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ON INERTIAL SUBGRADIENT EXTRAGRADIENT RULE FOR MONOTONE BILEVEL EQUILIBRIUM PROBLEMS

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Abstract. In a real Hilbert space, let the GSVI and CFPP represent a general system of variational inclusions and a common fixed point problem of countable nonexpansive mappings and an asymptotically nonexpansive mapping, respectively. In this paper, via a new inertial subgradient extragradient rule we introduce and analyze two iterative algorithms for solving the monotone bilevel equilibrium problem (MBEP) with the GSVI and CFPP as constraints. Some strong convergence theorems for the proposed algorithms are established under some mild assumptions. Our results improve and extend some corresponding results in the earlier and very recent literature.
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1. Introduction

Let $(\mathcal{H}, \|\cdot\|)$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let P_C be the nearest point projection from $\mathcal{H}$ onto C, where C is a nonempty closed convex subset of $\mathcal{H}$. A mapping $T : C \to C$ is known as being asymptotically nonexpansive, if there exists a sequence $\{\theta_k\} \subset [0,\infty)$ such that $\|T^k x - T^k y\| \leq (1+\theta_k)\|x-y\|\ \forall x, y \in C, k \geq 1$, with

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$\lim_{k\to\infty}\theta_k = 0$. In particular, whenever $\theta_k = 0\ \forall k \geq 1$, T is known as nonexpansive. We denote by $\mathrm{Fix}(T)$ the fixed point set of the mapping T and by $\mathcal{R}$ the set of all real numbers, respectively. Let A be a self-mapping on $\mathcal{H}$. Consider the classical variational inequality problem (VIP) of finding $x^* \in C$ s.t. $\langle Ax^*, y - x^*\rangle \geq 0\ \forall y \in C$. The solution set of the VIP is denoted by $\mathrm{VI}(C, A)$.

Suppose that the bifunction $\Phi : \mathcal{H}\times\mathcal{H} \to \mathcal{R}\cup\{+\infty\}$ satisfies $\Phi(x, x) = 0\ \forall x \in C$. Consider the equilibrium problem ($\mathrm{EP}(C, \Phi)$) for bifunction Φ on the constraint domain C, which is to find $x^* \in C$ such that $\Phi(x^*, y) \geq 0\ \forall y \in C$. The solution set of $\mathrm{EP}(C, \Phi)$ is denoted by $\mathrm{Sol}(C, \Phi)$. It is worth mentioning that the $\mathrm{EP}(C, \Phi)$ is a unified model of some problems, e.g., variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems, etc. To the most of our knowledge, the $\mathrm{EP}(C, \Phi)$ and its extended versions have been widely studied by many authors; see [1], [4], [5], [6], [7], [8], [10], [12], [13], [14], [15], [16], [17], [18], [19], [22], [25], [27], and references therein. In 2009, via a viscosity approximation approach, Chang et al. [15] introduced an iterative algorithm for finding a common solution of the common fixed point problem (CFPP) of countable nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$, the VIP for inverse-strongly monotone mapping A and the $\mathrm{EP}(C, \Phi)$ for bifunction Φ. Under some suitable assumptions, they proved the strong convergence of the proposed algorithm to a common solution.

Let $F_1, F_2 : \mathcal{H} \to \mathcal{H}$ be single-valued mappings and $B_1, B_2 : C \to 2^{\mathcal{H}}$ be multivalued mappings with $B_j y \neq \emptyset\ \forall y \in C, j = 1, 2$. Consider the general system of variational inclusions (GSVI), which is to find $(x^*, y^*) \in C \times C$ s.t.

$$\begin{cases} 0 \in \lambda_1(F_1 y^* + B_1 x^*) + x^* - y^*, \\ 0 \in \lambda_2(F_2 x^* + B_2 y^*) + y^* - x^*, \end{cases} \tag{1.1}$$

with constants $\lambda_1, \lambda_2 > 0$. In particular, if $B_1 = B_2 = \partial i_C$ where i_C is the indicator function of C defined by $i_C(x) = 0\ \forall x \in C$ and $i_C(x) = \infty\ \forall x \notin C$, then problem (1.1) reduces to the general system of variational inequalities (GSVI^+). It is worth pointing out that problem (1.1) has been transformed into a fixed point problem in the following way.

Suppose that the mappings $B_1, B_2 : C \to 2^{\mathcal{H}}$ both are maximal monotone. From [11, Lemma 2], we see that, for given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.1) if and only if $x^* \in \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the fixed-point set of the mapping $G := J_{\lambda_1}^{B_1}(I - \lambda_1 F_1)J_{\lambda_2}^{B_2}(I - \lambda_2 F_2)$, and $y^* = J_{\lambda_2}^{B_2}(I - \lambda_2 F_2)x^*$.

Let the Ω denote the common solution set of the fixed point problem (FPP) of an asymptotically nonexpansive mapping and the GSVI^+ for two inverse-strongly monotone mappings. In 2018, using a modified extragradient technique, Cai et al. [4] introduced a viscosity implicit rule for finding a common solution of the GSVI^+ and FPP. Under some appropriate restrictions they proved the strong convergence of the proposed algorithm to an element $x^* \in \Omega$. Subsequently, Ceng and Wen [14] proposed a hybrid extragradient-like implicit method with strong convergence for finding a common solution of the GSVI^+ and common fixed-point problem (CFPP).

On the other hand, Anh and An [1] introduced the monotone bilevel equilibrium problem (MBEP) with the fixed-point problem (FPP) constraint, i.e., a strongly monotone equilibrium problem $\mathrm{EP}(\varOmega, \Psi)$ over the common solution set $\varOmega$ of another monotone equilibrium problem $\mathrm{EP}(C, \Phi)$ and the fixed-point problem of $\mathcal{K}$-demicontractive mapping T:

$$\text{Find } x^* \in \varOmega \text{ such that } \Psi(x^*, y) \geq 0 \quad \forall y \in \varOmega, \tag{1.2}$$

where $\Psi : C \times C \to \mathcal{R} \cup \{+\infty\}$ such that $\Psi(x, x) = 0\ \forall x \in C$ and

$$\varOmega = \mathrm{Sol}(C, \Phi) \cap \mathrm{Fix}(T).$$

Pick the parameter sequences $\{\lambda_n\}$ and $\{\beta_n\}$ such that

$$\begin{cases} \{\lambda_n\} \subset (\underline{\alpha}, \overline{\alpha}) \subset \left(0, \min\left\{\dfrac{1}{2c_1}, \dfrac{1}{2c_2}\right\}\right), \ \lim\limits_{n\to\infty} \lambda_n = \lambda, \\ \beta_n \downarrow 0, \ 2\beta_n\eta - \beta_n^2 \varUpsilon^2 < 1, \ \sum\limits_{n=0}^{\infty} \beta_n = +\infty, \\ 0 < \tau < \min\{\eta, \varUpsilon\}, \ 0 < \beta_n < \min\left\{\dfrac{1}{\tau}, \dfrac{2\eta - 2\tau}{\varUpsilon^2 - \tau^2}, \dfrac{2\eta}{\varUpsilon^2}\right\}, \end{cases} \tag{1.3}$$

where $\varUpsilon$ is a constant associated with Ψ. The following modified subgradient extragradient method is proposed in [1, Algorithm 4.1], for finding a unique element of $\mathrm{Sol}(\varOmega, \Psi)$.

Algorithm 1.1.

Initial Step: Choose an initial point $x^0 \in C$ and $\{\alpha_n\} \subset [r, \bar{r}] \subset (0, 1 - \mathcal{K})$. The parameter sequences $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the conditions (1.3).

Iterative Steps: Compute x^{n+1} $(n \geq 0)$ as follows:

Step 1. Compute

$$v^n = \mathrm{argmin}\left\{\lambda_n \Phi(x^n, v) + \frac{1}{2}\|v - x^n\|^2 : v \in C\right\}$$

and

$$q^n = \mathrm{argmin}\left\{\lambda_n \Phi(v^n, z) + \frac{1}{2}\|z - x^n\|^2 : z \in C_n\right\},$$

where $C_n = \{y \in \mathcal{H} : \langle x^n - \lambda_n w^n - v^n, v^n - y\rangle \geq 0\}$ and $w^n \in \partial_2 \Phi(x^n, v^n)$.

Step 2. Compute

$$p^n = (1 - \alpha_n) q^n + \alpha_n T q^n$$

and

$$x^{n+1} = \mathrm{argmin}\left\{\beta_n \Psi(p^n, p) + \frac{1}{2}\|p - p^n\|^2 : p \in C\right\}.$$

Set $n := n + 1$ and return to Step 1.

It was proven in [1] that $\{x^n\}$ converges strongly to a unique element of $\mathrm{Sol}(\varOmega, \Psi)$ under some mild conditions. In what follows, let the CFPP indicate a common fixed-point problem of countable nonexpansive mappings and an asymptotically nonexpansive mapping. In this paper, via a new inertial subgradient extragradient rule we introduce and analyze two iterative algorithms for solving the monotone bilevel equilibrium problem (MBEP) with the GSVI and CFPP constraints, i.e., a strongly monotone equilibrium problem $\mathrm{EP}(\varOmega, \Psi)$ over the common solution set $\varOmega$ of another monotone equilibrium problem $\mathrm{EP}(C, \Phi)$, the GSVI and the CFPP. Some strong convergence results for the proposed algorithms are established under some mild assumptions. Our results improve and extend some corresponding results in the earlier and very recent literature; see e.g., [1], [9], [4], [15].

2. Preliminaries

We denote by " $\to$ " strong convergence and by " $\rightharpoonup$ " weak convergence in a real Hilbert space $\mathcal{H}$. Let $\emptyset \neq C \subset H$ be convex and closed. A bifunction $\Psi : C \times C \to \mathcal{R}$ is said to be

(i) η-strongly monotone, if $\Psi(x, y) + \Psi(y, x) \leq -\eta\|x - y\|^2 \ \forall x, y \in C$;

(ii) monotone, if $\Psi(x, y) + \Psi(y, x) \leq 0 \ \forall x, y \in C$;

(iii) Lipschitz-type continuous with constants $c_1, c_2 > 0$ (see [24]), if

$$\Psi(x, y) + \Psi(y, z) \geq \Psi(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2 \ \forall x, y, z \in C.$$

Also, recall that a mapping $F : C \to \mathcal{H}$ is said to be

(i) L-Lipschitz continuous or L-Lipschitzian if $\exists L > 0$ s.t.

$$\|Fx - Fy\| \leq L\|x - y\| \ \forall x, y \in C;$$

(ii) monotone if $\langle Fx - Fy, x - y\rangle \geq 0 \ \forall x, y \in C$;

(iii) pseudomonotone if $\langle Fx, y - x\rangle \geq 0 \Rightarrow \langle Fy, y - x\rangle \geq 0 \ \forall x, y \in C$;

(iv) η-strongly monotone if $\exists \eta > 0$ s.t. $\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2 \ \forall x, y \in C$;

(v) α-inverse-strongly monotone if $\exists \alpha > 0$ s.t.

$$\langle Fx - Fy, x - y\rangle \geq \alpha\|Fx - Fy\|^2 \ \forall x, y \in C.$$

It is clear that every inverse-strongly monotone mapping is monotone and Lipschitz continuous but the converse is not true. For each point $x \in \mathcal{H}$, we know that $\exists \mid$ (nearest point) $P_C x \in C$ s.t. $\|x - P_C x\| \leq \|x - y\| \ \forall y \in C$. The mapping P_C is said to be the metric projection of $\mathcal{H}$ onto C. Recall that the following statements hold (see [20]):

(i) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \ \forall x, y \in \mathcal{H}$;

(ii) $\langle x - P_C x, y - P_C x\rangle \leq 0 \ \forall x \in \mathcal{H}, y \in C$;

(iii) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \ \forall x \in \mathcal{H}, y \in C$;

(iv) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \ \forall x, y \in \mathcal{H}$;

(v) $\|sx + (1 - s)y\|^2 = s\|x\|^2 + (1 - s)\|y\|^2 - s(1 - s)\|x - y\|^2 \ \forall x, y \in \mathcal{H}, s \in [0, 1]$.

Definition 2.1. (see [26]) Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C, and $\{\zeta_i\}_{i=1}^{\infty}$ be a sequence in $[0, 1]$. $\forall k \geq 1$, $W_k : C \to C$ is the mapping formulated below:

$$\begin{cases} U_{k,k+1} = I, \ U_{k,k} = \zeta_k T_k U_{k,k+1} + (1 - \zeta_k)I, \\ U_{k,k-1} = \zeta_{k-1} T_{k-1} U_{k,k} + (1 - \zeta_{k-1})I, \\ \cdots \\ U_{k,i} = \zeta_i T_i U_{k,i+1} + (1 - \zeta_i)I, \\ \cdots \\ U_{k,2} = \zeta_2 T_2 U_{k,3} + (1 - \zeta_2)I, \\ W_k = U_{k,1} = \zeta_1 T_1 U_{k,2} + (1 - \zeta_1)I. \end{cases}$$

The W_k, which is called a W-mapping, is nonexpansive.

Lemma 2.1. (see [26]) Let $\{T_i\}_{i=1}^\infty$ be a family of nonexpansive operators with $\bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \neq \emptyset$ and $\{\zeta_i\}_{i=1}^\infty$ be a sequence in $(0,1]$. Then

(i) W_k is nonexpansive and $\mathrm{Fix}(W_k) = \bigcap_{i=1}^k \mathrm{Fix}(T_i)$ $\forall k \geq 1$;

(ii) the limit $\lim_{k\to\infty} U_{k,i}x$ exists for all $x \in C$ and $i \geq 1$;

(iii) the mapping W defined by

$$Wx := \lim_{k\to\infty} W_k x = \lim_{k\to\infty} U_{k,1}x \ \forall x \in C,$$

is nonexpansive with the fact that $\mathrm{Fix}(W) = \bigcap_{i=1}^\infty \mathrm{Fix}(T_i)$.

Lemma 2.2. (see [15]) Let $\{T_i\}_{i=1}^\infty$ be a sequence of nonexpansive self-mappings on C with $\bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \neq \emptyset$ and $\{\zeta_i\}_{i=1}^\infty$ be a sequence in $(0,l]$ for some $l \in (0,1)$. If D is any bounded subset of C, then

$$\lim_{k\to\infty} \sup_{x\in D} \|W_k x - Wx\| = 0.$$

Throughout this paper we always assume that $\{\zeta_i\}_{i=1}^\infty \subset (0,l]$ for some $l \in (0,1)$. Let $B : C \to 2^{\mathcal{H}}$ be a set-valued operator with $Bx \neq \emptyset$ $\forall x \in C$. B is said to be monotone if for each $x, y \in C$, one has $\langle u - v, x - y\rangle \geq 0$ $\forall u \in Bx, v \in By$. Also, B is said to be maximal monotone if $(I + \lambda B)C = \mathcal{H}$ for all $\lambda > 0$. For a monotone operator B, we define the mapping $J_\lambda^B : (I + \lambda B)C \to C$ by $J_\lambda^B = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such J_λ^B is called the resolvent of B for $\lambda > 0$.

Proposition 2.1. (see [11, Lemma 1]) Let $B : C \to 2^{\mathcal{H}}$ be maximal monotone. Then, for any given $\lambda > 0$,

$$\|J_\lambda^B x - J_\lambda^B y\|^2 \leq \langle x - y, J_\lambda^B x - J_\lambda^B y\rangle \ \forall x, y \in \mathcal{H}.$$

Lemma 2.3. (see [11, Lemma 4]) Let the mapping $F : \mathcal{H} \to \mathcal{H}$ be γ-inverse-strongly monotone. Then, for a given $\lambda \geq 0$,

$$\|(I - \lambda F)u - (I - \lambda F)v\|^2 \leq \|u - v\|^2 - \lambda(2\gamma - \lambda)\|Fu - Fv\|^2, \ \forall u, v \in \mathcal{H}.$$

In particular, if $0 \leq \lambda \leq 2\gamma$, then $I - \lambda F$ is nonexpansive.

Using Proposition 2.1 and Lemma 2.3, we immediately derive the following lemma.

Lemma 2.4. (see [11, Lemma 5]) Suppose that $B_1, B_2 : C \to 2^{\mathcal{H}}$ are two maximal monotone operators. Let the mappings $F_1, F_2 : \mathcal{H} \to \mathcal{H}$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : \mathcal{H} \to C$ be defined as

$$G := J_{\lambda_1}^{B_1}(I - \lambda_1 F_1) J_{\lambda_2}^{B_2}(I - \lambda_2 F_2).$$

Then G is nonexpansive for $0 \leq \lambda_1 \leq 2\alpha$ and $0 \leq \lambda_2 \leq 2\beta$.

The following inequality is an immediate consequence of the subdifferential inequality of the function $\|\cdot\|^2/2$.

Lemma 2.5. The inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \quad \forall x, y \in \mathcal{H}.$$

Lemma 2.6. (see [21]) Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero, i.e., if $\{u^k\}$ is a sequence in C such that $u^k \rightharpoonup u \in C$ and $(I-T)u^k \to 0$, then $(I-T)u = 0$, where I is the identity mapping of X.

The proposition below is very vital to analyze the convergence of the proposed algorithms.

Proposition 2.2. (see [23]) Let $\{\mathbf{\Gamma}_k\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\mathbf{\Gamma}_{k_j}\}$ of $\{\mathbf{\Gamma}_k\}$ which satisfies $\mathbf{\Gamma}_{k_j} < \mathbf{\Gamma}_{k_j+1}$ for each integer $j \ge 1$. Let the sequence $\{\tau(k)\}_{k \ge k_0}$ of integers be formulated as $\tau(k) = \max\{j \le k : \mathbf{\Gamma}_j < \mathbf{\Gamma}_{j+1}\}$, where integer $k_0 \ge 1$ such that $\{j \le k_0 : \mathbf{\Gamma}_j < \mathbf{\Gamma}_{j+1}\} \ne \emptyset$. Then, the following hold:

(i) $\tau(k_0) \le \tau(k_0+1) \le \cdots$ and $\tau(k) \to \infty$; and

(ii) $\mathbf{\Gamma}_{\tau(k)} \le \mathbf{\Gamma}_{\tau(k)+1}$ and $\mathbf{\Gamma}_k \le \mathbf{\Gamma}_{\tau(k)+1}$ $\forall k \ge k_0$.

On the other hand, the normal cone $N_C(x)$ of C at $x \in C$ is defined as

$$N_C(x) = \{z \in \mathcal{H} : \langle z, y - x\rangle \le 0 \ \forall y \in C\}.$$

The subdifferential of a convex function $g : C \to \mathcal{R} \cup \{+\infty\}$ at $x \in C$ is defined by

$$\partial g(x) = \{z \in \mathcal{H} : g(y) - g(x) \ge \langle z, y - x\rangle \ \forall y \in C\}.$$

Next we are devoted to finding a solution $x^* \in \mathrm{Sol}(\mathit{\Omega}, \Psi)$ of problem $\mathrm{EP}(\mathit{\Omega}, \Psi)$, with $\mathit{\Omega} = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi)$ and $T_0 := T$. Assume always that the following hold:

$\{T_i\}_{i=1}^{\infty}$ is a sequence of nonexpansive self-mappings on C and $T : \mathcal{H} \to C$ is an asymptotically nonexpansive mapping with a sequence $\{\theta_k\}$.

W_k is the W-mapping generated by $T_k, T_{k-1}, ..., T_1$ and $\zeta_k, \zeta_{k-1}, ..., \zeta_1$, where $\{\zeta_i\}_{i=1}^{\infty}$ is a sequence in $(0, l]$ for some $l \in (0, 1)$.

$B_1, B_2 : C \to 2^{\mathcal{H}}$ are two maximal monotone operators, and $F_1, F_2 : \mathcal{H} \to \mathcal{H}$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively.

$G : \mathcal{H} \to C$ is defined as $G := J_{\lambda_1}^{B_1}(I - \lambda_1 F_1) J_{\lambda_2}^{B_2}(I - \lambda_2 F_2)$ for $\lambda_1 \in (0, 2\alpha)$ and $\lambda_2 \in (0, 2\beta)$.

Choose the sequences $\{\beta_k\}, \{\gamma_k\}, \{\delta_k\} \subset (0,1)$ and $\{\alpha_k\}, \{\tau_k\}, \{s_k\} \subset (0, \infty)$ such that

(H1) $\beta_k + \gamma_k + \delta_k = 1$ $\forall k \ge 1$, $0 < \liminf\limits_{k\to\infty} \beta_k$ and $0 < \liminf\limits_{k\to\infty} \delta_k$;

(H2) $0 < \liminf\limits_{k\to\infty} \gamma_k \le \limsup\limits_{k\to\infty} \gamma_k < 1$ and $\sum\limits_{k=1}^{\infty} s_k = \infty$;

(H3) $\lim\limits_{k\to\infty} s_k = 0$, $\lim\limits_{k\to\infty} \tau_k / s_k = 0$, $\lim\limits_{k\to\infty} \theta_k / s_k = 0$ and $\sum\limits_{k=1}^{\infty} \theta_k < \infty$;

(H4) $\{\alpha_k\} \subset (\underline{\alpha}, \overline{\alpha}) \subset \left(0, \min\left\{\dfrac{1}{2c_1}, \dfrac{1}{2c_2}\right\}\right)$ and $\lim\limits_{k\to\infty} \alpha_k = \tilde{\alpha}$;

(H5) $2 s_k \nu - s_k^2 \mathit{\Upsilon}^2 < 1$, $0 < \lambda < \min\{\nu, \mathit{\Upsilon}\}$ and $0 < s_k < \min\left\{\dfrac{1}{\lambda}, \dfrac{2\nu - 2\lambda}{\mathit{\Upsilon}^2 - \lambda^2}, \dfrac{2\nu}{\mathit{\Upsilon}^2}\right\}$.

Algorithm 2.1. Initial Step: Let $\varepsilon > 0$ and $x^0, x^1 \in C$ be arbitrary. The sequences $\{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ in $(0,1)$, and positive sequences $\{\alpha_k\}, \{\tau_k\}, \{s_k\}$ satisfy conditions (H1)-(H5).

Iterative Steps: Calculate x^{k+1} as follows:

Step 1. Given the iterates x^{k-1} and x^k $(k \geq 1)$, choose ε_k such that $0 \leq \varepsilon_k \leq \overline{\varepsilon_k}$, where

$$\overline{\varepsilon_k} = \begin{cases} \min\left\{\varepsilon, \frac{\tau_k}{\|x^k - x^{k-1}\|}\right\} & \text{if } x^k \neq x^{k-1}, \\ \varepsilon & \text{otherwise.} \end{cases} \tag{2.1}$$

Step 2. Compute $t^k = x^k + \varepsilon_k(x^k - x^{k-1})$ and

$$y^k = \operatorname{argmin}\{\alpha_k \Phi(t^k, y) + \frac{1}{2}\|y - t^k\|^2 : y \in C\}.$$

Step 3. Chosen $w^k \in \partial_2\Phi(t^k, y^k)$, compute

$$C_k = \{y \in \mathcal{H} : \langle t^k - \alpha_k w^k - y^k, y^k - y\rangle \geq 0\}$$

and

$$z^k = \operatorname{argmin}\left\{\alpha_k \Phi(y^k, z) + \frac{1}{2}\|z - t^k\|^2 : z \in C_k\right\}.$$

Step 4. Compute $u^k = \beta_k x^k + \gamma_k W_k p^k + \delta_k T^k z^k$, with $p^k = J_{\lambda_1}^{B_1}(v^k - \lambda_1 F_1 v^k)$ and

$$v^k = J_{\lambda_2}^{B_2}(u^k - \lambda_2 F_2 u^k).$$

Step 5. Compute $x^{k+1} = \operatorname{argmin}\{s_k \Psi(u^k, t) + \frac{1}{2}\|t - u^k\|^2 : t \in C\}$.

Set $k := k + 1$ and return to Step 1.

Remark 2.1. From (2.1), it is readily known that $\lim\limits_{k\to\infty} \frac{\varepsilon_k}{s_k}\|x^k - x^{k-1}\| = 0$.

We need the following technical propositions in the sequel.

Proposition 2.3. (see [3, Theorem 2.1.3]) Let C be a convex subset of a real Hilbert space $\mathcal{H}$ and $g : C \to \mathcal{R} \cup \{+\infty\}$ be subdifferentiable. Then, $\hat{x}$ is a solution to the convex minimization problem $\min\{g(x) : x \in C\}$ if and only if $0 \in \partial g(\hat{x}) + N_C(\hat{x})$, where ∂g denotes the subdifferential of g.

Proposition 2.4. (see [2, Proposition 23]) Let X and Y be two sets, $\mathcal{G}$ be a set-valued map from Y to X, and W be a real valued function defined on $X \times Y$. Let the marginal function M be defined as

$$M(y) = \{x^* \in \mathcal{G}(y) : W(x^*, y) = \sup\{W(x, y) : x \in \mathcal{G}(y)\}\}.$$

If W and $\mathcal{G}$ are continuous, then M is upper semicontinuous (u.s.c.).

Next, we assume that two bifunctions $\Psi : C \times C \to \mathcal{R} \cup \{+\infty\}$ and $\Phi : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \cup \{+\infty\}$ satisfy the following conditions:

Ass$_\Phi$: (Φ_1) $\Omega = \bigcap_{i=0}^{\infty} \operatorname{Fix}(T_i) \cap \operatorname{Fix}(G) \cap \operatorname{Sol}(C, \Phi) \neq \emptyset$ with $T_0 := T$.

(Φ_2) Φ is monotone and Lipschitz-type continuous with constants $c_1, c_2 > 0$, and Φ is weakly continuous, i.e., $\{x^k \rightharpoonup \bar{x}$ and $y^k \rightharpoonup \bar{y}\} \Rightarrow \{\Phi(x^k, y^k) \to \Phi(\bar{x}, \bar{y})\}$.

Ass$_\Psi$: (Ψ_1) Ψ is ν-strongly monotone and weakly continuous.

(Ψ_2) $\forall i \in \{1, ..., m\}$, $\exists$ (mappings) $\bar{\Psi}_i, \hat{\psi}_i : C \times C \to \mathcal{H}$ such that

(a) $\bar{\Psi}_i(x, y) + \bar{\Psi}_i(y, x) = 0$ and $\|\bar{\Psi}_i(x, y)\| \leq \bar{L}_i\|x - y\|$ $\forall x, y \in C$;

(b) $\hat{\psi}_i(x, x) = 0$ and $\|\hat{\psi}_i(x, y) - \hat{\psi}_i(u, v)\| \leq \hat{L}_i\|(x - y) - (u - v)\|$ $\forall x, y, u, v \in C$;

(c) $\Psi(x, y) + \Psi(y, z) \geq \Psi(x, z) + \sum_{i=1}^{m}\langle\bar{\Psi}_i(x, y), \hat{\psi}_i(y, z)\rangle$ $\forall x, y, z \in C$.

(Ψ_3) For any sequence $\{y^k\} \subset C$ such that $y^k \to d$, we have

$$\limsup_{k\to\infty} \frac{|\Psi(d, y^k)|}{\|y^k - d\|} < +\infty.$$

It is easy to check that if Ψ satisfies the condition $\mathbf{Ass}_\Psi(\Psi_2)$, then

$$\Psi(x, y) + \Psi(y, z) \geq \Psi(x, z) - \frac{1}{2}\Upsilon\|x - y\|^2 - \frac{1}{2}\Upsilon\|y - z\|^2,$$

with

$$\Upsilon := \sum_{i=1}^{m} \bar{L}_i \hat{L}_i.$$

This means that Ψ is Lipschitz-type continuous with constants $c_1 = c_2 = \frac{1}{2}\Upsilon$.

3. Main results

We are now in a position to state and prove the first main result in this paper.

Theorem 3.1. Assume that $\{x^k\}$ is the sequence constructed by Algorithm 2.1. Let the bifunctions Ψ, Φ satisfy the assumptions $\mathbf{Ass}_\Phi$-$\mathbf{Ass}_\Psi$. Then, under the conditions (H1)-(H5), the sequence $\{x^k\}$ converges strongly to the unique solution x^* of the problem EP(Ω, Ψ) provided

$$T^k x^k - T^{k+1} x^k \to 0 \text{ and } \sum_{k=1}^{\infty} \varepsilon_k \|x^k - x^{k-1}\| < \infty.$$

Proof. First of all, by Lemma 2.1 we know that each W_k is a nonexpansive self-mapping on C. Also, note that the mapping $G : \mathcal{H} \to C$ is defined as

$$G = J_{\lambda_1}^{B_1}(I - \lambda_1 F_1) J_{\lambda_2}^{B_2}(I - \lambda_2 F_2),$$

where $\lambda_1 \in (0, 2\alpha)$ and $\lambda_2 \in (0, 2\beta)$. Then, by Lemma 2.4, we know that G is nonexpansive. Hence, by the Banach contraction mapping principle, we deduce from $\{\gamma_k\} \subset (0, 1)$ that for each $k \geq 1$, there exists a unique element $u^k \in C$ such that

$$u^k = \beta_k x^k + \gamma_k W_k G u^k + \delta_k T^k z^k. \tag{3.1}$$

Take an arbitrary point

$$q \in \Omega = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi).$$

Thanks to $\lim_{k\to\infty} \frac{\theta_k}{s_k} = 0$, we might assume that $\theta_k \leq \frac{1}{2}\lambda s_k \ \forall k \geq 1$. Next, we divide the proof into several claims.

Claim 1. We assert that

$$\|z^k - q\|^2 \leq \|t^k - q\|^2 - (1 - 2\alpha_k c_1)\|y^k - t^k\|^2 - (1 - 2\alpha_k c_2)\|z^k - y^k\|^2 \ \forall k \geq 1.$$

In fact, using Proposition 2.3, one obtains that for

$$y^k = \mathrm{argmin}\{\alpha_k \Phi(t^k, y) + \frac{1}{2}\|y - t^k\|^2 : y \in C\},$$

$\exists w^k \in \partial_2\Phi(t^k, y^k)$ s.t. $\alpha_k w^k + y^k - t^k \in -N_C(y^k)$. This hence arrives at

$$\langle \alpha_k w^k + y^k - t^k, x - y^k\rangle \geq 0 \ \forall x \in C.$$

According to $w^k \in \partial_2\Phi(t^k, y^k)$, one gets

$$\alpha_k[\Phi(t^k, x) - \Phi(t^k, y^k)] \geq \langle \alpha_k w^k, x - y^k\rangle \quad \forall x \in \mathcal{H}.$$

Thus, we get

$$\alpha_k[\Phi(t^k, x) - \Phi(t^k, y^k)] + \langle y^k - t^k, x - y^k\rangle \geq 0 \quad \forall x \in C. \tag{3.2}$$

It follows from $z^k \in C_k$ and the definition of C_k that

$$\langle t^k - \alpha_k w^k - y^k, v - y^k\rangle \leq 0,$$

and hence $\alpha_k\langle w^k, z^k - y^k\rangle \geq \langle t^k - y^k, z^k - y^k\rangle$. It is easy to see that

$$\alpha_k[\Phi(t^k, z^k) - \Phi(t^k, y^k)] \geq \alpha_k\langle w^k, z^k - y^k\rangle,$$

and hence

$$\alpha_k[\Phi(t^k, z^k) - \Phi(t^k, y^k)] \geq \langle t^k - y^k, z^k - y^k\rangle. \tag{3.3}$$

Using Proposition 2.3, one obtains that for

$$z^k = \operatorname{argmin}\{\alpha_k\Phi(y^k, y) + \frac{1}{2}\|y - t^k\|^2 : y \in C_k\},$$

$\exists h^k \in \partial_2\Phi(y^k, z^k)$ and $\exists r^k \in N_{C_k}(z^k)$ s.t. $\alpha_k h^k + z^k - t^k + r^k = 0$. So it follows that

$$\alpha_k\langle h^k, y - z^k\rangle \geq \langle t^k - z^k, y - z^k\rangle \ \forall y \in C_k,$$

and

$$\Phi(y^k, y) - \Phi(y^k, z^k) \geq \langle h^k, y - z^k\rangle \quad \forall y \in \mathcal{H}.$$

Putting $y = q \in C \subset C_k$ in two last inequalities and later adding them, one gets

$$\alpha_k[\Phi(y^k, q) - \Phi(y^k, z^k)] \geq \langle t^k - z^k, q - z^k\rangle.$$

Using the monotonicity of Φ, $q \in \mathrm{Sol}(C, \Phi)$ and $y^k \in C$, one gets

$$\Phi(y^k, q) \leq -\Phi(q, y^k) \leq 0.$$

Consequently, $-\alpha_k\Phi(y^k, z^k) \geq \langle t^k - z^k, q - z^k\rangle$. Combining this and the Lipschitz-type continuity of Φ,

$$\Phi(t^k, y^k) + \Phi(y^k, z^k) \geq \Phi(t^k, z^k) - c_1\|t^k - y^k\|^2 - c_2\|y^k - z^k\|^2,$$

one deduces from (3.3), ensures that

$$\langle t^k - z^k, z^k - q\rangle \geq \langle t^k - y^k, z^k - y^k\rangle - \alpha_k c_1\|t^k - y^k\|^2 - \alpha_k c_2\|y^k - z^k\|^2. \tag{3.4}$$

Therefore, applying the equality

$$\langle u, v\rangle = \frac{1}{2}(\|u + v\|^2 - \|u\|^2 - \|v\|^2) \ \forall u, v \in \mathcal{H},$$

for $\langle t^k - z^k, z^k - q\rangle$ and $\langle y^k - t^k, z^k - y^k\rangle$ in (3.4), we derive the desired claim.

Claim 2. We assert that

$$\|x^{k+1} - x\|^2 \leq \|u^k - x\|^2 - \|x^{k+1} - u^k\|^2 + 2s_k[\Psi(u^k, x) - \Psi(u^k, x^{k+1})] \ \forall x \in C.$$

In fact, thanks to

$$x^{k+1} = \operatorname{argmin}\{s_k \Psi(u^k, t) + \frac{1}{2}\|t - u^k\|^2 : t \in C\},$$

$\exists m^k \in \partial_2 \Psi(u^k, x^{k+1})$ s.t. $0 \in s_k m^k + x^{k+1} - u^k + N_C(x^{k+1})$. Using the normal cone N_C and subgradient m^k, one gets

$$\langle s_k m^k + x^{k+1} - u^k, x - x^{k+1}\rangle \geq 0 \ \forall x \in C$$

and

$$s_k[\Psi(u^k, x) - \Psi(u^k, x^{k+1})] \geq \langle s_k m^k, x - x^{k+1}\rangle \ \forall x \in C.$$

Adding these two inequalities, one has

$$2s_k[\Psi(u^k, x) - \Psi(u^k, x^{k+1})] + 2\langle x^{k+1} - u^k, x - x^{k+1}\rangle \geq 0 \quad \forall x \in C. \tag{3.5}$$

Setting $u = x^{k+1} - u^k$ and $v = x - x^{k+1}$ in the equality

$$\langle u, v\rangle = \frac{1}{2}(\|u + v\|^2 - \|u\|^2 - \|v\|^2) \ \forall u, v \in \mathcal{H},$$

one gets

$$2s_k[\Psi(x^{k+1}, x) - \Psi(u^k, x^{k+1})] + \|u^k - x\|^2 - \|x^{k+1} - u^k\|^2 - \|x^{k+1} - x\|^2 \geq 0 \ \forall x \in C.$$

Claim 3. We assert that if x^* is a solution of the MBEP with the GSVI and CFPP constraints, then

$$\|x^{k+1} - u_*^k\| \leq \eta_k \|u^k - x^*\| \leq (1 - \lambda s_k)\|u^k - x^*\|,$$

where

$$u_*^k = \operatorname{argmin}\{s_k \Psi(x^*, v) + \frac{1}{2}\|v - x^*\|^2 : v \in C\},$$

$$\eta_k = \sqrt{1 - 2s_k \nu + s_k^2 \varUpsilon^2}, \ 0 < \lambda < \min\{\nu, \varUpsilon\}, \ 0 < s_k < \min\left\{\frac{1}{\lambda}, \frac{2\nu - 2\lambda}{\varUpsilon^2 - \lambda^2}\right\},$$

and

$$\varUpsilon = \sum_{i=1}^m \bar{L}_i \hat{L}_i.$$

In fact, set

$$u_*^k = \operatorname{argmin}\{s_k \Psi(x^*, v) + \frac{1}{2}\|v - x^*\|^2 : v \in C\}.$$

Using the similar reasonings to those of (3.5), one also gets

$$s_k[\Psi(x^*, x) - \Psi(x^*, u_*^k)] + \langle u_*^k - x^*, x - u_*^k\rangle \geq 0 \quad \forall x \in C. \tag{3.6}$$

Putting $x = u_*^k \in C$ in (3.5) and $x = x^{k+1} \in C$ in (3.6), respectively, one obtains

$$\begin{aligned} 0 \leq\ & 2s_k[\Psi(u^k, u_*^k) - \Psi(u^k, x^{k+1}) + \Psi(x^*, x^{k+1}) - \Psi(x^*, u_*^k)] + \|u^k - x^*\|^2 \\ & - \|x^{k+1} - u^k - u_*^k + x^*\|^2 - \|x^{k+1} - u_*^k\|^2. \end{aligned} \tag{3.7}$$

From $\mathbf{Ass}_\Psi(\Psi_2)$ it follows that

$$\Psi(u^k, u_*^k) - \Psi(x^*, u_*^k) \leq \Psi(u^k, x^*) - \sum_{i=1}^m \langle \bar{\Psi}_i(u^k, x^*), \hat{\psi}_i(x^*, u_*^k)\rangle$$

and

$$\Psi(x^*, x^{k+1}) - \Psi(u^k, x^{k+1}) \le \Psi(x^*, u^k) - \sum_{i=1}^{m} \langle \bar{\Psi}_i(x^*, u^k), \hat{\psi}_i(u^k, x^{k+1}) \rangle.$$

Therefore, one has

$$\begin{aligned}\Psi(u^k, u_*^k) - \Psi(u^k, x^{k+1}) + \Psi(x^*, x^{k+1}) - \Psi(x^*, u_*^k) &\le \Psi(u^k, x^*) + \Psi(x^*, u^k) \\ &- \sum_{i=1}^{m} \langle \bar{\Psi}_i(u^k, x^*), \hat{\psi}_i(x^*, u_*^k) \rangle - \sum_{i=1}^{m} \langle \bar{\Psi}_i(x^*, u^k), \hat{\psi}_i(u^k, x^{k+1}) \rangle.\end{aligned}$$

Then, using $\mathbf{Ass}_\Psi(\Psi_2)$, and the ν-strong monotonicity of Ψ in $\mathbf{Ass}_\Psi(\Psi_1)$, one gets

$$\begin{aligned}&\Psi(u^k, u_*^k) - \Psi(u^k, x^{k+1}) + \Psi(x^*, x^{k+1}) - \Psi(x^*, u_*^k) \\ &\le -\nu \|u^k - x^*\|^2 + \sum_{i=1}^{m} \langle \bar{\Psi}_i(u^k, x^*), \hat{\psi}_i(u^k, x^{k+1}) - \hat{\psi}_i(x^*, u_*^k) \rangle \\ &\le -\nu \|u^k - x^*\|^2 + \varUpsilon \|u^k - x^*\| \|u^k - x^{k+1} - x^* + u_*^k\|. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), one has

$$\begin{aligned}0 &\le (1 - 2s_k\nu)\|u^k - x^*\|^2 + 2s_k \varUpsilon \|u^k - x^*\| \|u^k - x^{k+1} - x^* + u_*^k\| \\ &- \|x^{k+1} - u^k - u_*^k + x^*\|^2 - \|x^{k+1} - u_*^k\|^2 \\ &\le (1 - 2s_k\nu + s_k^2 \varUpsilon^2)\|u^k - x^*\|^2 - \|x^{k+1} - u_*^k\|^2.\end{aligned}$$

Using $0 < \lambda < \min\{\nu, \varUpsilon\}$ and $0 < s_k < \min\left\{\frac{1}{\lambda}, \frac{2\nu - 2\lambda}{\varUpsilon^2 - \lambda^2}\right\}$, we get

$$1 - \lambda s_k > \sqrt{1 - 2s_k\nu + s_k^2 \varUpsilon^2} = \eta_k \ge 0.$$

This attains the desired claim.

Claim 4. We assert that $\{x^k\}$ is bounded. In fact, setting

$$\begin{aligned}&X := C,\ Y := [0, 1],\ \mathcal{G}(s) := C\ \forall s \in Y, \\ &s := s_k,\ W(x, s) := -s\Psi(x^*, x) - \frac{1}{2}\|x - x^*\|^2\ \forall (x, s) \in X \times Y,\end{aligned}$$

we have that

$$M(s_k) = \operatorname{argmin}\left\{ s_k\Psi(x^*, x) + \frac{1}{2}\|x - x^*\|^2 : x \in C \right\} = \{u_*^k\}.$$

Note that M is continuous and $\lim_{k\to\infty} u_*^k = x^*$. By the continuity of Ψ on C, one gets

$$\lim_{k\to\infty} \Psi(x^*, u_*^k) = \Psi(x^*, x^*) = 0.$$

According to $\mathbf{Ass}_\Psi(\Psi_3)$, $\exists$ (constant) $\hat{M}(x^*) > 0$ s.t.

$$|\Psi(x^*, u_*^k)| \le \hat{M}(x^*)\|u_*^k - x^*\|\ \forall k \ge 1.$$

Setting $x = x^*$ in (3.6) and using $\Psi(x^*, x^*) = 0$, one has

$$\|u_*^k - x^*\|^2 \le s_k[-\Psi(x^*, u_*^k)] \le s_k\hat{M}(x^*)\|u_*^k - x^*\|\ \forall k \ge 1.$$

This hence ensures that $\|u_*^k - x^*\| \le s_k \hat{M}(x^*)\ \forall k \ge 1$. Also, in terms of Lemma 2.3 we know that $I - \lambda_1 F_1$ and $I - \lambda_2 F_2$ both are nonexpansive for $\lambda_1 \in (0, 2\alpha)$ and $\lambda_2 \in (0, 2\beta)$. Moreover, by Lemma 2.4, we know that G is nonexpansive. We write $y^* = J_{\lambda_2}^{B_2}(I - \lambda_2 F_2)x^*$. From [11, Lemma 2], we get $x^* = J_{\lambda_1}^{B_1}(I - \lambda_1 F_1)y^* = Gx^*$ and hence

$$\|p^k - x^*\| = \|Gu^k - x^*\| \le \|u^k - x^*\|. \tag{3.9}$$

Also, by Remark 2.1 we know that $\frac{\varepsilon_k}{s_k}\|x^k - x^{k-1}\| \to 0$ as $k \to \infty$, which hence implies that $\exists \widetilde{M}_0 > 0$ s.t. $\frac{\varepsilon_k}{s_k}\|x^k - x^{k-1}\| \le \widetilde{M}_0\ \forall k \ge 1$. Accordingly,

$$\|t^k - x^*\| = \|x^k - x^* + \varepsilon_k(x^k - x^{k-1})\| \le \|x^k - x^*\| + \widetilde{M}_0 s_k. \tag{3.10}$$

Using the result in Claim 1, from (3.10) we have

$$\|z^k - x^*\| \le \|t^k - x^*\| \le \|x^k - x^*\| + \widetilde{M}_0 s_k \quad \forall k \ge 1. \tag{3.11}$$

This along with the nonexpansivity of W_k, G and asymptotically nonexpansivity of T, yields

$$\begin{aligned}
\|u^k - x^*\|^2 &\le \beta_k(1+\theta_k)(\|x^k - x^*\| + \widetilde{M}_0 s_k)\|u^k - x^*\| + \gamma_k\|u^k - x^*\|^2 + \delta_k(1+\theta_k) \\
&\quad \times (\|x^k - x^*\| + \widetilde{M}_0 s_k)\|u^k - x^*\| \\
&= (1-\gamma_k)(1+\theta_k)(\|x^k - x^*\| + \widetilde{M}_0 s_k)\|u^k - x^*\| + \gamma_k\|u^k - x^*\|^2,
\end{aligned}$$

which hence leads to $\|u^k - x^*\| \le (1+\theta_k)(\|x^k - x^*\| + \widetilde{M}_0 s_k)$. Consequently,

$$\begin{aligned}
\|x^{k+1} - x^*\| &\le (1 - \lambda s_k)(1+\theta_k)(\|x^k - x^*\| + \widetilde{M}_0 s_k) + s_k \hat{M}(x^*) \\
&\le \max\left\{\|x^k - x^*\|, \frac{2[\widetilde{M}_0 + \hat{M}(x^*)]}{\lambda}\right\}.
\end{aligned} \tag{3.12}$$

By induction, we get

$$\|x^k - x^*\| \le \max\left\{\|x^1 - x^*\|, \frac{2[\widetilde{M}_0 + \hat{M}(x^*)]}{\lambda}\right\}\ \forall k \ge 1.$$

Thus, $\{x^k\}$ is bounded, and so are the sequences $\{p^k\}, \{t^k\}, \{u^k\}, \{v^k\}, \{y^k\}, \{z^k\}$.

Claim 5. We assert that if $x^{k_i} \rightharpoonup \hat{x}$, $t^{k_i} - x^{k_i} \to 0$ and $t^{k_i} - y^{k_i} \to 0$ for $\{k_i\} \subset \{k\}$, then $\hat{x} \in \mathrm{Sol}(C, \Phi)$. In fact, thanks to $t^{k_i} - x^{k_i} \to 0$ and $t^{k_i} - y^{k_i} \to 0$, one gets

$$\|x^{k_i} - y^{k_i}\| \le \|x^{k_i} - t^{k_i}\| + \|t^{k_i} - y^{k_i}\| \to 0\ (i \to \infty). \tag{3.13}$$

So it follows from $x^{k_i} \rightharpoonup \hat{x}$ that $t^{k_i} \rightharpoonup \hat{x}$ and $y^{k_i} \rightharpoonup \hat{x}$. Using $\{y^k\} \subset C$, $y^{k_i} \rightharpoonup \hat{x}$ and the weak closedness of C, we obtain that $\hat{x} \in C$. From (3.2), we get

$$\alpha_{k_i}\Phi(t^{k_i}, x) \ge \alpha_{k_i}\Phi(t^{k_i}, y^{k_i}) + \langle y^{k_i} - t^{k_i}, y^{k_i} - x\rangle\ \forall x \in C.$$

Taking the limit as $i \to \infty$ and using the conditions that

$$\lim_{k\to\infty} \alpha_k = \tilde{\alpha} > 0,$$

$\Phi(\hat{x}, \hat{x}) = 0$, $\{y^{k_i}\}$ is of boundedness and Φ is of weak continuity, we deduce that $\tilde{\alpha}\Phi(\hat{x}, x) \ge 0\ \forall x \in C$. This means that $\hat{x} \in \mathrm{sol}(C, \Phi)$.

Claim 6. We assert that $x^k \to x^*$, a unique solution of the MBEP with the GSVI and CFPP constraints. In fact, since each W_k and G both are nonexpansive and T is asymptotically nonexpansive, one obtains that

$$\begin{aligned}\|u^k - x^*\|^2 \le{}& \frac{\beta_k}{2}\|x^k - x^*\|^2 + \frac{1+\gamma_k}{2}\|u^k - x^*\|^2 + \frac{\delta_k}{2}\|z^k - x^*\|^2 + \frac{\theta_k \widetilde{M}}{2} \\ & - \frac{\beta_k}{2}\|x^k - u^k\|^2 - \frac{\delta_k}{2}\|T^k z^k - u^k\|^2,\end{aligned}$$

where

$$\sup_{k\ge 1}(2+\theta_k)(\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 \le \widetilde{M}$$

for some $\widetilde{M} > 0$. This hence arrives at

$$\begin{aligned}\|u^k - x^*\|^2 \le{}& \frac{1}{1-\gamma_k}[\beta_k\|x^k - x^*\|^2 + \delta_k\|z^k - x^*\|^2 + \theta_k\widetilde{M} - \beta_k\|x^k - u^k\|^2 \\ & - \delta_k\|T^k z^k - u^k\|^2]. \end{aligned} \tag{3.14}$$

Using the results in Claims 1 and 2 we infer from (3.11) and (3.14) that

$$\begin{aligned}\|x^{k+1} - x^*\|^2 \le{}& \frac{1}{1-\gamma_k}\{\beta_k\|x^k - x^*\|^2 + \delta_k[\|t^k - x^*\|^2 \\ & - (1-2\alpha_k c_1)\|y^k - t^k\|^2 - (1-2\alpha_k c_2)\|z^k - y^k\|^2] \\ & + \theta_k\widetilde{M} - \beta_k\|x^k - u^k\|^2 - \delta_k\|T^k z^k - u^k\|^2\} \\ & - \|x^{k+1} - u^k\|^2 + 2s_k[\Psi(u^k, x^*) - \Psi(u^k, x^{k+1})] \\ \le{}& (\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 - \frac{\delta_k}{1-\gamma_k}[(1-2\alpha_k c_1)\|y^k - t^k\|^2 \\ & + (1-2\alpha_k c_2)\|z^k - y^k\|^2] + \frac{\theta_k\widetilde{M}}{1-\gamma_k} \\ & - \frac{1}{1-\gamma_k}[\beta_k\|x^k - u^k\|^2 + \delta_k\|T^k z^k - u^k\|^2] \\ & - \|x^{k+1} - u^k\|^2 + s_k K, \end{aligned} \tag{3.15}$$

where $\sup_{k\ge 1}\{2|\Psi(u^k, x^*) - \Psi(u^k, x^{k+1})|\} \le K$ for some $K > 0$.

Next, we put $\mathbf{\Gamma}_k = \|x^k - x^*\|^2$ and demonstrate the convergence of $\{\mathbf{\Gamma}_k\}$ to zero in the following two aspects:

Aspect 1. Suppose that $\exists$ (integer) $k_0 \ge 1$ s.t. $\{\mathbf{\Gamma}_k\}$ is non-increasing. Then the limit $\lim_{k\to\infty}\mathbf{\Gamma}_k = \hbar < +\infty$ and $\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1} \to 0$ $(k \to \infty)$. From (3.15), we get

$$\begin{aligned}& \delta_k[(1-2\alpha_k c_1)\|y^k - t^k\|^2 + (1-2\alpha_k c_2)\|z^k - y^k\|^2] \\ & \quad + \beta_k\|x^k - u^k\|^2 + \delta_k\|T^k z^k - u^k\|^2 + \|x^{k+1} - u^k\|^2 \\ & \le \mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1} + \widetilde{M}_0 s_k(2\sqrt{\mathbf{\Gamma}_k} + \widetilde{M}_0 s_k) + \frac{\theta_k\widetilde{M}}{1-\gamma_k} + s_k K. \end{aligned} \tag{3.16}$$

Since $s_k \to 0$, $\theta_k \to 0$, $\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1} \to 0$,

$$0 < \liminf_{k\to\infty} \beta_k,\ 0 < \liminf_{k\to\infty} \delta_k \text{ and } 0 < \liminf_{k\to\infty}(1-\gamma_k),$$

we deduce from $\{\alpha_k\} \subset (\underline{\alpha}, \overline{\alpha}) \subset \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right)$ that

$$\lim_{k\to\infty} \|x^k - u^k\| = \lim_{k\to\infty} \|T^k z^k - u^k\| = 0, \tag{3.17}$$

$$\lim_{k\to\infty} \|y^k - t^k\| = \lim_{k\to\infty} \|z^k - y^k\| = \lim_{k\to\infty} \|x^{k+1} - u^k\| = 0. \tag{3.18}$$

We now claim that $\|u^k - p^k\| \to 0$ as $k \to \infty$. In fact, we put $y^* = J_{\lambda_2}^{B_2}(x^* - \lambda_2 F_2 x^*)$. Note that $p^k = Gu^k$. Using Lemma 2.3 one has

$$\|v^k - y^*\|^2 \le \|u^k - x^*\|^2 - \lambda_2(2\beta - \lambda_2)\|F_2 u^k - F_2 x^*\|^2, \tag{3.19}$$

$$\|p^k - x^*\|^2 \le \|v^k - y^*\|^2 - \lambda_1(2\alpha - \lambda_1)\|F_1 v^k - F_1 y^*\|^2. \tag{3.20}$$

Substituting (3.19) for (3.20), by (3.11) and (3.14) one gets

$$\begin{aligned}
\|p^k - x^*\|^2 &\le \|u^k - x^*\|^2 - \lambda_2(2\beta - \lambda_2)\|F_2 u^k - F_2 x^*\|^2 \\
&\quad - \lambda_1(2\alpha - \lambda_1)\|F_1 v^k - F_1 y^*\|^2 \\
&\le (\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 + \frac{\theta_k \widetilde{M}}{1-\gamma_k} \\
&\quad - \lambda_2(2\beta - \lambda_2)\|F_2 u^k - F_2 x^*\|^2 - \lambda_1(2\alpha - \lambda_1)\|F_1 v^k - F_1 y^*\|^2.
\end{aligned} \tag{3.21}$$

Also, substituting (3.21) for (3.15), one has

$$\begin{aligned}
&\|x^{k+1} - x^*\|^2 \le \|u^k - x^*\|^2 + s_k K \\
&\le \beta_k(1+\theta_k)^2(\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 + \gamma_k\|p^k - x^*\|^2 + \delta_k(1+\theta_k)^2\|z^k - x^*\|^2 + s_k K \\
&\le (\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 + \frac{\theta_k \widetilde{M}}{1-\gamma_k} - \gamma_k[\lambda_2(2\beta - \lambda_2)\|F_2 u^k - F_2 x^*\|^2 \\
&+ \lambda_1(2\alpha - \lambda_1)\|F_1 v^k - F_1 y^*\|^2] + s_k K,
\end{aligned}$$

which hence leads to

$$\begin{aligned}
&\gamma_k[\lambda_2(2\beta - \lambda_2)\|F_2 u^k - F_2 x^*\|^2 + \lambda_1(2\alpha - \lambda_1)\|F_1 v^k - F_1 y^*\|^2] \\
&\le \mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1} + \widetilde{M}_0 s_k(2\sqrt{\mathbf{\Gamma}_k} + \widetilde{M}_0 s_k) + \frac{\theta_k \widetilde{M}}{1-\gamma_k} + s_k K.
\end{aligned}$$

Since $\lambda_1 \in (0, 2\alpha)$, $\lambda_2 \in (0, 2\beta)$, $s_k \to 0$, $\theta_k \to 0$, $\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1} \to 0$,

$$\liminf_{k\to\infty} \gamma_k > 0 \text{ and } \liminf_{k\to\infty}(1-\gamma_k) > 0,$$

we get

$$\lim_{k\to\infty} \|F_2 u^k - F_2 x^*\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|F_1 v^k - F_1 y^*\| = 0. \tag{3.22}$$

On the other hand, we have

$$\|p^k - x^*\|^2 \le \|v^k - y^*\|^2 - \|v^k - p^k + x^* - y^*\|^2 + 2\lambda_1\|F_1 y^* - F_1 v^k\|\|p^k - x^*\|. \tag{3.23}$$

Similarly, we have

$$\|v^k-y^*\|^2 \le \|u^k-x^*\|^2-\|u^k-v^k+y^*-x^*\|^2+2\lambda_2\|F_2x^*-F_2u^k\|\|v^k-y^*\|. \tag{3.24}$$

Combining (3.23) and (3.24), by (3.11) and (3.14) one has

$$\begin{aligned}\|p^k-x^*\|^2 &\le \|u^k-x^*\|^2-\|u^k-v^k+y^*-x^*\|^2-\|v^k-p^k+x^*-y^*\|^2\\ &\quad+2\lambda_1\|F_1y^*-F_1v^k\|\|p^k-x^*\|+2\lambda_2\|F_2x^*-F_2u^k\|\|v^k-y^*\|\\ &\le (\|x^k-x^*\|+\widetilde{M}_0s_k)^2+\frac{\theta_k\widetilde{M}}{1-\gamma_k}\\ &\quad-\|u^k-v^k+y^*-x^*\|^2-\|v^k-p^k+x^*-y^*\|^2\\ &\quad+2\lambda_1\|F_1y^*-F_1v^k\|\|p^k-x^*\|+2\lambda_2\|F_2x^*-F_2u^k\|\|v^k-y^*\|. \end{aligned}\tag{3.25}$$

Substituting (3.25) for (3.15), from (3.11) one gets

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &\le \|u^k-x^*\|^2+s_kK \le (1-\gamma_k)[1+\theta_k(2+\theta_k)](\|x^k-x^*\|+\widetilde{M}_0s_k)^2\\ &\quad+\gamma_k[(\|x^k-x^*\|+\widetilde{M}_0s_k)^2+\frac{\theta_k\widetilde{M}}{1-\gamma_k}\\ &\quad-\|u^k-v^k+y^*-x^*\|^2-\|v^k-p^k+x^*-y^*\|^2+2\lambda_1\|F_1y^*-F_1v^k\|\\ &\quad\times\|p^k-x^*\|+2\lambda_2\|F_2x^*-F_2u^k\|\|v^k-y^*\|]+s_kK\\ &\le (\|x^k-x^*\|+\widetilde{M}_0s_k)^2+\frac{\theta_k\widetilde{M}}{1-\gamma_k}-\gamma_k[\|u^k-v^k+y^*-x^*\|^2\\ &\quad+\|v^k-p^k+x^*-y^*\|^2]\\ &\quad+2\lambda_1\|F_1y^*-F_1v^k\|\|p^k-x^*\|+2\lambda_2\|F_2x^*-F_2u^k\|\|v^k-y^*\|+s_kK.\end{aligned}$$

This hence yields

$$\begin{aligned}&\gamma_k[\|u^k-v^k+y^*-x^*\|^2+\|v^k-p^k+x^*-y^*\|^2]\\ &\le \mathbf{\Gamma}_k-\mathbf{\Gamma}_{k+1}+\widetilde{M}_0s_k(2\sqrt{\mathbf{\Gamma}_k}+\widetilde{M}_0s_k)+\frac{\theta_k\widetilde{M}}{1-\gamma_k}\\ &\quad+2\lambda_1\|F_1y^*-F_1v^k\|\|p^k-x^*\|+2\lambda_2\|F_2x^*-F_2u^k\|\|v^k-y^*\|+s_kK.\end{aligned}$$

Since $s_k\to 0$, $\theta_k\to 0$, $\mathbf{\Gamma}_k-\mathbf{\Gamma}_{k+1}\to 0$,

$$\liminf_{k\to\infty}\gamma_k>0 \text{ and } \liminf_{k\to\infty}(1-\gamma_k)>0,$$

one obtains from (3.22) that

$$\lim_{k\to\infty}\|u^k-v^k+y^*-x^*\|=\lim_{k\to\infty}\|v^k-p^k+x^*-y^*\|=0.$$

Thus,

$$\begin{aligned}\|u^k-Gu^k\|=\|u^k-p^k\| &\le \|u^k-v^k+y^*-x^*\|\\ &\quad+\|v^k-p^k+x^*-y^*\|\to 0 \quad (k\to\infty).\end{aligned}\tag{3.26}$$

Noticing $t^k=x^k+\varepsilon_k(x^k-x^{k-1})$, by Remark 2.1 we get

$$\|t^k-x^k\|=\varepsilon_k\|x^k-x^{k-1}\|\to 0\ (k\to\infty).$$

Also, note that
$$0=\beta_k(x^k-u^k)+\gamma_k(W_kp^k-u^k)+\delta_k(T^kz^k-u^k).$$
Since $\liminf\limits_{k\to\infty}\gamma_k>0$, from (3.17) and (3.26) we have
$$\begin{aligned}\|W_kp^k-u^k\|&=\frac{1}{\gamma_k}\|\beta_k(x^k-u^k)+\delta_k(T^kz^k-u^k)\|\\&\le\frac{1}{\gamma_k}(\|x^k-u^k\|+\|T^kz^k-u^k\|)\to 0\ (k\to\infty),\end{aligned}$$
and hence $\|W_ku^k-u^k\|\to 0\ (k\to\infty)$. Therefore,
$$\lim_{k\to\infty}\|t^k-x^k\|=0\quad\text{and}\quad\lim_{k\to\infty}\|W_ku^k-u^k\|=0.\tag{3.27}$$
Using (3.17) and (3.18), we get
$$\|x^k-x^{k+1}\|\le\|x^k-u^k\|+\|u^k-x^{k+1}\|\to 0\quad(k\to\infty),\tag{3.28}$$
$$\|z^k-t^k\|\le\|z^k-y^k\|+\|y^k-t^k\|\to 0\quad(k\to\infty).\tag{3.29}$$
Combining (3.17) and (3.26), one gets
$$\|x^k-Gx^k\|\le 2\|x^k-u^k\|+\|u^k-Gu^k\|\to 0\quad(k\to\infty).\tag{3.30}$$
We claim that $\|W_kx^k-x^k\|\to 0$ and $\|Tx^k-x^k\|\to 0$ as $k\to\infty$. In fact, using Lemma 2.1 (i) we obtain from (3.17) and (3.27) that
$$\|W_kx^k-x^k\|\le 2\|x^k-u^k\|+\|W_ku^k-u^k\|\to 0\quad(k\to\infty).\tag{3.31}$$
Combining (3.27) and (3.29), we have
$$\|x^k-z^k\|\le\|x^k-t^k\|+\|t^k-z^k\|\to 0\quad(k\to\infty).\tag{3.32}$$
Using (3.17) and (3.32), we deduce from the asymptotical nonexpansivity of T that
$$\|x^k-T^kx^k\|\le\|x^k-u^k\|+\|u^k-T^kz^k\|+(1+\theta_k)\|z^k-x^k\|\to 0\quad(k\to\infty).\tag{3.33}$$
This along with the assumption $\|T^kx^k-T^{k+1}x^k\|\to 0$, implies that
$$\|x^k-Tx^k\|\le(2+\theta_1)\|x^k-T^kx^k\|+\|T^kx^k-T^{k+1}x^k\|\to 0\quad(k\to\infty).\tag{3.34}$$
Next we claim that $\lim_{k\to\infty}\|x^k-x^*\|=0$. In fact, by the boundedness of $\{u^k\}$ and $\{x^k\}$, we know that $\exists$ (subsequence) $\{u^{k_i}\}\subset\{u^k\}$ s.t. $u^{k_i}\rightharpoonup\hat{x}\in C$ and
$$\liminf_{k\to\infty}[\Psi(x^*,u^k)+\Psi(u^k,x^{k+1})]=\lim_{i\to\infty}[\Psi(x^*,u^{k_i})+\Psi(u^{k_i},x^{k_i+1})].\tag{3.35}$$
From (3.17) and (3.18) it follows that $x^{k_i}\rightharpoonup\hat{x}$ and $x^{k_i+1}\rightharpoonup\hat{x}$. Then, by the result in Claim 5, we obtain that $\hat{x}\in\mathrm{Sol}(C,\Phi)$.

It is clear from (3.34) that $x^{k_i}-Tx^{k_i}\to 0$. Since Lemma 2.6 guarantees the demiclosedness of $I-T$ at zero, we get $\hat{x}\in\mathrm{Fix}(T)$. Also, since Lemma 2.6 guarantees the demiclosedness of $I-G$ at zero, from $x^{k_i}\rightharpoonup\hat{x}$ and $x^k-Gx^k\to 0$ (due to (3.30)) one has $\hat{x}\in\mathrm{Fix}(G)$. We claim that
$$\hat{x}\in\bigcap_{i=1}^{\infty}\mathrm{Fix}(T_i)=\mathrm{Fix}(W).$$

In fact, by Lemma 2.1 (iii) we know that W is a nonexpansive mapping satisfying

$$\mathrm{Fix}(W)=\bigcap_{i=1}^{\infty}\mathrm{Fix}(T_i).$$

Using (3.31) and Lemma 2.2, we get

$$\|x^k-Wx^k\|\le\|x^k-W_kx^k\|+\|W_kx^k-Wx^k\|\to 0\ (n\to\infty).$$

Since Lemma 2.6 ensures the demiclosedness of $I-W$ at zero, from $x^{k_i}\rightharpoonup\hat{x}$ it follows that

$$\hat{x}\in\mathrm{Fix}(W)=\bigcap_{i=1}^{\infty}\mathrm{Fix}(T_i).$$

Consequently,

$$\hat{x}\in\bigcap_{i=0}^{\infty}\mathrm{Fix}(T_i)\cap\mathrm{Fix}(G)\cap\mathrm{Sol}(C,\Phi)=\Omega.$$

In terms of (3.35), we have

$$\liminf_{k\to\infty}[\Psi(x^*,u^k)+\Psi(u^k,x^{k+1})]=\Psi(x^*,\hat{x})\ge 0. \tag{3.36}$$

Since Ψ is ν-strongly monotone, we have

$$\limsup_{k\to\infty}[\Psi(x^*,u^k)+\Psi(u^k,x^*)]\le\limsup_{k\to\infty}(-\nu\|u^k-x^*\|^2)=-\nu\hbar. \tag{3.37}$$

Combining (3.36) and (3.37), we obtain

$$\begin{aligned}&\limsup_{k\to\infty}[\Psi(u^k,x^*)-\Psi(u^k,x^{k+1})]\\&=\limsup_{k\to\infty}[\Psi(u^k,x^*)+\Psi(x^*,u^k)-\Psi(x^*,u^k)-\Psi(u^k,x^{k+1})]\\&\le\limsup_{k\to\infty}[\Psi(u^k,x^*)+\Psi(x^*,u^k)]-\liminf_{k\to\infty}[\Psi(x^*,u^k)+\Psi(u^k,x^{k+1})]\le-\nu\hbar.\end{aligned} \tag{3.38}$$

We claim $\hbar=0$. Conversely, in case $\hbar>0$, then we might assume that $\exists k_0\ge 1$ s.t.

$$\Psi(u^k,x^*)-\Psi(u^k,x^{k+1})\le-\frac{\nu\hbar}{2}\quad\forall k\ge k_0, \tag{3.39}$$

which together with (3.15), implies that for all $k\ge k_0$,

$$\begin{aligned}\|x^{k+1}-x^*\|^2&\le\frac{1}{1-\gamma_k}[\beta_k\|x^k-x^*\|^2+\delta_k\|t^k-x^*\|^2+\theta_k\widetilde{M}]\\&\quad+2s_k[\Psi(u^k,x^*)-\Psi(u^k,x^{k+1})]\\&\le\frac{1}{1-\gamma_k}\{\beta_k\|x^k-x^*\|^2+\delta_k[\|x^k-x^*\|^2+\varepsilon_k\|x^k\\&\quad-x^{k-1}\|(2\|x^k-x^*\|+\varepsilon_k\|x^k-x^{k-1})]+\theta_k\widetilde{M}\}-s_k\nu\hbar\\&\le\|x^k-x^*\|^2+\varepsilon_k\|x^k-x^{k-1}\|\widetilde{M}_1+\frac{\theta_k\widetilde{M}}{1-\gamma_k}-s_k\nu\hbar,\end{aligned} \tag{3.40}$$

where $\sup_{k\geq 1}\{2\|x^k - x^*\| + \varepsilon_k\|x^k - x^{k-1}\|\} \leq \widetilde{M}_1$ for some $\widetilde{M}_1 > 0$. So it follows that

$$\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k_0} \leq \sum_{j=k_0}^{k-1}\left(\varepsilon_j\|x^j - x^{j-1}\|\widetilde{M}_1 + \frac{\theta_j\widetilde{M}}{1-\gamma_j}\right) - \nu\hbar\sum_{j=k_0}^{k-1} s_j \quad \forall k \geq k_0. \tag{3.41}$$

Letting $k \to \infty$ in (3.41), we have

$$-\infty < \hbar - \mathbf{\Gamma}_{k_0} \leq \lim_{k\to\infty}\left[\sum_{j=k_0}^{k-1}\left(\varepsilon_j\|x^j - x^{j-1}\|\widetilde{M}_1 + \frac{\theta_j\widetilde{M}}{1-\gamma_j}\right) - \nu\hbar\sum_{j=k_0}^{k-1} s_j\right] = -\infty.$$

This reaches a contradiction. Therefore, $\lim_{k\to\infty}\mathbf{\Gamma}_k = 0$ and hence $\{x^k\}$ converges strongly to the unique solution x^* of the problem $\mathrm{EP}(\Omega, \Psi)$.

Aspect 2. Suppose that $\exists\{\mathbf{\Gamma}_{k_j}\} \subset \{\mathbf{\Gamma}_k\}$ s.t. $\mathbf{\Gamma}_{k_j} < \mathbf{\Gamma}_{k_j+1}$ $\forall j \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Let the mapping $\tau : \mathcal{N} \to \mathcal{N}$ be defined as

$$\tau(k) := \max\{j \leq k : \mathbf{\Gamma}_j < \mathbf{\Gamma}_{j+1}\}.$$

By Proposition 2.2, we get

$$\mathbf{\Gamma}_{\tau(k)} \leq \mathbf{\Gamma}_{\tau(k)+1} \quad \text{and} \quad \mathbf{\Gamma}_k \leq \mathbf{\Gamma}_{\tau(k)+1}. \tag{3.42}$$

Using the same reasonings as in (3.18) and (3.28), we deduce that

$$\lim_{k\to\infty}\|x^{\tau(k)+1} - u^{\tau(k)}\| = \lim_{k\to\infty}\|t^{\tau(k)} - y^{\tau(k)}\| = \lim_{k\to\infty}\|y^{\tau(k)} - z^{\tau(k)}\| = 0, \tag{3.43}$$

$$\lim_{k\to\infty}\|x^{\tau(k)+1} - x^{\tau(k)}\| = 0. \tag{3.44}$$

According to the boundedness of $\{u^k\}$, there exists a subsequence of $\{u^{\tau(k)}\}$ converging weakly to $\hat{x}$. We might assume that $u^{\tau(k)} \rightharpoonup \hat{x}$. Then, using the same reasonings as in Aspect 1, we can infer that

$$\hat{x} \in \Omega = \bigcap_{i=0}^{\infty}\mathrm{Fix}(T_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi).$$

From $u^{\tau(k)} \rightharpoonup \hat{x}$ and (3.43), we get $x^{\tau(k)+1} \rightharpoonup \hat{x}$.

Using the condition $\{\alpha_k\} \subset (\underline{\alpha}, \overline{\alpha}) \subset \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right)$, we have from (3.15) that

$$\begin{aligned}
&2s_{\tau(k)}[\Psi(u^{\tau(k)}, x^{\tau(k)+1}) - \Psi(u^{\tau(k)}, x^*)] \\
&\leq -\|x^{\tau(k)+1} - x^*\|^2 + \frac{1}{1-\gamma_{\tau(k)}}[\beta_{\tau(k)}\|x^{\tau(k)} - x^*\|^2 + \delta_{\tau(k)}\|t^{\tau(k)} - x^*\|^2 + \theta_{\tau(k)}\widetilde{M}] \\
&\leq \mathbf{\Gamma}_{\tau(k)} - \mathbf{\Gamma}_{\tau(k)+1} + \varepsilon_{\tau(k)}\|x^{\tau(k)} - x^{\tau(k)-1}\|\widetilde{M}_1 + \frac{\theta_{\tau(k)}\widetilde{M}}{1-\gamma_{\tau(k)}} \\
&\leq \varepsilon_{\tau(k)}\|x^{\tau(k)} - x^{\tau(k)-1}\|\widetilde{M}_1 + \frac{\theta_{\tau(k)}\widetilde{M}}{1-\gamma_{\tau(k)}},
\end{aligned}$$

which hence leads to

$$\Psi(u^{\tau(k)}, x^{\tau(k)+1}) - \Psi(u^{\tau(k)}, x^*) \\ \le \frac{\varepsilon_{\tau(k)}\|x^{\tau(k)} - x^{\tau(k)-1}\|}{s_{\tau(k)}} \cdot \frac{\widetilde{M_1}}{2} + \frac{\theta_{\tau(k)}}{s_{\tau(k)}} \cdot \frac{\widetilde{M}}{2(1-\gamma_{\tau(k)})}. \tag{3.45}$$

Since Ψ is ν-strongly monotone on C, we get

$$\nu\|u^{\tau(k)} - x^*\|^2 \le -\Psi(u^{\tau(k)}, x^*) - \Psi(x^*, u^{\tau(k)}). \tag{3.46}$$

Combining (3.45) and (3.46), we obtain from Remark 2.1, $\mathbf{Ass}_\Psi(\Psi_1)$ and $\hat{x} \in \Omega$ that

$$\begin{aligned}\nu\limsup_{k\to\infty}\|u^{\tau(k)} - x^*\|^2 &= \limsup_{k\to\infty}[-\frac{\varepsilon_{\tau(k)}\|x^{\tau(k)} - x^{\tau(k)-1}\|}{s_{\tau(k)}} \cdot \frac{\widetilde{M_1}}{2} - \frac{\theta_{\tau(k)}}{s_{\tau(k)}} \cdot \frac{\widetilde{M}}{2(1-\gamma_{\tau(k)})} \\ &\quad + \nu\|u^{\tau(k)} - x^*\|^2] \\ &\le \limsup_{k\to\infty}[-\Psi(u^{\tau(k)}, x^{\tau(k)+1}) - \Psi(x^*, u^{\tau(k)})] \\ &= -\Psi(\hat{x}, \hat{x}) - \Psi(x^*, \hat{x}) \le 0.\end{aligned}$$

Hence,

$$\lim_{k\to\infty}\|x^{\tau(k)} - x^*\|^2 = 0.$$

From (3.44) and $\mathbf{\Gamma}_k \le \mathbf{\Gamma}_{\tau(k)+1}$, we get

$$\begin{aligned}\|x^k - x^*\|^2 &\le \|x^{\tau(k)+1} - x^*\|^2 \\ &\le \|x^{\tau(k)} - x^*\|^2 + 2\|x^{\tau(k)+1} - x^{\tau(k)}\|\|x^{\tau(k)} - x^*\| + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2.\end{aligned}$$

So it follows from (3.44) that $x^k \to x^*$ as $k \to \infty$. This completes the proof. □

Next, we introduce another iterative algorithm by using the inertial subgradient extragradient rule.

Algorithm 3.1. Initial Step: Let $\varepsilon > 0$ and $x^0, x^1 \in C$ be arbitrary. The sequences $\{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ in $(0,1)$, and positive sequences $\{\alpha_k\}, \{\tau_k\}, \{s_k\}$ satisfy conditions (H1)-(H5).

Iterative Steps: Calculate x^{k+1} as follows:

Step 1. Given the iterates x^{k-1} and x^k $(k \ge 1)$, choose ε_k such that $0 \le \varepsilon_k \le \overline{\varepsilon_k}$, where

$$\overline{\varepsilon_k} = \begin{cases} \min\left\{\varepsilon, \frac{\tau_k}{\|x^k - x^{k-1}\|}\right\} & \text{if } x^k \ne x^{k-1}, \\ \varepsilon & \text{otherwise.}\end{cases}$$

Step 2. Compute $t^k = x^k + \varepsilon_k(x^k - x^{k-1})$ and

$$y^k = \operatorname{argmin}\left\{\alpha_k\Phi(t^k, y) + \frac{1}{2}\|y - t^k\|^2 : y \in C\right\}.$$

Step 3. Chosen $w^k \in \partial_2\Phi(t^k, y^k)$, compute

$$C_k = \{v \in \mathcal{H} : \langle t^k - \alpha_k w^k - y^k, v - y^k\rangle \le 0\}$$

and

$$z^k = \operatorname{argmin}\left\{\alpha_k\Phi(y^k, z) + \frac{1}{2}\|z - t^k\|^2 : z \in C_k\right\}.$$

Step 4. Compute $u^k = \beta_k x^k + \gamma_k W_k u^k + \delta_k T^k p^k$, with $p^k = J_{\lambda_1}^{B_1}(v^k - \lambda_1 F_1 v^k)$ and

$$v^k = J_{\lambda_2}^{B_2}(z^k - \lambda_2 F_2 z^k).$$

Step 5. Compute $x^{k+1} = \operatorname{argmin}\{s_k \Psi(u^k, t) + \frac{1}{2}\|t - u^k\|^2 : t \in C\}$.

Set $k := k + 1$ and return to Step 1.

Theorem 3.2. Assume that $\{x^k\}$ is the sequence constructed by Algorithm 3.1. Let the bifunctions Ψ, Φ satisfy the assumptions $\mathbf{Ass_\Phi}$-$\mathbf{Ass_\Psi}$. Then, under the conditions (H1)-(H5), the sequence $\{x^k\}$ converges strongly to the unique solution x^* of the problem EP(Ω, Ψ) provided

$$T^k x^k - T^{k+1} x^k \to 0 \text{ and } \sum_{k=1}^{\infty} \varepsilon_k \|x^k - x^{k-1}\| < \infty.$$

Proof. From Lemma 2.1 (i), it is readily known that each W_k is nonexpansive. Then, using the Banach contraction mapping principle, we know from $\{\gamma_k\} \subset (0,1)$ that $\forall k \geq 1$, $\exists \mid u^k \in C$ s.t. $u^k = \beta_k x^k + \gamma_k W_k u^k + \delta_k T^k p^k$. Take an arbitrary point

$$q \in \Omega = \bigcap_{i=0}^{\infty} \operatorname{Fix}(T_i) \cap \operatorname{Fix}(G) \cap \operatorname{Sol}(C, \Phi).$$

Noticing $\lim_{k\to\infty} \frac{\theta_k}{s_k} = 0$, we might assume that $\theta_k \leq \frac{1}{2}\lambda s_k$ $\forall k \geq 1$. Next we divide the proof into several claims below.

Claims 1-3. We assert that the results in Claims 1-3 of the proof of Theorem 3.1 are still valid. In fact, using the same reasonings as in the proof of Theorem 3.1, we obtain the desired claims.

Claim 4. We assert that $\{x^k\}$ is bounded. In fact, using the similar reasonings to those in the proof of Theorem 3.1, we obtain that the inequality (3.11) still holds. Since W_k and G are nonexpansive mappings, and T is asymptotically nonexpansive, we deduce from (3.11) and $p^k = Gz^k$ that

$$\begin{aligned}\|u^k - x^*\|^2 &\leq \beta_k \|x^k - x^*\|\|u^k - x^*\| + \gamma_k \|u^k - x^*\|^2 + \delta_k (1 + \theta_k)\|z^k - x^*\|\|u^k - x^*\| \\ &\leq (1 - \gamma_k)(1 + \theta_k)(\|x^k - x^*\| + \widetilde{M}_0 s_k)\|u^k - x^*\| + \gamma_k \|u^k - x^*\|^2,\end{aligned}$$

which hence yields $\|u^k - x^*\| \leq (1 + \theta_k)(\|x^k - x^*\| + \widetilde{M}_0 s_k)$. Consequently,

$$\begin{aligned}\|x^{k+1} - x^*\| &\leq \|x^{k+1} - u_*^k\| + \|u_*^k - x^*\| \leq (1 - \lambda s_k)\|u^k - x^*\| + \|u_*^k - x^*\| \\ &\leq (1 - \lambda s_k)(1 + \theta_k)(\|x^k - x^*\| + \widetilde{M}_0 s_k) + s_k \hat{M}(x^*) \\ &\leq \max\{\|x^k - x^*\|, \frac{2[\widetilde{M}_0 + \hat{M}(x^*)]}{\lambda}\}.\end{aligned}$$

By induction, we get

$$\|x^k - x^*\| \leq \max\left\{\|x^1 - x^*\|, \frac{2[\widetilde{M}_0 + \hat{M}(x^*)]}{\lambda}\right\} \ \forall k \geq 1.$$

Thus, $\{x^k\}$ is bounded, and so are the sequences $\{p^k\}, \{t^k\}, \{u^k\}, \{v^k\}, \{y^k\}, \{z^k\}$.

Claim 5. We assert that if $x^{k_i} \rightharpoonup \hat{x}$, $t^{k_i} - x^{k_i} \to 0$ and $t^{k_i} - y^{k_i} \to 0$ for $\{k_i\} \subset \{k\}$, then $\hat{x} \in \mathrm{Sol}(C, \Phi)$. In fact, using the same reasonings as in the proof of Theorem 3.1, we derive the desired claim.

Claim 6. We assert that $x^k \to x^*$, a unique solution of the MBEP with the GSVI and CFPP constraints.

In fact, set $\mathbf{\Gamma}_k = \|x^k - x^*\|^2$. Since each W_k and G are nonexpansive mappings and T is asymptotically nonexpansive, we obtain that

$$\begin{aligned}\|u^k - x^*\|^2 \le \frac{\beta_k}{2}\|x^k - x^*\|^2 + \frac{1+\gamma_k}{2}\|u^k - x^*\|^2 + \frac{\delta_k}{2}\|p^k - x^*\|^2 \\ + \frac{\theta_k \widetilde{M}}{2} - \frac{\beta_k}{2}\|x^k - u^k\|^2 - \frac{\delta_k}{2}\|T^k p^k - u^k\|^2,\end{aligned}$$

where $\sup\limits_{k\ge 1}(2+\theta_k)(\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 \le \widetilde{M}$ for some $\widetilde{M} > 0$. This implies that

$$\begin{aligned}\|u^k - x^*\|^2 \le \frac{1}{1-\gamma_k}[\beta_k\|x^k - x^*\|^2 + \delta_k\|p^k - x^*\|^2 \\ + \theta_k\widetilde{M} - \beta_k\|x^k - u^k\|^2 - \delta_k\|T^k p^k - u^k\|^2].\end{aligned} \tag{3.47}$$

By the results in Claims 1 and 2 we deduce from (3.11) and (3.47) that

$$\begin{aligned}&\|x^{k+1} - x^*\|^2 \le \|u^k - x^*\|^2 - \|x^{k+1} - u^k\|^2 + 2s_k[\Psi(u^k, x^*) - \Psi(u^k, x^{k+1})] \\ &\le \frac{1}{1-\gamma_k}\{\beta_k\|x^k - x^*\|^2 + \delta_k[\|t^k - x^*\|^2 - (1-2\alpha_k c_1)\|y^k - t^k\|^2 - (1-2\alpha_k c_2)\|z^k - y^k\|^2] \\ &\quad +\theta_k\widetilde{M} - \beta_k\|x^k - u^k\|^2 - \delta_k\|T^k p^k - u^k\|^2\} - \|x^{k+1} - u^k\|^2 \\ &\quad +2s_k[\Psi(u^k, x^*) - \Psi(u^k, x^{k+1})] \\ &\le (\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 - \frac{\delta_k}{1-\gamma_k}[(1-2\alpha_k c_1)\|y^k - t^k\|^2 + (1-2\alpha_k c_2)\|z^k - y^k\|^2] + \frac{\theta_k\widetilde{M}}{1-\gamma_k} \\ &\quad - \frac{1}{1-\gamma_k}[\beta_k\|x^k - u^k\|^2 + \delta_k\|T^k p^k - u^k\|^2] - \|x^{k+1} - u^k\|^2 + s_k K,\end{aligned} \tag{3.48}$$

where $\sup_{k\ge 1}\{2|\Psi(u^k, x^*) - \Psi(u^k, x^{k+1})|\} \le K$ for some $K > 0$.

Lastly, we show the convergence of $\{\mathbf{\Gamma}_k\}$ to zero in the following two aspects:

Aspect 1. Suppose that $\exists$ (integer) $k_0 \ge 1$ s.t. $\{\mathbf{\Gamma}_k\}$ is non-increasing. Then the limit

$$\lim_{k\to\infty} \mathbf{\Gamma}_k = \hbar < +\infty \text{ and } \lim_{k\to\infty}(\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1}) = 0.$$

From (3.48), we get

$$\begin{aligned}&\frac{\delta_k}{1-\gamma_k}[(1-2\alpha_k c_1)\|y^k - t^k\|^2 + (1-2\alpha_k c_2)\|z^k - y^k\|^2] \\ &+ \frac{1}{1-\gamma_k}[\beta_k\|x^k - u^k\|^2 + \delta_k\|T^k p^k - u^k\|^2] + \|x^{k+1} - u^k\|^2 \\ &\le \mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1} + \widetilde{M}_0 s_k(2\sqrt{\mathbf{\Gamma}_k} + \widetilde{M}_0 s_k) + \frac{\theta_k\widetilde{M}}{1-\gamma_k} + s_k K.\end{aligned}$$

From $\{\alpha_k\} \subset (\underline{\alpha}, \overline{\alpha}) \subset \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right)$, we find that

$$\lim_{k\to\infty} \|x^k - u^k\| = \lim_{k\to\infty} \|T^k p^k - u^k\| = 0, \tag{3.49}$$

$$\lim_{k\to\infty} \|y^k - t^k\| = \lim_{k\to\infty} \|z^k - y^k\| = \lim_{k\to\infty} \|x^{k+1} - u^k\| = 0. \tag{3.50}$$

Next we show that

$$\lim_{k\to\infty} \|x^k - x^*\| = 0.$$

In fact, using (3.21), we deduce from (3.47) and (3.48) that

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\le \|u^k - x^*\|^2 + s_k K \\
&\le \frac{1}{1-\gamma_k}[\beta_k \|x^k - x^*\|^2 + \delta_k \|p^k - x^*\|^2 + \theta_k \widetilde{M}] + s_k K \\
&\le (\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 + \frac{2\theta_k \widetilde{M}}{1-\gamma_k} - \frac{\delta_k}{1-\gamma_k}[\lambda_2(2\beta - \lambda_2)\|F_2 z^k - F_2 x^*\|^2 \\
&+ \lambda_1(2\alpha - \lambda_1)\|F_1 v^k - F_1 y^*\|^2] + s_k K.
\end{aligned}$$

Since $\lambda_1 \in (0, 2\alpha)$, $\lambda_2 \in (0, 2\beta)$, $s_k \to 0$, $\theta_k \to 0$, $\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1} \to 0$,

$$0 < \liminf_{k\to\infty} \delta_k \text{ and } 0 < \liminf_{k\to\infty} \gamma_k \le \limsup_{k\to\infty} \gamma_k < 1,$$

we get

$$\lim_{k\to\infty} \|F_2 z^k - F_2 x^*\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|F_1 v^k - F_1 y^*\| = 0. \tag{3.51}$$

On the other hand, using the same reasonings as those of (3.25) we get

$$\begin{aligned}
\|p^k - x^*\|^2 \le (\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 + \frac{\theta_k \widetilde{M}}{1-\gamma_k} \\
- \|z^k - v^k + y^* - x^*\|^2 - \|v^k - p^k + x^* - y^*\|^2 \\
+ 2\lambda_1 \|F_1 y^* - F_1 v^k\| \|p^k - x^*\| + 2\lambda_2 \|F_2 x^* - F_2 z^k\| \|v^k - y^*\|,
\end{aligned}$$

which along with (3.47) and (3.48), implies that

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\le \|u^k - x^*\|^2 + s_k K \\
&\le \frac{1}{1-\gamma_k}[\beta_k \|x^k - x^*\|^2 + \delta_k \|p^k - x^*\|^2 + \theta_k \widetilde{M}] + s_k K \\
&\le (\|x^k - x^*\| + \widetilde{M}_0 s_k)^2 + \frac{2\theta_k \widetilde{M}}{1-\gamma_k} \\
&- \frac{\delta_k}{1-\gamma_k}[\|z^k - v^k + y^* - x^*\|^2 + \|v^k - p^k + x^* - y^*\|^2] \\
&+ 2\lambda_1 \|F_1 y^* - F_1 v^k\| \|p^k - x^*\| + 2\lambda_2 \|F_2 x^* - F_2 z^k\| \|v^k - y^*\| + s_k K.
\end{aligned}$$

This immediately leads to

$$\begin{aligned}
&\frac{\delta_k}{1-\gamma_k}[\|z^k - v^k + y^* - x^*\|^2 + \|v^k - p^k + x^* - y^*\|^2] \\
&\le \mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1} + \widetilde{M}_0 s_k(2\sqrt{\mathbf{\Gamma}_k} + \widetilde{M}_0 s_k) \\
&\quad + \frac{2\theta_k \widetilde{M}}{1-\gamma_k} + 2\lambda_1\|F_1 y^* - F_1 v^k\|\|p^k - x^*\| + 2\lambda_2\|F_2 x^* - F_2 z^k\|\|v^k - y^*\| + s_k K.
\end{aligned}$$

Since $s_k \to 0$, $\theta_k \to 0$, $\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1} \to 0$,

$$0 < \liminf_{k\to\infty} \delta_k,\ 0 < \liminf_{k\to\infty} \gamma_k \le \limsup_{k\to\infty} \gamma_k < 1,$$

we deduce from (3.51) that

$$\lim_{k\to\infty} \|z^k - v^k + y^* - x^*\| = \lim_{k\to\infty} \|v^k - p^k + x^* - y^*\| = 0.$$

Thus,

$$\|z^k - Gz^k\| = \|z^k - p^k\| \le \|z^k - v^k + y^* - x^*\| + \|v^k - p^k + x^* - y^*\| \to 0\ (k \to \infty). \tag{3.52}$$

Using the similar reasonings to those of (3.27), we get

$$\lim_{k\to\infty} \|t^k - x^k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|W_k u^k - u^k\| = 0. \tag{3.53}$$

Using (3.49) and (3.50), we obtain that

$$\|x^k - x^{k+1}\| \le \|x^k - u^k\| + \|u^k - x^{k+1}\| \to 0\ (k \to \infty)$$

and

$$\|z^k - x^k\| \le \|z^k - y^k\| + \|y^k - t^k\| + \|t^k - x^k\| \to 0 \quad (k \to \infty). \tag{3.54}$$

Combining (3.52) and (3.54), we get

$$\|x^k - Gx^k\| \to 0\ (k \to \infty).$$

We show that $\|W_k x^k - x^k\| \to 0$ and $\|Tx^k - x^k\| \to 0$ as $k \to \infty$. In fact, using Lemma 2.1 (i) we deduce from (3.49) and (3.53) that

$$\|W_k x^k - x^k\| \le 2\|x^k - u^k\| + \|W_k u^k - u^k\| \to 0\ (k \to \infty).$$

Using (3.49), (3.52) and (3.54), we infer from the asymptotical nonexpansivity of T that

$$\begin{aligned}
\|x^k - T^k x^k| &\le \|x^k - u^k\| + \|u^k - T^k p^k\| + \|T^k p^k - T^k z^k\| + \|T^k z^k - T^k x^k\| \\
&\le \|x^k - u^k\| + \|u^k - T^k p^k\| + (1+\theta_k)[\|p^k - z^k\| + \|z^k - x^k\|] \to 0 \\
&(k \to \infty).
\end{aligned}$$

Using the same reasonings as those of (3.34) we have $\lim_{k\to\infty} \|x^k - Tx^k\| = 0$. Further, using the same arguments as in Aspect 1 of the proof of Theorem 3.1, we deduce that $\lim_{k\to\infty} \mathbf{\Gamma}_k = 0$, and hence $\{x^k\}$ converges strongly to the unique solution x^* of the problem $\mathrm{EP}(\Omega, \Psi)$.

Aspect 2. Suppose that $\exists\{\mathbf{\Gamma}_{k_j}\} \subset \{\mathbf{\Gamma}_k\}$ s.t. $\mathbf{\Gamma}_{k_j} < \mathbf{\Gamma}_{k_j+1}$ $\forall j \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Let the mapping $\tau : \mathcal{N} \to \mathcal{N}$ be defined as

$$\tau(k) := \max\{j \leq k : \mathbf{\Gamma}_j < \mathbf{\Gamma}_{j+1}\}.$$

In the remainder of the proof, using the same reasonings as in Aspect 2 of the proof of Theorem 3.1, we derive the desired claim. This completes the proof. □

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