\xi_2\xi_3 - l\xi_4^2$, where $k, l \geq 0$ with $k + l < 1$.

Example 2.4. $G(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 - k\xi_2 - l\xi_3 - m\max\{2\xi_4, \xi_1 + \xi_4\}$ where $k, l, m \geq 0$ with $k + l + 2m < 1$.

Example 2.5. $G(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1^2 - k\max\{\xi_2^2, \xi_3^2, \xi_4^2\} - l\max\{\xi_1\xi_3, \xi_2\xi_4\} - m\xi_3\xi_4$ where $k, l, m \geq 0$ with $k + l + m < 1$.

Example 2.6. $G(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1^3 - k\xi_1^2\xi_2 - l\xi_1\xi_3\xi_4 - m\xi_2\xi_3^2 - n\xi_3\xi_4^2$ where $k, l, m, n \geq 0$ with $k + l + m + n < 1$.

Example 2.7. $G(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 - k\max\{\xi_2, \frac{\xi_3 + \xi_4}{2}\}$ where $k \in [0, 1)$.

Example 2.8. $G(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 - (k\xi_2^n + l\xi_3^n + m\xi_4^n)^{\frac{1}{n}}$ where $k, l, m, n > 0$ with $k + l + m < 1$.

Before going to our main results, first we need the following lemma.

Lemma 2.1. *Let ω_δ be a wt-distance with weight δ over a b-metric space (X, ρ_δ). Let $I, J : X \to X$ be two mappings such that $J(X) \subseteq I(X)$. Further, let $\{x_r\}$ be a sequence in X such that $Jx_r = Ix_{r+1}$ for all $r \in \{0\} \cup \mathbb{N}$ with*

$$\lim_{r\to\infty} \omega_\delta(Ix_r, Ix_{r+1}) = 0, \lim_{r\to\infty} \omega_\delta(Ix_{r+1}, Ix_r) = 0. \tag{2.1}$$

If $\lim\limits_{s>r, r\to\infty} \omega_\delta(Ix_r, Ix_s) \neq 0$, then there exists $\tau > 0$ and two sub-sequences $\{r_t\}$ and $\{s_t\}$ of non-negative integers with $s_t > r_t > t$ such that

$$\frac{\tau}{\delta} \leq \limsup_{t\to\infty} \omega_\delta(Ix_{r_t}, Ix_{s_t}) \leq \delta\tau; \;\; \frac{\tau}{\delta} \leq \limsup_{t\to\infty} \omega_\delta(Ix_{r_t}, Ix_{s_t+1}) \leq \delta^2\tau;$$

$$\frac{\tau}{\delta} \le \limsup_{t\to\infty} \omega_\delta(Ix_{r_t+1}, Ix_{s_t}) \le \delta^2\tau; \frac{\tau}{\delta^2} \le \limsup_{t\to\infty} \omega_\delta(Ix_{r_t+1}, Ix_{s_t+1}) \le \delta^3\tau.$$

Proof. Let $\{Ix_r\}$ be a sequence in X such that $\lim_{s>r, r\to\infty} \omega_\delta(Ix_r, Ix_s) \neq 0$. Consequently, for all $M \in \mathbb{N}$ there exists a $\tau > 0$ such that $\omega_\delta(Ix_r, Ix_s) > \tau$ for all $s, r \in \mathbb{N}$ with $s > r > M$. Also, from (2.1), there exists a $m_0 \in \mathbb{N}$ such that

$$\omega_\delta(Ix_r, Ix_{r+1}) < \tau, \omega_\delta(Ix_{r+1}, Ix_r) < \tau, \text{ for all } r > m_0. \tag{2.2}$$

Now we pick up two sub sequences $\{x_{r_t}\}$ and $\{x_{s_t}\}$ such that $m_0 \le r_t < s_t < s_t + 1$ and

$$\omega_\delta(Ix_{r_t}, Ix_{s_t}) > \tau, \text{ for all } t. \tag{2.3}$$

Again, observe that,

$$\omega_\delta(Ix_{r_t}, Ix_{s_t-1}) \le \tau \text{ for all } t, \tag{2.4}$$

where s_t is taken as the smallest element $s \in \{r_t, r_t+1, r_t+2, \cdots,\}$ which satisfy (2.3). Here we observe that $r_t + 1 \le s_t$ for all t. Also, note that the situation $r_t + 1 = s_t$ is absurd because of (2.3), (2.4). Hence, we must have $r_t + 2 \le s_t$ for all t. Consequently, it gives that

$$r_t + 1 < s_t < s_t + 1, \text{ for all } t.$$

Now, applying triangle inequality and by using (2.3), (2.4), we get,

$$\begin{aligned} \tau &\le \omega_\delta(Ix_{r_t}, Ix_{s_t}) \\ &\le \delta[\omega_\delta(Ix_{r_t}, Ix_{s_t-1}) + \omega_\delta(Ix_{s_t-1}, Ix_{s_t})] \\ &\le \delta[\tau + \omega_\delta(Ix_{s_t-1}, Ix_{s_t})]. \end{aligned} \tag{2.5}$$

Considering lim sup as $t \to \infty$ in (2.5), and by using (2.1), we derive that

$$\tau \le \limsup_{t\to\infty} \omega_\delta(Ix_{r_t}, Ix_{s_t}) < \delta\tau. \tag{2.6}$$

From Definition 1.7, we obtain,

$$\omega_\delta(Ix_{r_t}, Ix_{s_t}) \le \delta[\omega_\delta(Ix_{r_t}, Ix_{s_t+1}) + \omega_\delta(Ix_{s_t+1}, Ix_{s_t})]. \tag{2.7}$$

Also, from Definition 1.7, we get,

$$\omega_\delta(Ix_{r_t}, Ix_{s_t+1}) \le \delta[\omega_\delta(Ix_{r_t}, Ix_{s_t}) + \omega_\delta(Ix_{s_t}, Ix_{s_t+1})]. \tag{2.8}$$

Considering lim sup as $t \to \infty$ in (2.7), (2.8) and by utilizing (2.1), (2.6), we derive that

$$\frac{\tau}{\delta} \le \limsup_{t\to\infty} \omega_\delta(Ix_{r_t}, Ix_{s_t+1}) \le \delta^2\tau. \tag{2.9}$$

Again, from Definition 1.7, we obtain that

$$\begin{aligned} \omega_\delta(Ix_{r_t}, Ix_{s_t}) &\le \delta[\omega_\delta(Ix_{r_t}, Ix_{r_t+1}) + \omega_\delta(Ix_{r_t+1}, Ix_{s_t})] \\ &\le \delta\omega_\delta(Ix_{r_t}, Ix_{r_t+1}) + \delta^2[\omega_\delta(Ix_{r_t+1}, Ix_{s_t+1}) + \omega_\delta(Ix_{s_t+1}, Ix_{s_t})], \end{aligned} \tag{2.10}$$

and

$$\omega_\delta(Ix_{r_t+1}, Ix_{s_t+1}) \le \delta[\omega_\delta(Ix_{r_t+1}, Ix_{r_t}) + \omega_\delta(Ix_{r_t}, Ix_{s_t+1})]. \tag{2.11}$$

Observe that by using (2.6), (2.10) and the following inequality

$$\omega_\delta(Ix_{r_t+1}, Ix_{s_t}) \le \delta[\omega_\delta(Ix_{r_t+1}, Ix_{r_t}) + \omega_\delta(Ix_{r_t}, Ix_{s_t})],$$

one can easily show that $\frac{\tau}{\delta} \leq \limsup_{t\to\infty} \omega_\delta(Ix_{r_t+1}, Ix_{s_t}) \leq \delta^2\tau$. Now, considering lim sup as $t \to \infty$ in (2.10), (2.11) and by utilizing (2.1), (2.9), we have

$$\frac{\tau}{\delta} \leq \limsup_{t\to\infty} \omega_\delta(Ix_{r_t+1}, Ix_{s_t+1}) \leq \delta^3\tau. \qquad \square$$

Note. The relations given in Lemma 2.1 for lim sup are also true for lim inf.

The following lemma is needed in the proof of our first main result.

Lemma 2.2. *Let ω be a w-distance over a metric space (X,ρ). Let $I, J : X \to X$ and α is a mapping from $X\times X$ into $\mathbb{R}_+$. Suppose that $C(I,J) \neq \emptyset$ and $\alpha(Il_1, Il_2) \geq 1$ for $l_1, l_2 \in C(I,J)$. Also, suppose that for all $x, y \in X$ with $\alpha(Ix, Iy) \geq 1$ the following relation holds:*

$$G(\theta(\omega(Jx, Jy)), \theta(\omega(Ix, Iy)), \theta(\omega(Ix, Jx)), \theta(\omega(Iy, Jy))) \leq 0, \tag{2.12}$$

where $G \in \Omega$, $\theta \in \Theta$. Then, the number of point of coincidence of the functions I and J is at most one.

Proof. Assume that $b_1 = Im = Jm$ and $b_2 = In = Jn$. Then, by our assumption $\alpha(Im, In) \geq 1$ and $\alpha(Im, Im) \geq 1$. Now, by using (2.12) with $\alpha(Im, Im) \geq 1$, we have

$$\begin{aligned} &G(\theta(\omega(Jm, Jm)), \theta(\omega(Im, Im)), \theta(\omega(Im, Jm)), \theta(\omega(Im, Jm))) \leq 0 \\ \Rightarrow\ &G(\theta(\omega(Im, Im)), \theta(\omega(Im, Im)), \theta(\omega(Im, Im)), \theta(\omega(Im, Im))) \leq 0. \end{aligned}$$

Now, by applying (G_{2a}) property, we have

$$\theta(\omega(Im, Im)) \leq \mu(\theta(\omega(Im, Im))).$$

It is clear from the above inequality that if $\theta(\omega(Im, Im)) > 0$, then we arrive at a contradiction. Thus we must have $\theta(\omega(Im, Im)) = 0$ which implies $\omega(Im, Im) = 0$. In a similar way one can show that $\omega(In, In) = 0$. Now, by using (2.12) with $\alpha(Im, In) \geq 1$, we have

$$\begin{aligned} &G(\theta(\omega(Jm, Jn)), \theta(\omega(Im, In)), \theta(\omega(Im, Jm)), \theta(\omega(In, Jn))) \leq 0 \\ &\Rightarrow G(\theta(\omega(Im, In)), \theta(\omega(Im, In)), \theta(\omega(Im, Im)), \theta(\omega(In, In))) \leq 0 \\ &\Rightarrow G(\theta(\omega(Im, In)), \theta(\omega(Im, In)), 0, 0) \leq 0. \end{aligned}$$

Thus, from (G_3), we have $\theta(\omega(Im, In)) = 0 \Rightarrow \omega(Im, In) = 0$. Now, by Lemma 1.1($l_1$), $\omega(Im, In) = 0$ and $\omega(Im, Im) = 0$ implies $Im = In$. Consequently,

$$b_1 = Jm = Im = In = Jn = b_2,$$

i.e., the number of point of coincidence of I and J is unique. $\square$

Now, we state our first main result on w-distance by using implicit relation through α-admissible mappings.

Theorem 2.1. *Let (X,ω) be a w-distance over a metric space (X,ρ). Let α be a mapping from the cross product of X into $\mathbb{R}_+$. Further, let I, J be two mappings from*

X into itself such that $J(X)\subseteq I(X)$. Suppose that for all $x,y\in X$ with $\alpha(Ix,Iy)\geq 1$ the following relation holds:

$$G(\theta(\omega(Jx,Jy)),\theta(\omega(Ix,Iy)),\theta(\omega(Ix,Jx)),\theta(\omega(Iy,Jy)))\leq 0, \tag{2.13}$$

where $G\in\Omega$, $\theta\in\Theta$. Also, suppose that the following conditions are satisfied:
(C1) there exists a x_0 such that $\alpha(Ix_0,Jx_0)\geq 1$ and $\alpha(Jx_0,Ix_0)\geq 1$;
(C2) J is $\alpha-$admissible w.r.t I, i.e., the pair $(I,J)\in\alpha_{\mathcal{A}}(X)$;
(C3) α has the transitivity property, i.e., $\alpha(x,y)\geq 1,\alpha(y,z)\geq 1$ implies $\alpha(x,z)\geq 1$;
(C4) I and J both are continuous mappings;
(C5) I,J are compatible mappings;
(C6) (X,ρ) is a complete metric space.
Then I and J have a coincidence point, i.e., $C(I,J)\neq\emptyset$.

Proof. By our assumption $(C1)$, there exists a point $x_0\in X$ such that $\alpha(Ix_0,Jx_0)\geq 1$. Since $J(X)\subseteq I(X)$, consequently we can always find a $x_1\in X$ such that $Jx_0=Ix_1$. Hence, we have $\alpha(Ix_0,Ix_1)\geq 1$. Again, by $(C2)$, and by using $J(X)\subseteq I(X)$, we can obtain $\alpha(Jx_0,Jx_1)=\alpha(Ix_1,Jx_1)\geq 1$. Since $J(X)\subseteq I(X)$, consequently we can find a $x_2\in X$ such that $\alpha(Ix_1,Ix_2)\geq 1$. Now if we continue in this way, then we can easily construct a sequence $\{x_r\}$ such that $Jx_r=Ix_{r+1}$ with $\alpha(Ix_r,Ix_{r+1})\geq 1$ for all $r\in\{0\}\cup\mathbb{N}$. Similarly, by using $\alpha(Jx_0,Ix_0)\geq 1$, we can show that $\alpha(Ix_{r+1},Ix_r)\geq 1$ for all $r\in\{0\}\cup\mathbb{N}$. If $Ix_r=Ix_{r+1}$ for some $r\in\mathbb{N}$, then we get $Ix_r=Jx_r$, i.e., I and J have a coincidence point at $x=x_r$. Hence, our proof is completed. Thus from here, we assume that $Ix_r\neq Ix_{r+1}$, for all $r\in\{0\}\cup\mathbb{N}$. First, observe that $A_r=\omega(Ix_r,Ix_{r+1})>0$ for all $n\in\{0\}\cup\mathbb{N}$. Suppose not, i.e., there exists a $r^{\#}\in\{0\}\cup\mathbb{N}$ such that $A_{r^\#}=\omega(Ix_{r^\#},Ix_{r^\#+1})=0$. Again, $\alpha(Ix_r,Ix_{r+1})\geq 1$ for all $r\in\{0\}\cup\mathbb{N}$. Then by taking $x=x_{r^\#},u=x_{r^\#+1}$ in (2.13), we have

$$G(\theta(\omega(Jx_{r^\#},Jx_{r^\#+1})),\theta(\omega(Ix_{r^\#},Ix_{r^\#+1})),\theta(\omega(Ix_{r^\#},Jx_{r^\#})),\theta(\omega(Ix_{r^\#+1},Jx_{r^\#+1})))\leq 0$$
$$\Rightarrow G(\theta(\omega(Ix_{r^\#+1},Ix_{r^\#+2})),\theta(\omega(Ix_{r^\#},Ix_{r^\#+1})),\theta(\omega(Ix_{r^\#},Ix_{r^\#+1})),\theta(\omega(Ix_{r^\#+1},Ix_{r^\#+2})))\leq 0.$$

Now, by applying (G_{2a}), we have

$$\omega(Ix_{r^\#+1},Ix_{r^\#+2})\leq\mu(\omega(Ix_{r^\#},Ix_{r^\#+1})). \tag{2.14}$$

But by our assumption $\omega(Ix_{r^\#},Ix_{r^\#+1})=0$. Correspondingly (2.14) implies the following

$$\omega(Ix_{r^\#+1},Ix_{r^\#+2})\leq\mu(\omega(Ix_{r^\#},Ix_{r^\#+1}))=\mu(0)=0.$$

Thus, we have $\omega(Ix_{r^\#+1},Ix_{r^\#+2})=0$. Next, we have the following

$$\omega(Ix_{r^\#},Ix_{r^\#+2})\leq\omega(Ix_{r^\#},Ix_{r^\#+1})+\omega(Ix_{r^\#+1},Ix_{r^\#+2}).$$

Putting the values of $\omega(Ix_{r^\#+1},Ix_{r^\#+2})$ and $\omega(Ix_{r^\#},Ix_{r^\#+1})$ in the above inequality, we get $\omega(Ix_{r^\#},Ix_{r^\#+2})=0$. Now, by applying Lemma 1.1(l_1) on $\omega(Ix_{r^\#},Ix_{r^\#+1})=0$ and $\omega(Ix_{r^\#},Ix_{r^\#+2})=0$, we obtain $Ix_{r^\#+1}=Ix_{r^\#+2}$, a contradiction. Consequently, we have $\omega(Ix_r,Ix_{r+1})>0$ for all $r\in\{0\}\cup\mathbb{N}$. Also, observe that no two consecutive terms of the sequence $\{\omega(Ix_{r+1},Ix_r)\}_{r=1}^{\infty}$ can not be equal with 0. Otherwise, it will contradict the assumption of $Ix_r\neq Ix_{r+1}$, for all r. Now, since $\alpha(Ix_r,Ix_{r+1})\geq 1$, considering $x=x_r$, $u=x_{r+1}$ in (2.13), we get

$$G(\theta(\omega(Jx_r,Jx_{r+1})),\theta(\omega(Ix_r,Ix_{r+1})),\theta(\omega(Ix_r,Jx_r)),\theta(\omega(Ix_{r+1},Jx_{r+1})))\leq 0$$
$$\Rightarrow G(\theta(\omega(Ix_{r+1},Ix_{r+2})),\theta(\omega(Ix_r,Ix_{r+1})),\theta(\omega(Ix_r,Ix_{r+1})),\theta(\omega(Ix_{r+1},Ix_{r+2})))\leq 0.$$

Now, by utilizing (G_{2a}), we have

$$\theta(\omega(Ix_{r+1}, Ix_{r+2})) \leq \mu(\theta(\omega(Ix_r, Ix_{r+1}))). \tag{2.15}$$

Clearly, from the above inequality we can conclude that $\{\omega(Ix_r, Ix_{r+1})\}_{r=1}^{\infty}$ is a decreasing sequence, bounded below by zero. Let $\lim_{r\to\infty} \omega(Ix_r, Ix_{r+1}) = a^*$. By considering limit as $r \to \infty$ in (2.15), we can show that $a^* = 0$. Next, we wish to show that $\lim_{r\to\infty} \omega(Ix_{r+1}, Ix_r) = 0$. Now, since $\alpha(Ix_{r+1}, Ix_r) \geq 1$, considering $x = x_{r+1}$, $u = x_r$ in (2.13), we get

$$G(\theta(\omega(Jx_{r+1}, Jx_r)), \theta(\omega(Ix_{r+1}, Ix_r)), \theta(\omega(Ix_{r+1}, Jx_{r+1})), \theta(\omega(Ix_r, Jx_r))) \leq 0$$
$$\Rightarrow G(\theta(\omega(Ix_{r+2}, Ix_{r+1})), \theta(\omega(Ix_{r+1}, Ix_r)), \theta(\omega(Ix_{r+1}, Ix_{r+2})), \theta(\omega(Ix_r, Ix_{r+1}))) \leq 0.$$

Now, since $\{\omega(Ix_r, Ix_{r+1})\}_{r=1}^{\infty}$ is a decreasing sequence and the function G is decreasing in it's third co-ordinate, i.e., we have

$$G(\theta(\omega(Ix_{r+2}, Ix_{r+1})), \theta(\omega(Ix_{r+1}, Ix_r)), \theta(\omega(Ix_r, Ix_{r+1})), \theta(\omega(Ix_r, Ix_{r+1}))) \leq 0.$$

Now, by applying (G_{2b}) condition with the fact that $\omega(Ix_r, Ix_{r+1}) > 0$ for all r, we have

$$\text{either } \theta(\omega(Ix_{r+2}, Ix_{r+1})) \leq \mu(\theta(\omega(Ix_{r+1}, Ix_r))) \ -(A)$$
$$\text{or } \theta(\omega(Ix_{r+2}, Ix_{r+1})) \leq \mu(\theta(\omega(Ix_r, Ix_{r+1}))) \ -(B).$$

Suppose (A) holds for all $r \in \{0\} \cup \mathbb{N}$, then

$$\theta(\omega(Ix_{r+2}, Ix_{r+1})) \leq \mu^r(\theta(\omega(Ix_1, Ix_0))).$$

Otherwise, suppose there exists a $r^* \in \{0\} \cup \mathbb{N}$ for which (B) holds. Then, we can derive the following

$$\theta(\omega(Ix_{r+2}, Ix_{r+1})) \leq \mu^r(\theta(\omega(Ix_0, Ix_1))), \text{ for all } r \geq r^*.$$

Thus, from any situation we get that $\theta(\omega(Ix_{r+2}, Ix_{r+1})) \to 0$ as $r \to \infty$ implies $\omega(Ix_{r+2}, Ix_{r+1}) \to 0$ as $r \to \infty$. Our next aim is to show that the sequence $\{Ix_r\}$ is a Cauchy sequence, i.e., $\lim_{s>r, r\to\infty} \omega(Ix_r, Ix_s) = 0$. Suppose not, i.e., $\lim_{s>r, r\to\infty} \omega(Ix_r, Ix_s) \neq 0$. Consequently, by Lemma 2.1, there exists $\tau > 0$ and two sub-sequences $\{r_t\}$ and $\{s_t\}$ of non-negative integers with $s_t > r_t > t$ such that

$$\lim_{t\to\infty} \omega(Ix_{r_t}, Ix_{s_t}) = \tau; \ \lim_{t\to\infty} \omega(Ix_{r_t}, Ix_{s_t+1}) = \tau; \ \lim_{t\to\infty} \omega(Ix_{r_t+1}, Ix_{s_t+1}) = \tau. \tag{2.16}$$

Now, by utilizing $(C3)$ with $\alpha(Ix_r, Ix_{r+1}) \geq 1$ for all r, we derive that $\alpha(Ix_r, Ix_s) \geq 1$ for all $r, s \in \mathbb{N}$ with $r < s$. Now, considering $x = x_{r_t}, u = x_{s_t}$ in (2.13), we obtain the following

$$G(\theta(\omega(Jx_{r_t}, Jx_{s_t})), \theta(\omega(Ix_{r_t}, Ix_{s_t})), \theta(\omega(Ix_{r_t}, Jx_{r_t})), \theta(\omega(Ix_{s_t}, Jx_{s_t}))) \leq 0$$
$$\Rightarrow G(\theta(\omega(Ix_{r_t+1}, Ix_{s_t+1})), \theta(\omega(Ix_{r_t}, Ix_{s_t})), \theta(\omega(Ix_{r_t}, Ix_{r_t+1})), \theta(\omega(Ix_{s_t}, Ix_{s_t+1}))) \leq 0.$$

Now, by applying continuity of G, θ with (2.16), and taking limit as $t \to \infty$, we obtain

$$G(\theta(\tau), \theta(\tau), 0, 0) \leq 0.$$

Clearly, above relation contradicts $(G3)$, since $\theta(\tau) > 0$. Thus we must have $\lim_{s>r, r\to\infty} \omega(Ix_r, Ix_s) = 0$. Hence, by Lemma 1.1$(l_4)$ $\{Ix_r\}$ is a Cauchy sequence.

Again, by our assumption (X, ρ) is a complete metric space. Consequently, there exists a $x^* \in X$ such that $Ix_r \to x^*$, i.e.,

$$\lim_{r\to\infty} \rho(Ix_r, x^*) = \lim_{r\to\infty} \rho(Jx_r, x^*).$$

Also, by using continuity of the mappings I, J, we get

$$\begin{aligned} &Ix_r \to x^* \text{ implies } J(Ix_r) \to Jx^*, \\ &Jx_r \to x^* \text{ implies } I(Jx_r) \to Ix^*. \end{aligned} \tag{2.17}$$

Consequently, we obtain

$$\begin{aligned} \rho(Ix^*, Jx^*) &\le [\rho(Ix^*, IJx_r) + \rho(IJx_r, Jx^*)] \\ &\le \rho(Ix^*, IJx_r) + [\rho(IJx_r, JIx_r) + \rho(JIx_r, Jx^*)]. \end{aligned} \tag{2.18}$$

Making $n \to \infty$ in (2.18), we have $\rho(Ix^*, Jx^*) = 0$ implies $Ix^* = Jx^*$. Therefore, x^* becomes a coincidence point of I and J. Hence, we get $\mathcal{C}(I, J) \neq \emptyset$. □

Next, we move to our second theorem, where we drop the assumption of continuity.

Theorem 2.2. *Suppose all the hypotheses of Theorem 2.1 are satisfied except* $(C4), (C5), (C6)$. *Suppose that the following two conditions are satisfied:*

$(C7)$ $\inf\{\omega(Ix, Ix^*) + \omega(Ix, Jx) : x \in X\} > 0$ *for every* x^* *with* $Ix^* \neq Jx^*$;

$(C8)$ (IX, ρ) *is a complete sub-space of* (X, ρ).

Then, I and J have a coincidence point, $\mathcal{C}(I, J) \neq \emptyset$.

Proof. Proceeding similarly as in Theorem 2.1, we can construct a sequence $\{Ix_r\}$ such that $Ix_{r+1} = Jx_r$ with $\{Ix_r\}$ is a Cauchy-sequence in (X, ρ). Since (IX, ρ) is a complete sub-space of (X, ρ), consequently there exists a $x^* \in X$ such that $Ix_r \to Ix^*$. Next, we wish to show that x^* is a coincidence point of I, J. We now show this by using the method contradiction, i.e., suppose $Ix^* \neq Jx^*$. Since $\{Ix_r\}$ is a Cauchy sequence, consequently for each $\sigma > 0$, we can always find a $L_\sigma \in \mathbb{N}$ such that for each $r > L_\sigma$, we get $\omega(Ix_{L_\sigma}, Ix_r) < \sigma$. Again, by using the definition of w-distance over metric space, we obtain

$$\omega(Ix_{L_\sigma}, Ix^*) < \liminf_{r\to\infty} \omega(Ix_{L_\sigma}, Ix_r) < \sigma,$$

which shows that $\omega(Ix_{L_\sigma}, Ix^*) < \sigma$. Now let us consider $\sigma = \frac{1}{s}$, where $s \in \mathbb{N}$ and $L_\sigma = r_s$. Then, we get

$$\lim_{s\to\infty} \omega(Ix_{r_s}, Ix^*) = 0.$$

Since we assume $Ix^* \neq Jx^*$, consequently by applying $(C7)$, we get

$$\begin{aligned} 0 &< \inf\{\omega(Ix, Ix^*) + \omega(Ix, Jx) : x \in X\} \\ &\le \inf\{\omega(Ix_{r_s}, Ix^*) + \omega(Ix_{r_s}, Jx_{r_s}) : r_s \in \mathbb{N}\} \\ &= \inf\{\omega(Ix_{r_s}, Ix^*) + \omega(Ix_{r_s}, Ix_{r_s+1}) : r_s \in \mathbb{N}\} \\ &\to 0 \text{ as } s \to \infty, \end{aligned}$$

which contradicts our assumption $(C7)$. Therefore, we must have $Ix^* = Jx^*$. Hence, our proof is completed. □

Our next theorem deals with the uniqueness of the point of coincidence of the functions I, J.

Theorem 2.3. *Suppose that all the hypotheses of Theorem 2.1 and Theorem 2.2 are satisfied. Also, suppose that if $\mathcal{C}(I,J) \neq \emptyset$, then $\alpha(Il_1, Il_2) \geq 1$ for all $l_1, l_2 \in \mathcal{C}(I,J)$. Then, the number of point of coincidence of the functions I, J is unique. Furthermore, if I and J are weakly compatible, then I and J have unique common fixed point x^* with $\omega(x^*, x^*) = 0$.*

Proof. From Theorem 2.1(respectively Theorem 2.2), it is clear that $\mathcal{C}(I,J) \neq \emptyset$. Also, by our assumption, we have $\alpha(Il_1, Il_2) \geq 1$ whenever $l_1, l_2 \in \mathcal{C}(I,J)$. Since, $l_1, l_2 \in X$ with $\alpha(Il_1, Il_2) \geq 1$, consequently the point (l_1, l_2) satisfy (2.13). Now, by applying Lemma 2.2, we obtain that the number of point of coincidence of the functions I and J is unique. Let x^* be the unique point of coincidence. Hence, there exists a $u \in X$ such that $x = Iu = Ju$. Again, we have the following,

$$Ix^* = IJu = JIu = Jx^*,$$

since I and J are weakly compatible. Thus, we have $x^* = Ix^* = Jx^*$, which shows that x^* is a fixed point of the mappings I, J and that is unique also. Hence, our proof is completed. □

2.1.1. *Well-posedness and limit shadowing property over w-distance by using implicit relation and α-admissible mappings.* In this section, we discuss the well-posedness property. To do this, first we recall a class of function, defined by Gordji *et al.*[25], $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that θ satisfies all the condition given in Definition 1.11 together with sub-additivity property, i.e., $\theta(t+s) \leq \theta(t) + \theta(s),$ for all $t, s \in \mathbb{R}_+$.
We write $\tilde{\Theta}$ to denote the above mentioned class of functions.
The study of well-posedness property for a fixed point problem, has become very popular among the researchers in the field of mathematics, see for example [12], [44], [48],[50]. Now we state the definition of well-posedness in generalize sense. Here, we will write "c.f.p" to mean "common fixed point".

Definition 2.2. Let (X, ω) be a w-distance over a metric space (X, ρ). Let $I, J : X \to X$ and $\alpha : X \times X \to \mathbb{R}_+$ be three given mappings. Then I and J are said to be well-posed in generalized sense w.r.t c.f.p problem if
(C_1) I and J have a unique c.f.p x^* in X;
(C_2) for any sequence $\{x_r\}$ in X with $\alpha(Ix^*, Ix_r) \geq 1$ such that

$$\begin{aligned} \lim_{r\to\infty} \omega(x_r, Ix_r) = \lim_{r\to\infty} \omega(Ix_r, x_r) = 0; \\ \lim_{r\to\infty} \omega(x_r, Jx_r) = \lim_{r\to\infty} \omega(Jx_r, x_r) = 0, \end{aligned} \tag{2.19}$$

implies $\lim_{r\to\infty} \rho(x_r, x^*) = 0$.

Next, we state the definition of limit shadowing property in generalized sense.

Definition 2.3. Let (X, ω) be a w-distance over a metric space (X, ρ). Let $I, J : X \to X$ and $\alpha : X \times X \to \mathbb{R}_+$ be three given mappings. Then I and J are said to satisfy limit shadowing property in generalized sense w.r.t c.f.p problem if for any sequence $\{x_r\}$ in X with $\alpha(Ix^*, Ix_r) \geq 1$ satisfying (2.19), implies that there exists a $k \in X$ such that $\lim_{r\to\infty} \rho(I^r k, x_r) = \lim_{r\to\infty} \rho(J^r k, x_r) = 0$.

Theorem 2.4. *Let (X, ω) be a w-distance over a metric space (X, ρ). Let $I, J : X \to X$ with $J(X) \subseteq I(X)$ and $\alpha : X \times X \to \mathbb{R}_+$ be three given mappings such that*

$$G(\theta(\omega(Jx, Jy)), \theta(\omega(Ix, Iy)), \theta(\omega(Ix, Jx)), \theta(\omega(Iy, Jy))) \leq 0,$$

for all $x, y \in X$ with $\alpha(Ix, Iy) \geq 1$, where $G \in \Omega$, $\theta \in \tilde{\Theta}$. Assume that all the hypotheses of Theorem 2.3 are satisfied. Then, I and J are said to satisfy well-posedness as well as limit shadowing property in generalized sense with respect to c.f.p problem.

Proof. Clearly Theorem 2.3 guarantees that the mappings I and J have a unique c.f.p. Let x^* be such unique c.f.p of I and J. Next, we consider a sequence $\{x_r\}$ in X such that (2.19) holds with $\alpha(Ix^*, Ix_r) \geq 1$ for all $r \in \mathbb{N}$. Then, by the condition of Theorem 2.4, we have

$$\begin{aligned} &G(\theta(\omega(Jx^*, Jx_r)), \theta(\omega(Ix^*, Ix_r)), 0, \theta(\omega(Ix_r, Jx_r))) \leq 0 \\ &\Rightarrow G(\theta(\omega(x^*, Jx_r)), \theta(\omega(x^*, Ix_r)), 0, \theta(\omega(Ix_r, Jx_r))) \leq 0. \end{aligned}$$

Now, by applying condition $(G4)$ of Definition 2.1, we have

$$\theta(\omega(x^*, Jx_r)) \leq \gamma \max\{\theta(\omega(x^*, Ix_r)), \theta(\omega(Ix_r, Jx_r))\}, \text{ where } \gamma \in (0, 1).$$

Again, we have,

$$\omega(x^*, x_r) \leq \omega(x^*, Jx_r) + \omega(Jx_r, x_r).$$

Since θ is non-decreasing and satisfies sub-additivity property, consequently we have,

$$\begin{aligned} &\theta(\omega(x^*, x_r)) \\ &\leq \theta(\omega(x^*, Jx_r) + \omega(Jx_r, x_r)) \\ &\leq \theta(\omega(x^*, Jx_r)) + \theta(\omega(Jx_r, x_r)) \\ &\leq \gamma[\theta(\omega(x^*, Ix_r)) + \theta(\omega(Ix_r, Jx_r))] + \theta(\omega(Jx_r, x_r)) \\ &\leq \gamma[\theta(\omega(x^*, x_r) + \omega(x_r, Ix_r)) + \theta(\omega(Ix_r, x_r) + \omega(x_r, Jx_r))] + \theta(\omega(Jx_r, x_r)) \\ &\leq \gamma\theta(\omega(x^*, x_r)) + \gamma\theta(\omega(x_r, Ix_r)) + \gamma\theta(\omega(Ix_r, x_r)) + \gamma\theta(\omega(x_r, Jx_r)) + \theta(\omega(Jx_r, x_r)). \end{aligned}$$

Thus, we have the following

$$\begin{aligned} \theta(\omega(x^*, x_r)) \leq &\frac{\gamma}{1-\gamma}\theta(\omega(x_r, Ix_r)) + \frac{\gamma}{1-\gamma}\theta(\omega(Ix_r, x_r)) + \frac{\gamma}{1-\gamma}\theta(\omega(x_r, Jx_r)) \\ &+ \frac{1}{1-\gamma}\theta(\omega(Jx_r, x_r)). \end{aligned}$$

Using (2.19), and making $r \to \infty$ in the above inequality, we obtain

$$\lim_{r\to\infty} \theta(\omega(x^*, x_r)) = 0 \Rightarrow \theta(\lim_{r\to\infty} \omega(x^*, x_r)) = 0, \text{ since } \theta \text{ is continuous.}$$

Again, by using property of θ, we get $\lim_{r\to\infty} \omega(x^*, x_r) = 0$. Also, from Theorem 2.3, we have $\omega(x^*, x^*) = 0$. Thus, from Lemma 1.1(l_2), we get $x_r \to x^*$ as $r \to \infty$, i.e., $\lim_{r\to\infty} \rho(x^*, x_r) = 0$. Hence, our proof is completed. □

2.2. $(\eta, \theta, \mathcal{Z}, \phi)_\beta$**-contraction.** Before going to our new theorems, first we introduce the following class of functions.
Let Φ denotes the collection of all such functions $\phi : \mathbb{R}_+^6 \to \mathbb{R}_+$ such that it satisfies the following conditions:

(1) ϕ is a continuous mapping;
(2) $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = 0$ if and only if $t_i = t_j$ for $i \neq j$ with $i, j \in \{1, 2, 3, 4, 5, 6\}$.

Next, we give some examples of functions which belongs to the class Φ. Here we take $K = \{1, 2, 3, 4, 5, 6\}$.

Example 2.9. $\phi_1(t_1, t_2, t_3, t_4, t_5, t_6) = \sinh(\Pi_{i,j \in K, i \neq j} |t_i - t_j|)$.

Example 2.10. $\phi_2(t_1, t_2, t_3, t_4, t_5, t_6) = L \min\{|t_i - t_j| : i, j \in K \text{ with } i \neq j\}$.

Example 2.11. $\phi_3(t_1, t_2, t_3, t_4, t_5, t_6) = e^{\Pi_{i,j \in K, i \neq j} |t_i - t_j|} - 1$.

Example 2.12. $\phi_4(t_1, t_2, t_3, t_4, t_5, t_6) = \cosh(\Pi_{i,j \in K, i \neq j} |t_i - t_j|) - 1$.

Example 2.13. $\phi_5(t_1, t_2, t_3, t_4, t_5, t_6) = ln(1 + \Pi_{i,j \in K, i \neq j} |t_i - t_j|)$.

From now we write $\phi_{\omega_\delta}^{(I,J)}[x, y]$ to mean

$$\phi_{\omega_\delta}^{(I,J)}[x, y] = \phi(\omega_\delta(Ix, Ix), \omega_\delta(Iy, Iy), \omega_\delta(Ix, Jx), \omega_\delta(Iy, Jy), \omega_\delta(Ix, Jy), \omega_\delta(Iy, Jx)),$$

where $x, y \in X$, (X, ω_δ) be a wt-distance over a b-metric space (X, ρ_δ) with $\phi \in \Phi$ and $I, J : X \to X$ be two given mappings.

In our next theorem (i.e. Theorem 2.5), we introduce a new kind of contraction (which is $(\eta, \theta, \mathcal{Z}, \phi)_\beta$-contraction) in the context of wt-distance over a b-metric space by involving the concept of α-admissible mappings associated with δ. Before going to that, first we give a brief mathematical background for constructing such $(\eta, \theta, \mathcal{Z}, \phi)_\beta$-contraction. Here, we first look into *Jaggi*-contraction [30], which is:

$$\begin{aligned} d(Jx, Jy) &\leq a_1 \frac{d(x, Jx) d(y, Jy)}{d(x, y)} + a_2 d(x, y) \\ &\leq (a_1 + a_2) \max\{d(x, y), d(y, Jy)\} \max\{\frac{d(x, Jx)}{d(x, y)}, 1\} \\ &\leq \beta \max\{d(x, y), d(x, Jx) d(y, Jy), d(x, Jy)\} \max\{\frac{d(x, Jx)}{d(x, y)}, 1\}, \end{aligned} \tag{2.20}$$

and next, we look into *Dass-Gupta* contraction[23], which is:

$$\begin{aligned} d(Jx, Jy) &\leq a_1 \frac{d(y, Jy)(1 + d(x, Jx))}{1 + d(x, y)} + a_2 d(x, y) \\ &\leq (a_1 + a_2) \max\{d(x, y), d(y, Jy)\} \max\{\frac{1 + d(x, Jx)}{1 + d(x, y)}, 1\} \\ &\leq \beta \max\{d(x, y), d(x, Jx) d(y, Jy), d(x, Jy)\} \max\{\frac{1 + d(x, Jx)}{1 + d(x, y)}, 1\}, \end{aligned} \tag{2.21}$$

where $\beta(= a_1 + a_2) \in (0,1)$. If we observe in (2.20), the second maximum term it is coming $\max\{\frac{d(x,Jx)}{d(x,y)}, 1\}$ for *Jaggi*-contraction and $\max\{\frac{1+d(x,Jx)}{1+d(x,y)}, 1\}$ for *Dass-Gupta* contraction in (2.21). This motivates us to construct some functions which we have taken as "$\mathcal{Z}(\tau, s)$" function in our next theorem (i.e. Theorem 2.5). Since we have to extend and generalize the main result of Liu *et al.* [47], where the authors have taken integral type functions. Now, if we apply integral type condition on *Jaggi* and *Dass-Gupta* contraction, then we have the following

$$\begin{aligned}&\int_0^{d(Jx,Jy)} \varphi(s)ds\\ &\leq \beta \max\{\int_0^{d(x,y)} \varphi(s)ds, \int_0^{d(x,Jx)} \varphi(s)ds, \int_0^{d(y,Jy)} \varphi(s)ds, \int_0^{d(x,Jy)} \varphi(s)ds\} \max\{\frac{d(x,Jx)}{d(x,y)}, 1\},\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}&\int_0^{d(Jx,Jy)} \varphi(s)ds\\ &\leq \beta \max\{\int_0^{d(x,y)} \varphi(s)ds, \int_0^{d(x,Jx)} \varphi(s)ds, \int_0^{d(y,Jy)} \varphi(s)ds, \int_0^{d(x,Jy)} \varphi(s)ds\} \max\{\frac{1+d(x,Jx)}{1+d(x,y)}, 1\},\end{aligned} \tag{2.23}$$

where φ is a Lebesgue integrable function. We know that every integral type function is an altering distance function but not conversely. Due to this fact, we have considered an altering distance function "θ" in the contraction of Theorem 2.5. Observe that in [13], authors have considered Berinde type generalized contraction. The contraction, given by inequality (9) in [13], is known as Berinde type contraction due to the addition of the last term which is "$LN(x,y)$", where $N(x,y) = \min\{d_m^p(x,Tx), d_m^p(y,Ty), d_m^p(x,Ty), d_m^p(y,Tx)\}$. Motivated by this, we have added the function "$\phi_{\omega_\delta}^{(I,J)}[x,y]$" in the $(\eta, \theta, \mathcal{Z}, \phi)_\beta$-contraction. One can see that we have given $\phi_2(t_1,t_2,t_3,t_4,t_5,t_6) = L\min\{|t_i - t_j| : i,j \in K \text{ with } i \neq j\}$ (Example 2.10) as an example of our newly introduced class of functions "Φ" which is inspired from "$LN(x,y)$" term. Now, we move to our next theorem.

Theorem 2.5. *Let (X,ω_δ) be a wt-distance over a b-metric space (X,ρ_δ). Let α be a mapping from the cross product of X into $\mathbb{R}_+$. Further, let I, J be two mappings from X into itself such that $J(X) \subseteq I(X)$. Suppose that for all $x, y \in X$ with $\alpha(Ix,Iy) \geq \delta$ the following relation holds:*

$$\begin{aligned}\eta(\theta(\delta^\varrho \omega_\delta(Jx,Jy))) \leq &\beta\eta(\max\{\theta(\omega_\delta(Ix,Iy)), \theta(\omega_\delta(Ix,Jx)), \theta(\omega_\delta(Iy,Jy)), \theta(\frac{\omega_\delta(Ix,Jy)}{2\delta})\})\cdot\\ &\max\{\mathcal{Z}(\omega_\delta(Ix,Jx), \omega_\delta(Ix,Iy)), 1\} + \phi_{\omega_\delta}^{(I,J)}[x,y],\end{aligned} \tag{2.24}$$

where $\eta, \theta \in \Theta$, $\beta \in [0,1)$, $\varrho > 1$, $\phi \in \Phi$ and $\mathcal{Z}$ is a continuous function from $\mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ such that $\mathcal{Z}(\tau, s) \leq 1$ for all $\tau, s \in \mathbb{R}_+$ with $\tau \leq s$. Also, suppose that the following conditions are satisfied:

$(D1)$ there exists a x_0 such that $\alpha(Ix_0,Jx_0) \geq \delta$ and $\alpha(Jx_0,Ix_0) \geq \delta$;

$(D2)$ J is $\alpha-$admissible w.r.t I associated with δ, i.e., the pair $(I,J) \in \alpha_{\mathcal{A}}^\delta(X)$;

$(D3)$ α has the transitivity property, i.e., $\alpha(x,y) \geq \delta, \alpha(y,z) \geq \delta$ implies $\alpha(x,z) \geq \delta$;

$(D4)$ I and J both are continuous mappings;
$(D5)$ I, J are compatible mappings;
$(D6)$ (X, ρ_δ) is a complete b-metric space.
Then, I and J have a coincidence point, i.e., $\mathcal{C}(I, J) \neq \emptyset$.

Proof. From condition $(D1)$ there exist a x_0 such that $\alpha(Ix_0, Jx_0) \geq \delta$ and $\alpha(Jx_0, Ix_0) \geq \delta$. Similarly as in Theorem 2.1 we can show that $\alpha(Ix_r, Ix_{r+1}) \geq \delta$ and $\alpha(Ix_{r+1}, Ix_r) \geq \delta$ for all $r \in \mathbb{N}$. Also, by using $(D3)$ we can show that $\alpha(Ix_r, Ix_s) \geq \delta$ for all $r, s \in \mathbb{N}$. As in Theorem 2.1, we can show that $Ix_r \neq Ix_{r+1}$, for all $r \in \mathbb{N}$, otherwise our proof is completed. First, observe that $A_r = \omega_\delta(Ix_r, Ix_{r+1}) > 0$ for all $r \in \mathbb{N}$. If not, i.e., there exists a $r^* \in \mathbb{N}$ such that $A_{r^*} = \omega_\delta(Ix_{r^*}, Ix_{r^*+1}) = 0$. Again, using $\alpha(Ix_r, Ix_{r+1}) \geq \delta$, and taking $x = x_{r^*}, u = x_{r^*+1}$ in (2.24), we obtain

$$\begin{aligned}
&\eta(\theta(\delta^{\varrho}\omega_\delta(Jx_{r^*}, Jx_{r^*+1}))) \\
&\leq \beta\eta(\max\{\theta(\omega_\delta(Ix_{r^*}, Ix_{r^*+1})), \theta(\omega_\delta(Ix_{r^*}, Jx_{r^*})), \theta(\omega_\delta(Ix_{r^*+1}, Jx_{r^*+1})), \theta(\frac{\omega_\delta(Ix_{r^*}, Jx_{r^*+1})}{2\delta})\}) \cdot \\
&\max\{\mathcal{Z}(\omega_\delta(Ix_{r^*}, Jx_{r^*}), \omega_\delta(Ix_{r^*}, Ix_{r^*+1})), 1\} + \phi^{(I,J)}_{\omega_\delta}[x_{r^*}, x_{r^*+1}].
\end{aligned}$$

After doing some simple calculation, and keeping in mind that $\eta, \theta \in \Theta$, $\phi \in \Phi$, $\mathcal{Z}(\tau, s) \leq 1$ for $\tau \leq s$, and $\frac{\omega_\delta(Ix_{r^*}, Ix_{r^*+2})}{2\delta} \leq \omega_\delta(Ix_{r^*+1}, Ix_{r^*+2})$, we can derive the following

$$\eta(\theta(\delta^{\varrho}\omega_\delta(Ix_{r^*+1}, Ix_{r^*+2}))) \leq \beta\eta(\theta(\omega_\delta(Ix_{r^*+1}, Ix_{r^*+2}))).$$

Clearly, we get a contradiction, if we assume $\theta(\omega_\delta(Ix_{r^*+1}, Ix_{r^*+2})) > 0$. Thus, we have $\omega_\delta(Ix_{r^*+1}, Ix_{r^*+2}) = 0$. Now, by applying same argument that have been given in Theorem 2.1 to show $Ix_{r^*+1} = Ix_{r^*+2}$, a contradiction. Also, no two consecutive terms of $\{\omega_\delta(Ix_{r+1}, Ix_r)\}$ can not be equal with zero, similar type argument follows from Theorem 2.1. Now we divide the proof the proof into two cases.

Case 1. In this case we consider $\delta = 1$. Here we divide the proof into several steps.
Step 1. First, we show that $\lim_{r\to\infty} \omega_\delta(Ix_r, Ix_{r+1}) = 0$.
Now, we have $\alpha(Ix_r, Ix_{r+1}) \geq 1$, for all $r \in \mathbb{N}$. Thus, considering $x = x_r, u = x_{r+1}$ in (2.24), we have

$$\begin{aligned}
&\eta(\theta(\omega(Jx_r, Jx_{r+1}))) \\
&\leq \beta\eta(\max\{\theta(\omega(Ix_r, Ix_{r+1})), \theta(\omega(Ix_r, Jx_r)), \theta(\omega(Ix_{r+1}, Jx_{r+1})), \theta(\frac{\omega(Ix_r, Jx_{r+1})}{2})\}) \cdot \\
&\max\{\mathcal{Z}(\omega(Ix_r, Jx_r), \omega(Ix_r, Ix_{r+1})), 1\} + \phi^{(I,J)}_{\omega}[x_r, x_{r+1}].
\end{aligned} \tag{2.25}$$

Again, we have

$$\theta(\frac{\omega(Ix_r, Jx_{r+1})}{2}) \leq \theta(\max\{\omega(Ix_r, Ix_{r+1}), \omega(Ix_{r+1}, Jx_{r+2})\}),$$

and $\phi^{(I,J)}_{\omega}[x_r, x_{r+1}] = 0$. Thus, from (2.25) we have

$$\eta(\theta(\omega(Ix_{r+1}, Ix_{r+2}))) \leq \beta\eta(\max\{\theta(\omega(Ix_r, Ix_{r+1})), \theta(\omega(Ix_{r+1}, Ix_{r+2}))\}) \ (A_1).$$

If $\max\{\theta(\omega(Ix_r, Ix_{r+1})), \theta(\omega(Ix_{r+1}, Ix_{r+2})\} = \theta(\omega(Ix_{r+1}, Ix_{r+2})$, then we get a contradiction from (A_1). Thus, from (A_1) we obtain,

$$\eta(\theta(\omega(Ix_{r+1}, Ix_{r+2}))) \leq \beta\eta(\theta(\omega(Ix_r, Ix_{r+1}))) < \eta(\theta(\omega(Ix_r, Ix_{r+1}))) \ (A_2).$$

Thus, from (A_2), we have $\{\omega(Ix_r, Ix_{r+1})\}_{r=0}^{\infty}$ is a non-increasing sequence, and one can easily show that $\lim_{r\to\infty} \omega(Ix_r, Ix_{r+1}) = 0$.

Step 2. Our next step is to show that $\lim_{r\to\infty} \omega(Ix_{r+1}, Ix_r) = 0$, and we use method of contradiction to prove it. First we show that $\liminf_{r\to\infty} \omega(Ix_{r+1}, Ix_r) = 0$. Let us assume on the contrary, i.e., $\liminf_{r\to\infty} \omega(Ix_{r+1}, Ix_r) = b^* > 0$. Next, we put $\omega(Ix_{r+1}, Ix_r) = b_r$. First, suppose that $b^* \in (0, \infty)$. Now, if we put $x = x_{r+1}, u = x_r$ in (2.24), then we have

$$\begin{aligned}
&\eta(\theta(\omega(Ix_{r+2}, Ix_{r+1})))\\
&\leq \beta\eta(\max\{\theta(\omega(Ix_{r+1}, Ix_r)), \theta(\omega(Ix_{r+1}, Ix_{r+2})), \theta(\omega(Ix_r, Ix_{r+1})), \theta(\frac{\omega(Ix_{r+1}, Ix_{r+1})}{2})\})\cdot\\
&\max\{\mathcal{Z}(\omega(Ix_{r+1}, Ix_{r+2}), \omega(Ix_{r+1}, Ix_r)), 1\} + \phi_\omega^{(I,J)}[x_{r+1}, x_r].
\end{aligned}$$

Since $\{\omega(Ix_r, Ix_{r+1})\}_{r=0}^{\infty}$ is a decreasing sequence, θ is a non-decreasing function and utilizing the property of ϕ, we obtain

$$\begin{aligned}
\eta(\theta(\omega(Ix_{r+2}, Ix_{r+1}))) \leq &\beta\eta(\max\{\theta(\omega(Ix_{r+1}, Ix_r)), \theta(\omega(Ix_r, Ix_{r+1}))\})\cdot\\
&\max\{\mathcal{Z}(\omega(Ix_{r+1}, Ix_{r+2}), \omega(Ix_{r+1}, Ix_r)), 1\} \ (A_3).
\end{aligned}$$

Now we consider lim inf in both sides of (A_3), we get

$$\begin{aligned}
&\eta(\theta(\liminf_{r\to\infty} \omega(Ix_{r+2}, Ix_{r+1})))\\
&\leq \beta\eta(\max\{\theta(\liminf_{r\to\infty} \omega(Ix_{r+1}, Ix_r)), \theta(\liminf_{r\to\infty} \omega(Ix_r, Ix_{r+1}))\})\cdot\\
&\max\{\mathcal{Z}(\liminf_{r\to\infty} \omega(Ix_{r+1}, Ix_{r+2}), \liminf_{r\to\infty} \omega(Ix_{r+1}, Ix_r)), 1\} \ (A_4).
\end{aligned}$$

Clearly, from (A_4), we have

$$\begin{aligned}
&\eta(\theta(b^*)) \leq \beta\eta(\max\{\theta(b^*), \theta(0)\}) \cdot \max\{\mathcal{Z}(0, b^*), 1\}\\
&\Rightarrow \eta(\theta(b^*)) \leq \beta\eta(\theta(b^*)),
\end{aligned}$$

which is a contradiction. Hence $b^* \notin (0, \infty)$. Next, we suppose that $b^* = \infty$. Consequently, there exists a sub-sequence $\{b_{r_i}\}$ of $\{b_r\}$ such that $\lim_{r\to\infty} b_{r_i} = \infty$ with $b_{r_i} > \omega(Ix_{r_i}, Ix_{r_i+1})$, for $i = 1, 2, 3, \cdots$, since $\lim_{r\to\infty} \omega(Ix_r, Ix_{r+1}) = 0$. Also, $b_{r_i} > \omega(Ix_{r_i}, Ix_{r_i+1})$ implies $b_{r_i} > \omega(Ix_{r_i+1}, Ix_{r_i+2})$, since $\{\omega(Ix_r, Ix_{r+1})\}_{r=0}^{\infty}$ is a decreasing sequence. Thus, from (A_3), we have

$$\begin{aligned}
\eta(\theta(\omega(Ix_{r_i+2}, Ix_{r_i+1}))) \leq &\beta\eta(\max\{\theta(\omega(Ix_{r_i+1}, Ix_{r_i})), \theta(\omega(Ix_{r_i}, Ix_{r_i+1}))\})\cdot\\
&\max\{\mathcal{Z}(\omega(Ix_{r_i+1}, Ix_{r_i+2}), \omega(Ix_{r_i+1}, Ix_{r_i})), 1\}.
\end{aligned}$$

Consequently, we have

$$\omega(Ix_{r_i+2}, Ix_{r_i+1}) \leq \omega(Ix_{r_i+1}, Ix_{r_i}) \ (A_5).$$

Thus, from (A_5) it is clear that $\lim_{r\to\infty} \omega(Ix_{r_i+2}, Ix_{r_i+1}) = \infty$ otherwise if it has a sub-sequence whose sub-sequential limit is $l \in [0, \infty)$, then it will contradicts the fact that $\liminf_{r\to\infty} b_r = \infty$. Again, inequality (A_5) can be written as

$$\omega(Ix_{r_i+2}, Ix_{r_i+1}) - \omega(Ix_{r_i+1}, Ix_{r_i}) \leq 0.$$

Now, taking $i \to \infty$ in the above inequality, we obtain $\infty - \infty \leq 0$, a meaningless expression. Thus $\liminf_{r\to\infty} b_r \neq \infty$. Consequently, we must have $\liminf_{r\to\infty} b_r = 0$.

Step 3. Next we show that $\limsup_{r\to\infty} b_r = 0$. Let us assume on the contrary, i.e., $\limsup_{r\to\infty} b_r > 0$. Let D denotes the collection of all sub-sequential limits whose value is > 0 and $\sup D = \limsup_{r\to\infty} b_r$. Now, for $\epsilon > 0$, we can always construct a sub-sequence of $\{b_r\}$, say $\{b_{r_m}\}$, such that each term of $\{b_{r_m}\}$ belongs to ϵ-neighbourhood 0 since $\liminf_{r\to\infty} b_r = 0$, whereas each term of $\{b_{r_m+1}\}$ lies in the ϵ-neighbourhood of a point, which belongs to D for all $m \in \mathbb{N}$. Now putting $x = x_{r_m+1}, u = x_{r_m}$ in (2.24), we have

$$\begin{aligned}&\eta(\theta(\omega(Ix_{r_m+2}, Ix_{r_m+1})))\\ &\leq \beta\eta(\max\{\theta(\omega(Ix_{r_m+1}, Ix_{r_m})), \theta(\omega(Ix_{r_m+1}, Ix_{r_m+2})), \theta(\omega(Ix_{r_m}, Ix_{r_m+1})),\\ &\theta(\frac{\omega(Ix_{r_m+1}, Ix_{r_m+1})}{2})\}) \cdot \max\{\mathcal{Z}(\omega(Ix_{r_m+1}, Ix_{r_m+2}), \omega(Ix_{r_m+1}, Ix_{r_m})), 1\}\\ &+ \phi_\omega^{(I,J)}[x_{r_m+1}, x_{r_m}].\end{aligned}$$

Since $\{\omega(Ix_r, Ix_{r+1})\}_{r=0}^\infty$ is a decreasing sequence, θ is a non-decreasing function and utilizing the property of ϕ, we obtain

$$\begin{aligned}\eta(\theta(\omega(Ix_{r_m+2}, Ix_{r_m+1}))) \leq &\beta\eta(\max\{\theta(\omega(Ix_{r_m+1}, Ix_{r_m})), \theta(\omega(Ix_{r_m}, Ix_{r_m+1}))\})\cdot\\ &\max\{\mathcal{Z}(\omega(Ix_{r_m+1}, Ix_{r_m+2}), \omega(Ix_{r_m+1}, Ix_{r_m})), 1\}.\end{aligned}$$

Clearly, above inequality can be written as the following

$$\begin{aligned}&\eta(\theta(b_{r_m+1}))\\ &\leq \beta\eta(\max\{\theta(b_{r_m}), \theta(\omega(Ix_{r_m}, Ix_{r_m+1}))\}) \cdot \max\{\mathcal{Z}(\omega(Ix_{r_m+1}, Ix_{r_m+2}), b_{r_m}), 1\} \quad (A_6).\end{aligned}$$

Now as $m \to \infty$ in (A_6) observe that L.H.S of (A_6) is always > 0, whereas R.H.S tends to 0. Clearly, this is a contradiction to our assumption, i.e., $\limsup_{r\to\infty} b_r > 0$. Thus, we must have $\limsup_{r\to\infty} b_r = 0$. Hence, we conclude that $\lim_{r\to\infty} b_r = 0$.

Step 4. In this step, we show that $\{Ix_r\}$ is a Cauchy sequence. To do this we use method of contradiction. Suppose that $\{Ix_r\}$ is not a Cauchy sequence. Then, there exists $\tau > 0$ for which we can find two sub-sequence $\{r_t\}$ and $\{s_t\}$ with $s_t > r_t > t$ such that $\omega(Ix_{r_t}, Ix_{s_t}) \geq \tau$ and $\omega(Ix_{r_t}, Ix_{s_t-1}) < \tau$. Now we put $x = x_{r_t}, u = x_{s_t}$ in (2.24), we obtain

$$\begin{aligned}\eta(\theta(\omega(Ix_{r_t+2}, Ix_{s_t+1}))) \leq &\beta\eta(\max\{\theta(\omega(Ix_{r_t}, Ix_{s_t})), \theta(\omega(Ix_{r_t}, Ix_{r_t+1})), \theta(\omega(Ix_{s_t}, Ix_{s_t+1})),\\ &\theta(\frac{\omega(Ix_{r_t}, Ix_{s_t+1})}{2})\}) \cdot \max\{\mathcal{Z}(\omega(Ix_{r_t}, Ix_{r_t+1}), \omega(Ix_{r_t}, Ix_{s_t})), 1\}\\ &+ \phi_\omega^{(I,J)}[x_{r_t}, x_{s_t}].\end{aligned} \tag{2.26}$$

Now if we put $\delta = 1$ in Lemma 2.1, then we obtain

$$\lim_{t\to\infty} \omega(Ix_{r_t}, Ix_{s_t}) = \tau;\ \lim_{t\to\infty} \omega(Ix_{r_t}, Ix_{s_t+1}) = \tau;\ \lim_{t\to\infty} \omega(Ix_{r_t+1}, Ix_{s_t+1}) = \tau. \tag{2.27}$$

Also, note that $\lim_{r\to\infty} \omega(Ix_r, Ix_r) = 0$, since

$$\omega(Ix_r, Ix_r) \leq \omega(Ix_r, Ix_{r+1}) + \omega(Ix_{r+1}, Ix_r).$$

Consequently, making $t \to \infty$ in (2.26), we have

$$\eta(\theta(\tau)) \leq \beta\eta(\theta(\tau)),$$

a contradiction as $\beta \in [0,1)$. Thus $\{Ix_r\}$ is a Cauchy sequence by Lemma 1.1(l_4).
Case 2. In this case, we consider $\delta > 1$. Here we take $x = x_0, u = x_1$ in (2.24), we have

$$\begin{aligned}&\eta(\theta(\delta^\varrho \omega_\delta(Jx_0, Jx_1)))\\ &\leq \beta\eta(\max\{\theta(\omega_\delta(Ix_0, Ix_1)), \theta(\omega_\delta(Ix_0, Jx_0)), \theta(\omega_\delta(Ix_1, Jx_1)), \theta(\frac{\omega_\delta(Ix_0, Jx_1)}{2\delta})\})\cdot\\ &\max\{\mathcal{Z}(\omega_\delta(Ix_0, Jx_0), \omega_\delta(Ix_0, Ix_1)), 1\} + \phi^{(I,J)}_{\omega_\delta}[x_0, x_1].\end{aligned}$$

From the above inequality, we have

$$\eta(\theta(\delta^\varrho \omega_\delta(Ix_1, Ix_2))) \leq \beta\eta(\max\{\theta(\omega_\delta(Ix_0, Ix_1)), \theta(\omega_\delta(Ix_1, Ix_2))\}). \tag{2.28}$$

Clearly, if $\max\{\theta(\omega_\delta(Ix_0, Ix_1)), \theta(\omega_\delta(Ix_1, Ix_2))\} = \theta(\omega_\delta(Ix_1, Ix_2))$, then we arrive at a contradiction from (2.28). Thus, from (2.28), we obtain

$$\omega_\delta(Ix_1, Ix_2) \leq \frac{1}{\delta^\varrho}\omega_\delta(Ix_0, Ix_1) = \sigma\omega_\delta(Ix_0, Ix_1), \text{ putting } \sigma = \frac{1}{\delta^\varrho} \in [0,1).$$

Continuing in this way, one can show that

$$\begin{aligned}&\omega_\delta(Ix_n, Ix_{n+1})\\ &\leq \sigma\omega_\delta(Ix_{n-1}, Ix_n) \leq \sigma^2\omega_\delta(Ix_{n-2}, Ix_{n-1}) \leq \cdots \leq \sigma^n\omega_\delta(Ix_0, Ix_1)\ (A_7).\end{aligned}$$

Now, by using (A_7) together with triangular inequality of Definition 1.7, one can easily show that $\{Ix_r\}_{r=0}^\infty$ is a Cauchy sequence. Thus from any case we have $\{Ix_r\}_{r=0}^\infty$ is a Cauchy sequence. Since (X, ρ_δ) is a complete b-metric space, consequently there exists a $x^* \in X$ such that $Ix_r \to x^*$ as $r \to \infty$. Now we proceed exactly in a similar way using triangular inequality for b-metric space, we can show that $Ix^* = Jx^*$. Hence, we have $\mathcal{C}(I, J) \neq \emptyset$. □

Next, we state a theorem without proof where we drop the assumption of continuity.

Theorem 2.6. *Suppose all the hypotheses of Theorem 2.5 are satisfied except* $(D4), (D5), (D6)$. *Suppose that the following two conditions are satisfied:*

- $(D7)$ $\inf\{\omega_\delta(Ix, Ix^*) + \omega_\delta(Ix, Jx) : x \in X\} > 0$ *for every* x^* *with* $Ix^* \neq Jx^*$;
- $(D8)$ (IX, ρ_δ) *is a complete sub-space of* (X, ρ_δ).

Then, I and J have a coincidence point, i.e., $\mathcal{C}(I, J) \neq \emptyset$.

3. Caristi type (α-Ω-$\mathcal{Z}$)-contraction

Motivated by the result of Caristi[19], in this section we introduce another new kind of contraction, i.e., Caristi type (α-Ω-$\mathcal{Z}$)-contraction in the setting of wt-distance by using α-admissible mappings. Authors studied Caristi type fixed point theorem in different context (see, for example, [4], [2], [29], [34], [39]).

Definition 3.1. Let J be a given mapping from X into X and α be a mapping from X^2 into $\mathbb{R}_+$. Also, suppose that $\mathcal{Z}$ is a continuous mapping from $\mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ such that $\mathcal{Z}(\tau, s) \leq 1$ for all $\tau, s \in \mathbb{R}_+$ with $\tau \leq s$ and Ω be a mapping from

X into $\mathbb{R}_+$. Then, J is said to be generalized Caristi type (α-Ω-$\mathcal{Z}$)-contraction over wt-distance ω_δ if $\omega_\delta(x, Jx) > 0$ implies

$$\omega_\delta(Jx, Jy) \leq (\Omega(x) - \Omega(Jx))E(x, y)\max\{\mathcal{Z}(\omega_\delta(x, Jx), \omega_\delta(x, y)), 1\},$$

for all $x, y \in X$ with $\alpha(x, y) \geq \delta$, where $E(x, y) = \max\{\omega_\delta(x, y), \omega_\delta(x, Jx), \omega_\delta(y, Jy)\}$.

Theorem 3.1. *Let (X, ω_δ) be a wt-distance over a complete b-metric space (X, ρ_δ). Let $J : X \to X$ be a generalized Caristi type (α-Ω-$\mathcal{Z}$)-contraction. Suppose that the following conditions are satisfied:*
(G_1) *J is α-admissible w.r.t δ;*
(G_2) *there exists a $x_0 \in X$ such that $\alpha(x_0, Jx_0) \geq \delta$ and $\omega_\delta(J^r x_0, J^r x_0) = 0$ for all $r \in \{0\} \cup \mathbb{N}$;*
(G_3) *either J is continuous or*
(G_4) $\inf\{\omega_\delta(x, y) + \omega_\delta(x, Jx) : x \in X\} > 0$ *for every y with $y \neq Jy$.*
Then, J has a fixed point x^* in X. Furthermore, if $\alpha(x^*, x^*) \geq \delta$ then $\omega_\delta(x^*, x^*) = 0$.

Proof. By condition (G_2) there exists a point $x_0 \in X$ such that $\alpha(x_0, Jx_0) \geq \delta$. Also, by (G_2), we can easily define a sequence $\{x_r\}$ such that $x_{r+1} = Jx_r = J^{r+1}x_0$ with $\omega_\delta(J^r x_0, J^r x_0) = 0$ for all $r \in \{0\} \cup \mathbb{N}$. Suppose there exists a r^* such that $x_{r^*} = x_{r^*+1}$, then $x_{r^*} \in X$ is a fixed point of J, and we have nothing to show. Hence, from now now, we assume that $x_r \neq x_{r+1}$ for all $r \in \{0\} \cup \mathbb{N}$. Since J is α-admissible associated with δ, consequently we have $\alpha(x_r, Jx_{r+1}) \geq \delta$ for all r. Observe that

$$\omega_\delta(x_r, x_{r+1}) = \omega_\delta(x_r, Jx_r) = \omega_\delta(J^r x_0, J^{r+1}x_0) > 0.$$

Otherwise if $\omega_\delta(x_r, x_{r+1}) = 0$ then $\omega_\delta(J^r x_0, J^r x_0) = 0$ implies $x_r = x_{r+1}$, a contradiction. Let us put $\gamma_r = \omega_\delta(x_{r-1}, x_r)$. Since J is a Caristi type (α-Ω-$\mathcal{Z}$)-contraction, thus we have

$$\begin{aligned}\gamma_{r+1} &= \omega_\delta(x_r, x_{r+1}) \\ &= \omega_\delta(Jx_{r-1}, Jx_r) \\ &\leq (\Omega(x_{r-1}) - \Omega(Jx_{r-1}))E(x_{r-1}, x_r)\max\{\mathcal{Z}(\omega_\delta(x_{r-1}, x_r), \omega_\delta(x_{r-1}, x_r)), 1\} \\ &\leq (\Omega(x_{r-1}) - \Omega(x_r))\max\{\omega_\delta(x_{r-1}, x_r), \omega_\delta(x_r, x_{r+1})\}.\end{aligned} \tag{3.1}$$

Now, suppose that

$$\max\{\omega_\delta(x_{r-1}, x_r), \omega_\delta(x_r, x_{r+1})\} = \omega_\delta(x_r, x_{r+1})$$

for some r. Then, from (3.1), we have

$$\omega_\delta(x_r, x_{r+1}) \leq (\Omega(x_{r-1}) - \Omega(x_r))\omega_\delta(x_r, x_{r+1}), \tag{3.2}$$

$$\Rightarrow 1 + \Omega(x_r) \leq \Omega(x_{r-1}).$$

Clearly, $\{\Omega(x_r)\}_{r=1}^{\infty}$ is a decreasing sequence of positive real numbers and hence $\Omega(x_r) \to a^*$ as $r \to \infty$, where $a^* \in \mathbb{R}_+$. Now, from (3.2), we have $1 \leq \Omega(x_{r-1}) - \Omega(x_r)$ and taking limit as $r \to \infty$, we have $1 \leq 0$, a contradiction. Thus we must have

$\max\{\omega_\delta(x_{r-1}, x_r), \omega_\delta(x_r, x_{r+1})\} = \omega_\delta(x_{r-1}, x_r)$ for all r. Therefore from (3.2), we obtain

$$\begin{aligned} &\omega_\delta(x_r, x_{r+1}) \leq (\Omega(x_{r-1}) - \Omega(x_r))\omega_\delta(x_{r-1}, x_r) \\ &\Rightarrow \frac{\omega_\delta(x_r, x_{r+1})}{\omega_\delta(x_{r-1}, x_r)} \leq (\Omega(x_{r-1}) - \Omega(x_r)) \\ &\Rightarrow 0 < \frac{\gamma_{r+1}}{\gamma_r} \leq (\Omega(x_{r-1}) - \Omega(x_r)). \end{aligned} \tag{3.3}$$

Clearly, (3.3) implies that $\{\Omega(x_r)\}_{r=1}^{\infty}$ is a non increasing sequence of positive real numbers and consequently $\Omega(x_r) \to a^*$ as $r \to \infty$, where $a^* \in \mathbb{R}_+$. Thus, we have

$$\sum_{r=1}^{s} \frac{\gamma_{r+1}}{\gamma_r} \leq \sum_{r=1}^{s} [\Omega(x_{r-1}) - \Omega(x_r)] = \Omega(x_0) - \Omega(x_s) \to \Omega(x_0) - a^* \text{ as } s \to \infty.$$

Thus $\sum_{r=1}^{\infty} \frac{\gamma_{r+1}}{\gamma_r}$ is a convergent series and hence by the property of a convergent series we have

$$\lim_{r\to\infty} \frac{\gamma_{r+1}}{\gamma_r} = 0. \tag{3.4}$$

Consequently, for $\lambda \in (0,1)$, there exists a $r_0 \in \mathbb{N}$ such that $\frac{\gamma_{r+1}}{\gamma_r} \leq \lambda$ for all $r \geq r_0$ implies $\gamma_{r+1} \leq \lambda\gamma_r$. Further, we can choose λ in such a way such that $\lambda\delta < 1$.Thus we have $\gamma_{r_0+r} \leq \lambda^r \gamma_{r_0}$, where $r \in \mathbb{N}$. Consequently, using triangular inequality of Definition 1.7, for $r, s \in \mathbb{N}$ with $r < s$ one can can easily show that $\omega_\delta(x_{r_0+r}, x_{r_0+s}) \to 0$ as $r \to \infty$. Hence, $\{x_r\}$ is a Cauchy sequence and by completeness there exists a x^* such that $x_r \to x^*$ as $r \to \infty$. Next, we wish to show that x^* is a fixed point of J. We prove this into two cases.

Case-1: Suppose that (G_3) holds, i.e., J is a continuous mapping. Then, we have

$$x^* = \lim_{r\to\infty} x_{r+1} = \lim_{r\to\infty} Jx_r = J(\lim_{r\to\infty} x_r) = Jx^*.$$

Case-2: Now we suppose that (G_4) holds. Assume that $x^* \neq Jx^*$. Again, $x_r \to x^*$ as $r \to \infty$ and $\omega_\delta(x_r, x_s) \to 0$ as $r \to \infty$, where $s > r$. Thus for each $\tau > 0$, there exists a $R_\tau \in \mathbb{N}$ such that for $r > R_\tau$ implies $\omega_\delta(x_{R_\tau}, x_r) < \tau$. Also, by using property-2 of Definition 1.7, we have

$$\omega_\delta(x_{R_\tau}, x^*) \leq \varliminf_{r\to\infty} \delta\omega_\delta(x_{R_\tau}, x_r) < \tau \text{ as } r \to \infty.$$

Hence, $\omega_\delta(x_{R_\tau}, x^*) < \tau$. Now, if we put $\tau = \frac{1}{h}$ and $R_\tau = r_h$, then we get $\lim_{h\to\infty} \omega_\delta(x_{r_h}, x^*) = 0$. Since, $x^* \neq Jx^*$, i.e., we have

$$\begin{aligned} 0 &< \inf\{\omega_\delta(x, x^*) + \omega_\delta(x, Jx) : x \in X\} \\ &\leq \inf\{\omega_\delta(x_{r_h}, x^*) + \omega_\delta(x_{r_h}, Jx_{r_h})\} \\ &\to 0 \text{ as } h \to \infty, \end{aligned}$$

a contradiction. Thus, we must have $x^* = Jx^*$. Furthermore, if $\alpha(x^*, x^*) \geq \delta$ then $\omega_\delta(x^*, x^*) = 0$ follows easily. □

Now, we give an example to support Caristi type (α-Ω-$\mathcal{Z}$)-contraction.

Example 3.1. Let $X = \{1, 2, 3, 4\}$. Define the function $J : X \to X$ by $J(1) = 1$, $J(2) = 3$, $J(3) = 1$, $J(4) = 4$. Let $\rho_\delta : X \times X \to \mathbb{R}_+$ be a mapping given by,

$$\rho_\delta(x, y) = |x - y|^2.$$

Clearly the mapping ρ_δ is a b-metric space with $\delta = 2$. Also let $\omega_\delta : X \times X \to \mathbb{R}_+$ be a mapping defined in the following way,

$$\omega_\delta(1,1) = \omega_\delta(2,2) = \omega_\delta(3,3) = 0, \omega_\delta(4,4) = 1, \omega_\delta(3,2) = 5, \omega_\delta(2,3) = 4, \omega_\delta(1,3) = 8,$$

$$\omega_\delta(3,1) = 10, \omega_\delta(1,2) = 2, \omega_\delta(2,1) = 3, \omega_\delta(4,x) = \omega_\delta(x,4) = 2 \text{ for all } x \in \{1,2,3\}.$$

Then it can be easily observed that ω_δ is not a b-metric space, and also it is not a w-distance. To check this, we observe the following facts.
Since $\omega_\delta(1,3) \neq \omega_\delta(3,1)$, so it not a b-metric and also note $\omega_\delta(1,3) \nleq \omega_\delta(1,2) + \omega_\delta(2,3)$. Consequently it is not a w-distance. Here ω_δ is a wt-distance with $\delta = 2$. Let $\Omega : X \to \mathbb{R}_+$ and $\alpha : X \times X \to \mathbb{R}_+$ be two mappings, given by $\Omega(1) = 0, \Omega(2) = 5, \Omega(3) = 2, \Omega(4) = 1$ and

$$\alpha(x, y) = \begin{cases} 3, & \text{if } x, y \in \{1, 2, 3\}, \\ 0, & \text{otherwise.} \end{cases}$$

Consider $\mathcal{Z}(\tau, s) = \frac{\tau}{1+s}$. Note that $\omega_\delta(x, Jx) > 0$ implies $x = 2, 3, 4$. Here $\omega_\delta(x, Jx) > 0$ with $\alpha(x, y) \geq \delta$ implies (x, y) equals to $(2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$. Again, it can be easily verified that at that points the mapping J satisfies Caristi type (α-Ω-$\mathcal{Z}$)-contraction. Take $x_0 = 3$, then $\alpha(x_0, Jx_0) \geq \delta$ and $\omega_\delta(J^r x_0, J^r x_0) = 0$, for all $r \in \{0\} \cup \mathbb{N}$. Also, $\inf\{\rho_\delta(x, x^*) + \rho_\delta(x, Jx) : x \in X\} > 0$ for each $x^* \in X$ with $x^* \neq Jx^*$ holds. Consequently, 1 is a fixed point of J. But here we observe two things. First of all Banach contraction principle is not valid as $\omega_\delta(J2, J3) = \omega_\delta(3, 1) = 10, \omega_\delta(2, 3) = 4$, and secondly the Caristi type (α-Ω-$\mathcal{Z}$)-contraction becomes invalid for all those points where $\alpha(x, u) \ngeq \delta$. For example take $x = 4, u = 1$.

4. Application

In this section, we apply our new findings to obtain a solution of a non-linear integral equation of Fredholm type, given by

$$\varphi(\tau) = \int_c^d B(\tau, \mu) E(\tau, \varphi(\mu)) d\mu, \tag{4.1}$$

where $B : [c, d]^2 \to \mathbb{R}_+$ and $E : [c, d] \times \mathbb{R} \to \mathbb{R}$ are two continuous functions. Denote $X = \mathcal{C}[c, d]$, the collection of all continuous functions from $[c, d]$ into $\mathbb{R}$. Define $J : X \to X$ as

$$(J\varphi)(\tau) = \int_c^d B(\tau, \mu) E(\tau, \varphi(\mu)) d\mu, \tag{4.2}$$

for all $\varphi \in X$ and $\mu \in [c, d]$. Let us consider a complete b-metric space ρ_δ on X with a wt-distance ω_δ as

$$\rho_\delta(x, y) = \sup_{\tau \in [c,d]} |x(\tau) - y(\tau)|^\lambda$$

and

$$\omega_\delta(x,y) = \sup_{\tau\in[c,d]} (|x(\tau)|^\lambda + |y(\tau)|^\lambda),$$

where $\lambda \in \mathbb{N}\backslash\{1\}$ and $\delta = 2^{\lambda-1}$. Now we observe that φ is a solution of the integral equation (4.1) if it is a fixed point of the operator J. Let us denote

$$\mathcal{M}_\xi^J(x,y) = \max\{\xi(\omega_\delta(x,y)), \xi(\omega_\delta(x,Jx)), \xi(\omega_\delta(y,Jy)), \xi(\frac{\omega_\delta(x,Jy)}{2\delta})\},$$

where $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and strictly non-decreasing function satisfying

$$(1)\ \xi(\tau) = 0 \Leftrightarrow \tau = 0; (2)\ \xi(\tau) < \tau, \text{ for } \tau > 0,$$

and we write "**A**" to denote all such functions.

Theorem 4.1. *Let us choose a Fredholm integral equation given by (4.1). Assume that the following assertions are satisfied:*
(C_1) *B is a continuous and measurable function from $[c,d]^2$ into $\mathbb{R}_+$ at $\tau \in [c,d]$ for each $\mu \in [c,d]$ such that*

$$\int_c^d B(\tau,\mu)^\sigma d\mu \leq \Omega,$$

where $\Omega \in (0,\infty)$ and σ is a real number, given by $\sigma = \frac{\lambda}{\lambda-1}$;
(C_2) *$E : [c,d]\times\mathbb{R} \to \mathbb{R}_+$ is non-decreasing function in its second variable as well as continuous such that for all $x,y \in X$ with $x(\mu) \leq y(\mu)$ for all $\mu \in [c,d]$, we get*

$$[E(\tau,x(\mu)) + E(\tau,y(\mu))]^\lambda \leq \Delta(\tau,\mu)\xi(|x(\mu)|^\lambda + |y(\mu)|^\lambda),$$

*where $\xi \in$ **A**, and $\Delta : [c,d]^2 \to \mathbb{R}_+$ is a continuous function such that*

$$\sup_{\tau\in[c,d]} \Big(\int_c^d \Delta(\tau,\mu)d\mu\Big) < \frac{2^{\frac{-3\lambda^2++3\lambda-1}{\lambda}}}{\Omega^{\lambda-1}};$$

(C_3) *$\inf\{\omega_\delta(x,y) + \omega_\delta(x,Jx) : x \in X\} > 0$ for each $y \in X$ with $y \neq Jy$;*
(C_4) *there exists a $x_0 \in X$ such that $x_0(\tau) \leq \int_c^d B(\tau,\mu)E(\tau,x_0(\mu))d\mu$ for $\tau \in [c,d]$.*
Then (4.1) has a solution in X.

Proof. First, we define a mapping as follows

$$\alpha(x,y) = \begin{cases} \sigma_1, & \text{if } x(\tau) \leq y(\tau) \text{ for all } \tau \in [c,d], \\ \sigma_2, & \text{otherwise,} \end{cases}$$

where $\sigma_1 \in [2^{\lambda-1},\infty)$ and $\sigma_2 \in [0,2^{\lambda-1})$. It can be easily checked that J is an α-admissible mapping associate with δ due to the fact that E is a non-decreasing function in its second variable. Also, α enjoys transitivity property. Next, we see that condition (C_4) implies $\alpha(x,Jx_0) \geq \sigma_1$. Let $\alpha(x,y) \geq \sigma_1 \geq \delta$, i.e., $x(\tau) \leq y(\tau)$

for all $\tau \in [c,d]$. Next, we have

$$
\begin{aligned}
&|(Jx)(\tau)|^{\lambda} + |(Jy)(\tau)|^{\lambda} \\
&\leq (|(Jx)(\tau)| + |(Jy)(\tau)|)^{\lambda} \text{ [since } \lambda \in \mathbb{N} \setminus \{1\}] \\
&= \Big(| \int_c^d B(\tau,\mu) E(\tau, x(\mu)) d\mu | + | \int_c^d B(\tau,\mu) E(\tau, u(\mu)) d\mu | \Big)^{\lambda} \\
&\leq \Big(\int_c^d |B(\tau,\mu)||E(\tau, x(\mu))| d\mu + \int_c^d |B(\tau,\mu)||E(\tau, y(\mu))| d\mu \Big)^{\lambda} \\
&\leq \Big(\int_c^d B(\tau,\mu)[|E(\tau, x(\mu))| + |E(\tau, y(\mu))|] d\mu \Big)^{\lambda} \\
&\leq \Big[\Big(\int_c^d B(\tau,\mu)^{\sigma} d\mu \Big)^{\frac{1}{\sigma}} \Big(\int_c^d [|E(\tau, x(\mu))| + |E(\tau, y(\mu))|]^{\lambda} d\mu \Big)^{\frac{1}{\lambda}} \Big]^{\lambda} \\
&\leq \Omega^{\frac{\lambda}{\sigma}} \Big(\int_c^d [|E(\tau, x(\mu))| + |E(\tau, y(\mu))|]^{\lambda} d\mu \Big) \\
&\leq \Omega^{\lambda-1} \Big(\int_c^d \Delta(\tau,\mu) \xi(|x(\mu)|^{\lambda} + |y(\mu)|^{\lambda}) d\mu \Big) \\
&\leq \Omega^{\lambda-1} \Big(\int_c^d \Delta(\tau,\mu) \xi(\omega_{\delta}(x,y)) d\mu \Big) \\
&\leq \Omega^{\lambda-1} \Big(\int_c^d \Delta(\tau,\mu) \mathcal{M}_{\xi}^{J}(x,y) d\mu \Big) \\
&\leq \Omega^{\lambda-1} \cdot \mathcal{M}_{\xi}^{J}(x,y) \cdot \sup_{\tau \in [c,d]} \Big(\int_c^d \Delta(\tau,\mu) d\mu \Big) \\
&\leq \Omega^{\lambda-1} \cdot \mathcal{M}_{\xi}^{J}(x,y) \cdot \frac{2^{\frac{-3\lambda^2 + +3\lambda - 1}{\lambda}}}{\Omega^{\lambda-1}} \\
&\leq 2^{\frac{-3\lambda^2 + +3\lambda - 1}{\lambda}} \cdot \mathcal{M}_{\xi}^{J}(x,y).
\end{aligned}
$$

Thus, from the last inequality, we obtain

$$
\begin{aligned}
&\Big(\xi(\delta^3 \omega_{\delta}(Jx, Jy)) \Big)^{\lambda} \\
&\leq \Big(\delta^3 \omega_{\delta}(Jx, Jy) \Big)^{\lambda} \\
&= \Big(2^{2\lambda-3} \omega_{\delta}(Jx, Jy) \Big) \\
&\leq \Big(2^{2\lambda-3} \cdot 2^{\frac{-3\lambda^2 + +3\lambda - 1}{\lambda}} \cdot \mathcal{M}_{\xi}^{J}(x,y) \Big) \\
&\leq 2^{-1} \cdot \Big(\mathcal{M}_{\xi}^{J}(x,y) \Big)^{\lambda} \cdot (1) \\
&= 2^{-1} \cdot \Big(\mathcal{M}_{\xi}^{J}(x,y) \Big)^{\lambda} \cdot \max\{a, 1\}, \text{ where } a \in (0,1).
\end{aligned}
$$

Now, in Theorem 2.6, consider I as an "identity mapping", $\eta(t) = t^\lambda$, $\theta(t) = \xi(t)$, $\beta = \frac{1}{2}$, $\mathcal{Z}(\tau, s) = a$, where $a \in (0, 1)$, $\varrho = 3$, $\phi_{\omega_\delta}^{(I,J)}[x, y] = \phi_2(t_1, t_2, t_3, t_4, t_5, t_6)$ (Example 2.10) with $L = 0$. Also, it can be easily checked that all the conditions of Theorem 2.6 are satisfied, i.e., J has a fixed point, i.e., integral equation (4.1) has a solution. □

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