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A NEW APPROACH TO FIXED POINT PROPERTY OF NONEXPANSIVE MULTIVALUED MAPPINGS

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Abstract. The purpose of this paper is to establish a general fixed point theorem for mean nonexpansive multivalued mappings defined on weakly compact and convex subsets of Banach spaces with property (D). The main result obtained in this paper significantly extends and improves many of the well-known results in the existing literature.

Key Words and Phrases: Fixed point, mean nonexpansive multivalued mapping, property (D), (DL)-condition, Banach space.
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1. Introduction

Recently, fixed point theory has become an important tool in different fields of pure and applied science, such as biology, chemistry, economics, engineering and physics. The study of existence of fixed points of nonexpansive mappings, initiated in 1965 independently by Browder [3, 4], Göhde [20] and Kirk [23], is one of dynamic research subject in nonlinear functional analysis. A mapping T on a nonempty subset E of a Banach space X is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$.

The study of fixed points for contractive and nonexpansive multivalued mappings using the Hausdorff metric was initiated by Markin [28] and Nadler [29]. From then on, many researchers have studied the possibility of extending classical fixed point theorems for nonexpansive singlevalued mappings to the setting of nonexpansive multivalued mappings. The technique of employing the asymptotic centers and their Chebyshev radii in fixed point theory was first discovered by Edelstein [16] and by using this fact Lim [27] proved the existence of fixed points for nonexpansive multivalued mappings in uniformly convex Banach spaces. One of the most general fixed point theorems for nonexpansive multivalued self-mappings was obtained by Kirk and Massa [24] in 1990, proving the existence of fixed points in Banach spaces for which the asymptotic center of each sequence in a closed bounded convex subset is nonempty and compact.

Motivated by the method of proof in [24], Domínguez and Lorenzo [14] studied some connections between asymptotic centers and the geometry of certain Banach spaces. Dhompongsa et al. [9] noticed that the main tool used in the proof of the results in [15] is a relationship concerning the Chebyshev radius of the asymptotic center of a bounded sequence and the asymptotic radius of the sequence. Consequently, they introduced the so-called Domínguez-Lorenzo condition, or briefly (DL)-condition, which guarantees the existence of fixed points for nonexpansive multivalued mappings. The condition was weakened later by Dhompongsa et al. in the work [8] when they introduced another condition called property (D). Many researchers established that, under various geometric properties of a Banach space often measured by different geometric constants, either the (DL)-condition or property (D) is guaranteed, which implies the fixed point property for nonexpansive multivalued mappings and normal structure in reflexive Banach spaces. In this setting the following implications have been obtained by various authors:

(a) Domínguez and Lorenzo [14]: $\Delta_{X,\beta}(\varepsilon) > 0$ for some $\varepsilon > 0$ implies (DL)-condition.
(b) Dhompongsa et al. [8]: $C_{NJ}(X) < 1 + \frac{WCS(X)^2}{4}$ implies property (D).
(c) Domínguez and Gavira [13]: $\xi_X(\beta) < \frac{1}{1-\beta}$ for some $\beta \in (0,1)$ implies (DL)-condition for separable subsets of X;
$r_X(1) > 0$ (equivalently, $\Delta_0(X) < 1$ or $\zeta_X(\beta) < \frac{1}{1-\beta}$ for some $\beta \in (0,1)$) implies (DL)-condition.
(d) Saejung [30]: $\varepsilon_0(X) < WCS(X)$ implies property (D) and each of the conditions $C_{NJ}(X) < \frac{4}{1+\mu(X)^2}$ or $\varepsilon_0(X) < 2 + WORTH$ implies (DL)-condition.
(e) Kaewkhao [22]: Each of the conditions $J(X) < 1 + \frac{1}{\mu(X)}$ or $C_{NJ}(X) < 1 + \frac{1}{\mu(X)^2}$ implies (DL)-condition.
(f) Gavira [17]: Each of the conditions $\rho'_X(0) < \frac{M(X)}{2}$ or $C_{NJ}(X) < 1 + \frac{M(X)^2}{4}$ or $J(X) < 1 + \frac{1}{R(1,X)}$ or $\rho'_X(0) < \frac{1}{\mu(X)}$ implies (DL)-condition.
(g) Kaewcharoen [21]: $C_{NJ}(X) < 1 + \frac{N(X)^2}{4}$ implies (DL)-condition.
(h) Dinarvand [12]: $A_2(X) < 1 + \frac{WCS(X)}{2}$ implies property (D) and $A_2(X) < 1 + \frac{N(X)}{2}$ implies (DL)-condition. Furthermore, each of the conditions $\rho'_X(0) < \frac{1}{A_2(X)}$ or $A_2(X) < 1 + \frac{M(X)}{2}$ or $T(X) < 1 + \frac{1}{R(1,X)}$ or $A_2(X) < 1 + \frac{1}{\mu(X)}$ or $T(X) < 1 + \frac{1}{\mu(X)}$ implies (DL)-condition.

For a good study related to the afore-mentioned conditions, we refer the reader to [10, 11, 31] and the references mentioned therein.

The purpose of this paper is to extend the classical fixed point theorem for nonexpansive multivalued mappings proved by Dhompongsa et al. [8]. Thanks to this result, we generalize and improve all the above mentioned results and many others.

2. Preliminaries

We start this section by reviewing some concepts and results which will be used in what follows.

Let X be a Banach space and E be a nonempty subset of X. We shall denote by 2^E the family of all subsets of E, $P(E)$ the family of all nonempty closed subsets of E, $CB(E)$ the family of all nonempty closed bounded subsets of E, $K(E)$ the family of all nonempty compact subsets of E and $KC(E)$ the family of all nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $CB(X)$ defined by

$$H(A,B) = \max\Big\{ \sup_{x\in A} \operatorname{dist}(x,B),\ \sup_{y\in B} \operatorname{dist}(y,A)\Big\},$$

or equivalently,

$$H(A,B) = \inf\Big\{\varepsilon > 0 \ :\ A \subset N(\varepsilon,B) \text{ and } B \subset N(\varepsilon,A)\Big\},$$

for all $A,B \in CB(X)$, where for $x \in X$ and $C \subset X$, $\operatorname{dist}(x,C) := \inf\{\|x-z\| \ :\ z \in C\}$ is the distance from the point x to the subset C and

$$N(\varepsilon,C) := \{x \in X \ :\ d(x,c) < \varepsilon \text{ for some } c \in C\}.$$

A multivalued mapping $T : E \to CB(X)$ is said to be k-contractive if there exists a constant $k \in [0,1)$ such that

$$H(Tx,Ty) \le k\|x-y\| \qquad (x,y \in E),$$

and T is said to be nonexpansive if

$$H(Tx,Ty) \le \|x-y\| \qquad (x,y \in E).$$

Recall that the Hausdorff measures of noncompactness of a nonempty bounded subset A of E is defined as the number

$$\chi_E(A) = \inf\Big\{\varepsilon > 0 \ :\ A \text{ can be covered by finitely many balls centered at points in } E \text{ with radii} \le \varepsilon\Big\}.$$

It must be noted that this measure depends on E and it is, in general, different from $\chi := \chi_X$. Furthermore, if E is a convex closed set, it is easy to check that the usual arguments to prove $\chi_X(A) = \chi_X(\overline{\text{co}}(A))$ (see [2, Theorem 2.4]) equally well apply to prove $\chi_E(A) = \chi_E(\overline{\text{co}}(A))$ for any nonempty bounded subset $A \subset E$. Furthermore, if E is separable, for any nonempty bounded subset A of E there exists $B \subset A$ such that $\chi_E(B) = \chi_E(A)$ and B is χ_E-minimal, i.e. $\chi_E(B) = \chi_E(D)$ for any infinite subset D of B (for the definition and properties of χ_E-minimal sets, see [25, Chapter 8]).

A multivalued mapping $T : E \to 2^X$ is called k-χ_E-condensing if for each bounded subset B of E with $\chi_E(B) > 0$, we have

$$\chi_E(T(B)) \le k\chi_E(B),$$

where $T(B) = \cup_{x\in B} Tx$.

A multivalued mapping $T : E \to 2^X$ is said to be upper semicontinuous on E if $\{x \in E \ :\ Tx \subset V\}$ is open in E whenever $V \subset X$ is open and T is said to be lower semicontinuous if $T^{-1}(V) = \{x \in E \ :\ Tx \cap V \neq \emptyset\}$ is open in E whenever $V \subset X$ is open. T is said to be continuous if it is both upper and lower semicontinuous. As another different type of continuity for multivalued mappings, a multivalued mapping

$T : E \to CB(X)$ is said to be continuous on E with respect to the Hausdorff metric H if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$. It is easy to see that both definitions of continuity are equivalent if Tx is compact for every $x \in E$ (see [1, 6]). A point $x \in E$ is called a fixed point of a multivalued mapping T if $x \in Tx$.

Recall that the inward set of E at $x \in E$ is defined by

$$I_E(x) = \Big\{x + \lambda(y - x) \,:\, \lambda \geq 0,\, y \in E\Big\}.$$

Obviously, $E \subset I_E(x)$ and also one can see that $I_E(x)$ is a convex set as E does.

Theorem 2.1. ([7]) *Let X be a Banach space and $\emptyset \neq C \subset X$ be compact convex. Let $F : C \to 2^X$ be upper semicontinuous with closed convex values. If $Fx \cap \overline{I_C(x)} \neq \emptyset$ on C, then F has a fixed point.*

In 1975, Zhang [32] introduced an interesting generalization of nonexpansive mappings as follows.

Definition 2.2. ([32]) Let T be a mapping on a nonempty subset E of a Banach space X. We say that T is a mean nonexpansive mapping if for each $x, y \in E$ and $\alpha > 0, \beta \geq 0$ with $\alpha + \beta \leq 1$,

$$\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|.$$

In this interesting paper, Zhang [32] proved that T has normal structure and has a unique fixed point in E, where E is a weakly compact closed convex subset of X. In connection with this celebrated work, Chen et al. [5] introduced the multivalued analogs of the nonexpansive type mappings defined in [32] in the following manner.

Definition 2.3. ([5]) Let E be a nonempty subset of a Banach space X and $T : E \to CB(X)$ be a multivalued mapping. We say that T is a mean nonexpansive multivalued mapping if for each $x, y \in E$ and $\alpha > 0, \beta \geq 0$ with $\alpha + \beta \leq 1$,

$$H(Tx, Ty) \leq \alpha\|x - y\| + \beta \operatorname{dist}(x, Ty).$$

If $\alpha = 1$ and $\beta = 0$, then T is said to be a nonexpansive multivalued mapping. If $0 < \alpha < 1$ and $\beta = 0$, then T is said to be a contractive multivalued mapping.

The following theorem, which is an extension of Nadler's fixed point theorem, is one of the main tools for proving our results.

Theorem 2.4. ([5]) *Let X be a Banach space and $T : X \to CB(X)$ be a mean nonexpansive multivalued mapping with $\alpha + \beta < 1$. Then T has a fixed point.*

Next, we set out some useful results concerning the asymptotic centers. Let E be a nonempty bounded closed subset of a Banach space X and $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r\big(E, \{x_n\}\big)$ and the asymptotic center $A\big(E, \{x_n\}\big)$ of $\{x_n\}$ in E are defined by

$$r\big(E, \{x_n\}\big) = \inf\Big\{\limsup_{n\to\infty} \|x_n - x\| \,:\, x \in E\Big\},$$

and

$$A\big(E, \{x_n\}\big) = \Big\{x \in E \,:\, \limsup_{n\to\infty} \|x_n - x\| = r\big(E, \{x_n\}\big)\Big\},$$

respectively. It is known that if E is nonempty weakly compact and convex, then so is $A\big(E,\{x_n\}\big)$ (see [19]).

The sequence $\{x_n\}$ is called regular with respect to E if $r\big(E,\{x_n\}\big) = r\big(E,\{x_{n_i}\}\big)$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to E if $A\big(E,\{x_n\}\big) = A\big(E,\{x_{n_i}\}\big)$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 2.5. (Goebel [18], Lim [27]) *Let E be a nonempty bounded closed subset of a Banach space X and $\{x_n\}$ be a bounded sequence in X. Then the following assertions hold:*

(i) *There exists a subsequence of $\{x_n\}$ which is regular with respect to E.*
(ii) *If E is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform with respect to E.*

Let C be a nonempty bounded subset of X. The Chebyshev radius of C relative to E is defined by

$$r_E(C) = \inf\big\{r_x(C) \,:\, x \in E\big\},$$

where $r_x(C) = \sup\{\|x-y\| \,:\, y \in C\}$.

The following two properties of Banach spaces were introduced and used to guarantee the existence of fixed points for multivalued nonexpansive mappings (see [8, 9]).

Definition 2.6. ([8]) A Banach space X is said to satisfy property (D) if there exists $\lambda \in [0,1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \subset E$ which is regular and asymptotically uniform with respect to E, and any sequence $\{y_n\} \subset A\big(E,\{x_n\}\big)$ which is regular and asymptotically uniform with respect to E, we have

$$r\big(E,\{y_n\}\big) \le \lambda\, r\big(E,\{x_n\}\big).$$

Definition 2.7. ([9]) A Banach space X is said to satisfy the (DL)-condition if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset E of X and for every sequence $\{x_n\}$ in E which is regular with respect to E, we have

$$r_E\big(A\big(E,\{x_n\}\big)\big) \le \lambda\, r\big(E,\{x_n\}\big).$$

It is interesting to remark at this point that property (D) is weaker than the (DL)-condition. Also, it is worthwhile to mention that property (D) and the (DL)-condition are stronger than weak normal structure and also imply the existence of fixed points for nonexpansive multivalued mappings.

3. Main Results

We begin this section by giving a lemma that will constitute a main tool in proving our results.

Lemma 3.1. *Let E be a nonempty bounded convex subset of a Banach space X and assume that $T : E \to CB(E)$ is a mean nonexpansive multivalued mapping. Then there exists a sequence $\{x_n\}$ in E such that*

$$\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0.$$

Proof. Let $z_0 \in E$. For each $n \in \mathbb{N}$, we define $T_n : E \to CB(E)$ by

$$T_n(x) = \frac{1}{n}z_0 + \left(1 - \frac{1}{n}\right)Tx \qquad (x \in E).$$

Then T_n is a mean nonexpansive multivalued mapping with $\alpha + \beta < 1$. According to Theorem 2.4, T_n has a fixed point $x_n \in E$. Therefore, for each $n \in \mathbb{N}$,

$$x_n \in \frac{1}{n}z_0 + \left(1 - \frac{1}{n}\right)Tx_n$$

This implies that there exists $y_n \in Tx_n$ such that

$$x_n = \frac{1}{n}z_0 + \left(1 - \frac{1}{n}\right)y_n. \tag{3.1}$$

Since $y_n \in Tx_n$ and E is bounded, it follows that

$$\begin{aligned}\operatorname{dist}(x_n, Tx_n) &\le \|x_n - y_n\| = \frac{1}{n}\|z_0 - y_n\| \\ &\le \frac{1}{n}\operatorname{diam}(E).\end{aligned}$$

Hence, $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$ which completes the proof.

Our next result can be considered as an extension of [14, Proposition 4.1].

Theorem 3.2. *Let E be a nonempty bounded closed convex subset of a Banach space X and $T : E \to K(E)$ be a mean nonexpansive continuous multivalued mapping. If $\{x_n\}$ is a sequence in E such that* $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$, *then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*

$$Tx \cap A \neq \emptyset, \qquad \textit{for all } x \in A := A(E, \{z_n\}).$$

Proof. Since T is a self-mapping, we can build a subsequence $\{z_n\}$ of $\{x_n\}$ which is regular and asymptotically uniform with respect to E (see [19]).
We now let $r := r(E, \{z_n\})$ and $A := A(E, \{z_n\})$ and take $x \in A$. Hence, we obtain

$$\begin{aligned}\operatorname{dist}(z_n, Tx) &\le \operatorname{dist}(z_n, Tz_n) + H(Tz_n, Tx) \\ &\le \operatorname{dist}(z_n, Tz_n) + \alpha\|z_n - x\| + \beta\operatorname{dist}(z_n, Tx).\end{aligned}$$

Therefore,

$$\operatorname{dist}(z_n, Tx) \le \frac{1}{1-\beta}\operatorname{dist}(z_n, Tz_n) + \frac{\alpha}{1-\beta}\|z_n - x\|. \tag{3.2}$$

Since Tx is compact, there exists a point $w_n \in Tx$ such that

$$\|z_n - w_n\| = \operatorname{dist}(z_n, Tx).$$

This along with (3.2) implies that

$$\|z_n - w_n\| \leq \frac{1}{1-\beta}\operatorname{dist}(z_n, Tz_n) + \frac{\alpha}{1-\beta}\|z_n - x\|. \tag{3.3}$$

Again by the compactness of Tx, we may assume that $\{w_n\}$ converges to a point $w \in Tx$. Hence, by applying our assumption and (3.3), we have

$$\begin{aligned}\limsup_{n\to\infty} \|z_n - w\| &\leq \frac{\alpha}{1-\beta}\limsup_{n\to\infty} \|z_n - x\| \\ &\leq \limsup_{n\to\infty} \|z_n - x\| \\ &= r.\end{aligned}$$

This means that $w \in A$ and therefore $Tx \cap A \neq \emptyset$ for all $x \in A$.

Now we are ready to prove an analogous result to the Kirk-Massa theorem [24] for mean nonexpansive multivalued mappings.

Theorem 3.3. *Let E be a nonempty bounded closed convex subset of a Banach space X and $T : E \to K(E)$ be a mean nonexpansive continuous multivalued mapping. If the asymptotic center in E of each bounded sequence of X is nonempty and compact, then T has a fixed point.*

Proof. Let $\{x_n\}$ be the sequence in E defined by (3.1). According to Lemma 3.1, we obtain that $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$. By use of Theorem 3.2, there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that

$$Tx \cap A \neq \emptyset, \qquad \text{for all } x \in A := A(E, \{z_n\}).$$

Because of our assumption, A is nonempty and compact. Now, we define a mapping $\widetilde{T} : A \to KC(A)$ by

$$\widetilde{T}x := Tx \cap A \qquad \text{for all } x \in A.$$

We claim that $\widetilde{T}$ is upper semicontinuous. Indeed, let $\{u_n\} \subset A$ such that $\lim_{n\to\infty} u_n = u$ and $v_n \in \widetilde{T}u_n$ such that $\lim_{n\to\infty} v_n = v$. As A is compact, it follows that $v \in A$. From the continuity of T, we have

$$\operatorname{dist}(v, Tu) \leq \operatorname{dist}(v, Tu_n) + H(Tu_n, Tu) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $v \in Tu$ and therefore $v \in \widetilde{T}u$.

Since $Tx \cap A = \widetilde{T}x$ is a compact convex set we can apply the Kakutani-Bohnenblust-Karlin theorem [19] to obtain a fixed point for $\widetilde{T}$ and hence for T. This completes the proof.

The next result provides a relationship between mean nonexpansive and $(\alpha+\beta)$-χ_E-condensing multivalued mappings with $\alpha + \beta < 1$.

Theorem 3.4. *Let E be a nonempty weakly compact convex separable subset of a Banach space X and $T : E \to K(E)$ be a mean nonexpansive multivalued mapping with $\alpha + \beta < 1$. Then T is $(\alpha+\beta)$-χ_E-condensing.*

Proof. Let A be a bounded subset of E. Since E is separable, it follows that there exists a χ_E-minimal subset $B \subset T(A)$ such that $\chi_E(B) = \chi_E(T(A))$. We can assume that B is countable, i.e. $B = \{y_n : n \in \mathbb{N}\}$. Since E is separable, taking a subsequence, we can assume that $\lim_{n\to\infty} \|y_n - x\|$ exists for all $x \in E$. Then following similar arguments as those given in the proof of Theorem 3.2 in [15], we have

$$\chi_E(\{y_n : n \in \mathbb{N}\}) = r(E, \{y_n\}).$$

Let $\{x_n\}$ be the sequence in A defined by (3.1). By applying Lemma 3.1, we obtain $\lim_{n\to\infty} \mathrm{dist}(x_n, Tx_n) = 0$. Taking again a subsequence we can assume that the set $\{x_n : n \in \mathbb{N}\}$ is χ_E-minimal and $\lim_{n\to\infty} \|x_n - x\|$ exists for every $x \in E$.
On the other hand, similar to the first part of the proof, we can show that

$$\chi_E(\{x_n : n \in \mathbb{N}\}) = r(E, \{x_n\}).$$

Let $u \in A(E, \{x_n\})$, i.e. $\lim_{n\to\infty} \|x_n - u\| = \chi_E(\{x_n : n \in \mathbb{N}\})$. By using compactness of Tu, there exists a point $u_n \in Tu$ such that

$$\|y_n - u_n\| = \mathrm{dist}(y_n, Tu). \tag{3.4}$$

Again by the compactness of Tu and taking a subsequence, we can assume that $\{u_n\}$ converges strongly to a point $v \in Tu$. Hence, by applying (3.4), we have

$$\begin{aligned}
\chi_E(T(A)) = r(E, \{y_n\}) &\leq \lim_{n\to\infty} \|y_n - v\| = \lim_{n\to\infty} \|y_n - u_n\| \\
&= \limsup_{n\to\infty} \mathrm{dist}(y_n, Tu) \leq \limsup_{n\to\infty} H(Tx_n, Tu) \\
&\leq \limsup_{n\to\infty} \big(\alpha\|u - x_n\| + \beta \mathrm{dist}(u, Tx_n)\big) \\
&\leq \limsup_{n\to\infty} \big(\alpha\|u - x_n\| + \beta\|u - x_n\| + \beta \mathrm{dist}(x_n, Tx_n)\big) \\
&\leq (\alpha + \beta) \limsup_{n\to\infty} \|u - x_n\| \\
&= (\alpha + \beta) \lim_{n\to\infty} \|u - x_n\| = (\alpha + \beta)\chi_E(\{x_n : n \in \mathbb{N}\}) \\
&\leq (\alpha + \beta)\chi_E(A).
\end{aligned}$$

This completes the proof.

Next, we present a theorem which will play a crucial role in the proof of our main result.

Theorem 3.5. *Let E be a nonempty weakly compact convex separable subset of a Banach space X and $T : E \to KC(E)$ be a mean nonexpansive continuous multivalued mapping with $\alpha + \beta < 1$. If A is a closed convex subset of E such that $Tx \cap A \neq \emptyset$ for all $x \in A$, then T has a fixed point in A.*

Proof. By Theorem 3.4, the mapping T is $(\alpha + \beta)$-χ_E-condensing. One can see that the arguments in the proof of [7, Theorem 11.5] equally well work for the measure χ_E when we assume that $T(E) \subset E$. By applying Theorem 2.1 and following similar arguments as those given in the proof of Theorem 3.3 in [15], we conclude that T has a fixed point in A. This completes the proof.

To prove our main result, we make use of the following lemma.

Lemma 3.6. *Let E be a nonempty subset of a Banach space X and $T : E \to P(E)$ be a mean nonexpansive multivalued mapping. Then for all $x, y \in E$, we have*

$$\mathrm{dist}(x, Tx) \leq 2\|x - y\| + (1 + \beta)\mathrm{dist}(y, Ty),$$

where $\alpha > 0$ and $\beta \geq 0$ with $\alpha + \beta \leq 1$.
Proof. Let $x, y \in E$ and $\alpha > 0$ and $\beta \geq 0$ with $\alpha + \beta \leq 1$. Thus, we have

$$\begin{aligned}
\operatorname{dist}(x, Tx) &\leq \|x - y\| + \operatorname{dist}(y, Tx) \\
&\leq \|x - y\| + \operatorname{dist}(y, Ty) + H(Tx, Ty) \\
&\leq \|x - y\| + \operatorname{dist}(y, Ty) + \alpha\|x - y\| + \beta \operatorname{dist}(x, Ty) \\
&\leq \|x - y\| + \operatorname{dist}(y, Ty) + \alpha\|x - y\| + \beta\|x - y\| + \beta \operatorname{dist}(y, Ty) \\
&\leq 2\|x - y\| + (1 + \beta)\operatorname{dist}(y, Ty).
\end{aligned}$$

This completes the proof.

We are now in the position to state the main result of this paper.

Theorem 3.7. *Let E be a nonempty weakly compact convex subset of a Banach space X which satisfies property (D). If $T : E \to KC(E)$ is a mean nonexpansive continuous multivalued mapping, then T has a fixed point.*
Proof. Since T is a self-mapping, we can assume that E is separable (otherwise, by [26] we can construct a closed convex subset of E that is invariant under T). Let $\{x_n^0\}$ be the sequence in E defined by (3.1). Taking into account Lemma 3.1, we conclude that $\lim_{n\to\infty} \operatorname{dist}(x_n^0, Tx_n^0) = 0$. From the boundedness of $\{x_n^0\}$, we can assume by using Lemma 2.5 that $\{x_n^0\}$ is regular and asymptotically uniform with respect to E. Denote $A^0 = A(E, \{x_n^0\})$. In view of Theorem 3.2, we can assume that $Tx \cap A^0 \neq \emptyset$ for all $x \in A^0$. Fix $z_1 \in A^0$ and define a mean nonexpansive multivalued mapping $T_n : E \to KC(E)$ by

$$T_n(x) = \frac{1}{n} z_1 + \left(1 - \frac{1}{n}\right) Tx, \qquad (x \in E).$$

Convexity of A^0 implies that $T_n(x) \cap A^0 \neq \emptyset$ for all $x \in A^0$. According to Theorem 3.5, it follows that T_n has a fixed point in A^0, say x_n^1. Therefore, we can take a sequence of iteration $\{x_n^1\}$ in A^0 satisfying $\lim_{n\to\infty} \operatorname{dist}(x_n^1, Tx_n^1) = 0$. Again by Lemma 2.5, we assume that $\{x_n^1\}$ in A^0 is regular and asymptotically uniform with respect to A^0. Since X satisfies property (D) with a corresponding $\lambda \in [0, 1)$, we obtain

$$r(E, \{x_n^1\}) \leq \lambda\, r(E, \{x_n^0\}).$$

Continuing this process, we can build the sequence $\{x_n^m\}$ in $A^{m-1} := A(E, \{x_n^{m-1}\})$ which is regular and asymptotically uniform with respect to E such that for each $m \geq 1$,

$$\lim_{n\to\infty} \operatorname{dist}(x_n^m, Tx_n^m) = 0,$$

and

$$r(E, \{x_n^m\}) \leq \lambda\, r(E, \{x_n^{m-1}\}).$$

Consequently,

$$r(E, \{x_n^m\}) \leq \lambda\, r(E, \{x_n^{m-1}\}) \leq \cdots \leq \lambda^m r(E, \{x_n^0\}).$$

Let $\{x_m = x_{m+1}^{m+1}\}$ be the diagonal sequence of the sequences $\{x_n^m\}$. Now, we assert that $\{x_m\}$ is a Cauchy sequence. For each $m, n \geq 1$, we have

$$\|x_m^m - x_n^m\| \leq \|x_m^m - x_k^{m-1}\| + \|x_k^{m-1} - x_n^m\| \qquad \text{for all } k.$$

Hence, we obtain

$$\|x_m^m - x_n^m\| \leq \limsup_{k\to\infty} \|x_m^m - x_k^{m-1}\| + \limsup_{k\to\infty} \|x_k^{m-1} - x_n^m\| \leq 2r(E, \{x_k^{m-1}\}).$$

Therefore, for each n, we get

$$\begin{aligned}\|x_{m-1} - x_m\| &\leq \|x_{m-1} - x_n^m\| + \|x_n^m - x_m\| \\ &= \|x_m^m - x_n^m\| + \|x_n^m - x_{m+1}^{m+1}\| \\ &\leq 2r(E, \{x_k^{m-1}\}) + \|x_n^m - x_{m+1}^{m+1}\|.\end{aligned}$$

By taking the superior limit as $n \to \infty$, we have

$$\|x_{m-1} - x_m\| \leq 2r(E, \{x_n^{m-1}\}) + r(E, \{x_n^m\}) \leq 3\lambda^{m-1} r(E, \{x_n^0\}).$$

Since $\lambda < 1$, it follows that $\{x_m\}$ is a Cauchy sequence. Thus, there exists a point $x \in E$ such that $\{x_m\}$ converges to x. Finally, we show that x is a fixed point of T. By Lemma 3.6, for each $m \geq 1$, we have

$$\operatorname{dist}(x_m, Tx_m) \leq 2\|x_n^m - x_{m+1}^{m+1}\| + (1+\beta)\operatorname{dist}(x_n^m, Tx_n^m).$$

By taking the superior limit as $n \to \infty$, we obtain

$$\operatorname{dist}(x_m, Tx_m) \leq 2 \limsup_{n\to\infty} \|x_n^m - x_{m+1}^{m+1}\| = 2r(E, \{x_n^m\}) \leq 2\lambda^m r(E, \{x_n^0\}).$$

Since $\lambda < 1$, we conclude that $\lim_{m\to\infty} \operatorname{dist}(x_m, Tx_m) = 0$. Again Lemma 3.6 implies that

$$\operatorname{dist}(x, Tx) \leq 2\|x - x_m\| + (1+\beta)\operatorname{dist}(x_m, Tx_m),$$

from which it follows that $\operatorname{dist}(x, Tx) = 0$. Since Tx is closed, it follows that $x \in Tx$ and this completes the proof.

As mentioned before, Theorem 3.7 includes all known results concerning the existence of fixed points for nonexpansive multivalued mappings in particular classes of Banach spaces.

Corollary 3.8. *Let E be a nonempty bounded closed convex subset of a Banach space X such that one of the geometric conditions (a) to (h) is satisfied. If $T : E \to KC(E)$ is a mean nonexpansive continuous multivalued mapping, then T has a fixed point.*

As an immediate consequence of Theorem 3.7, we deduce the following results due to Dhompongsa et al. [8].

Corollary 3.9. ([8, Theorem 3.6]) *Let E be a nonempty weakly compact convex subset of a Banach space X which satisfies property (D). If $T : E \to KC(E)$ is a nonexpansive multivalued mapping, then T has a fixed point.*

Corollary 3.10. ([8, Theorem 3.6]) *Let E be a nonempty weakly compact convex subset of a Banach space X which satisfies the (DL)-condition. If $T : E \to KC(E)$ is a nonexpansive multivalued mapping, then T has a fixed point.*

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