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VARIATIONAL INCLUSION PROBLEM AND TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPING: GRAPH CONVERGENCE, ALGORITHMS AND APPROXIMATION OF COMMON SOLUTIONS

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Abstract. In this paper, under some new appropriate conditions imposed on the parameter and mappings involved in the resolvent operator associated with an (H,η)-monotone operator, its Lipschitz continuity is proved and an estimate of its Lipschitz constant is computed. This paper is also concerned with the establishment of a new equivalence relationship between the graph convergence of a sequence of (H,η)-monotone operators and their associated resolvent operators, respectively, to a given (H,η)-monotone operator and its associated resolvent operator. A new iterative scheme for approximating a common element of the set of solutions of a variational inclusion problem and the set of fixed points of a given total asymptotically nonexpansive mapping is constructed. As an application of the obtained equivalence conclusion concerning graph convergence, under some suitable conditions, the strong convergence of the sequence generated by our suggested iterative algorithm to a common element of the above-mentioned two sets is proved. Our results improve and generalize the corresponding results of recent works.

Key Words and Phrases: Total $(\{a_n\},\{b_n\},\phi)$-asymptotically nonexpansive mapping, (H,η)-monotone operator, variational inclusion problem, fixed point problem, resolvent method, convergence analysis.
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1. Introduction

During the last decades, the theory of variational inequalities has been intensively considered by many authors and a great deal of papers have been devoted to the existence of solutions for different classes of variational inequality problems, see, for example, [6, 8] and the references therein. Because of its applications in different areas of science, social science, engineering and management, the study of various extensions of variational inequality problems has received a great deal of interest from the scientific community. One of the most important generalizations of variational inequalities is the so-called variational inclusions, see, for example, [2, 9, 14, 15, 18, 17, 19, 22, 23, 24, 25, 32, 39] and the references therein. The importance of theory as well as the applications of the variational inclusion inequality problem in a huge variety of scientific fields were one of the main motivations of researchers for constructing and developing of various methods for solving different classes of variational inclusion inequality problems in the framework of different spaces. Among the methods appeared in the literature, the resolvent operator method as an extension of the projection method. For more related details, the readers are referred to [4, 3, 10, 11, 12, 17, 19, 9, 2, 18, 23, 35] and the references contained therein. In order to solve and analyze variant classes of variational inequality and inclusion problems we need to generalize and extend the notion of maximal monotonicity in the context of different spaces. Huang and Fang [23, 17] defined the concept of maximal η-monotone operator and the resolvent operator associated with such an operator in the setting of Hilbert spaces, also they define H-monotone operator and the resolvent operator associated with such an operator for solving a class of variational inclusion problems involving H-monotone operators. In 2005, Fang et al. [19] succeeded to introduce other extension of maximal monotone operator the so-called (H,η)-monotone operator which can be viewed as a unifying framework for the classes of maximal monotone operators, maximal η-monotone operators and H-monotone operators. The connections between monotone mappings and nonexpansive mappings lead to a special theory of graph convergence. It establishes an equivalence between the graph convergence of a sequence of maximal monotone operators and their associated resolvent operators, respectively, to a given maximal monotone operator and its associated resolvent operator. See [7, 9, 2, 24, 35].

Since the appearance of the notion of nonexpansive mapping, due to the existence of a strong connection between monotone and accretive operators, two classes of operators which arise naturally in the theory of differential equations, and the class of nonexpansive mappings, the fixed point theory of nonexpansive mappings has rapidly grown into an important field of study in both pure and applied mathematics. It has become one of the most essential tools in nonlinear functional analysis. For this reason, during the past few decades, many authors have shown interest in extending the notion of nonexpansive mapping, and the study of the fixed point theory for generalized nonexpansive mappings has also attracted increasing attention. Recently, Alber et al. [1] introduced the concept of total asymptotically nonexpansive mapping, which is more general than asymptotically nonexpansive mapping and some another generalized nonexpansive mappings existing in the literature. See [12, 13, 29, 20, 1, 16, 27, 28, 34].

The paper is structured as follows. Section 2 provides the basic definitions and preliminaries concerning (H,η)-monotone operators along with some new examples. In the end of this section, the Lipschitz continuity of the resolvent operator associated with an (H,η)-monotone operator under some new appropriate conditions is proved and an estimate of its Lipschitz constant is also computed. In Sect.3, we first recall some background material on some classes of generalized nonexpansive mappings and provide a new example for illustration relation between the class of total asymptotically nonexpansive mappings and the class of asymptotically nonexpansive mappings. In this section, the well-known class of variational inclusion problems (VIP) is considered and with the goal of finding a point lying in the intersection of the set of solutions of the VIP and the set of fixed points of a given total asymptotically nonexpansive mapping, a new iterative algorithm is proposed. Finally, in Sect.4, the notions of graph convergence and the resolvent operator associated with an (H,η)-monotone operator are first used and a new equivalence relationship between the graph convergence of a sequence of (H,η)-monotone operators and their associated resolvent operators, respectively, to a given (H,η)-monotone operator and its associated resolvent operator is established. In the end, as an application of the obtained equivalence relationship, the strong convergence of the sequence generated by our suggested iterative algorithm to a common element of the above two sets are proved.

2. Preliminary notations and results

Throughout this paper, we assume that X is a real Hilbert space endowed with a norm $\|.\|$ and an inner product $\langle .,.\rangle$. For a given multi-valued mapping $M : X \rightrightarrows X$,

(i) the set Range(M) defined by

$$\text{Range}(M) = \{y \in X : \exists x \in X : (x,y) \in M\} = \bigcup_{x \in X} M(x)$$

is called the range of M;

(ii) the set Graph(M) defined by

$$\text{Graph}(M) = \{(x,u) \in X \times X : u \in M(x)\},$$

is called the graph of M.

In what follows, we recall some concepts and known results which will be used in the sequel.

Definition 2.1. Let $T : X \to X$ and $\eta : X \times X \to X$ be the operators. T is said to be

(i) η-monotone if,

$$\langle T(x) - T(y), \eta(x,y)\rangle \geq 0, \quad \forall x, y \in X;$$

(ii) strictly η-monotone if, T is η-monotone and equality holds if and only if $x = y$;

(iii) r-strongly η-monotone if, there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), \eta(x,y)\rangle \geq r\|x - y\|^2, \quad \forall x, y \in X;$$

(iv) θ-Lipschitz continuous if, there exists a constant $\theta > 0$ such that

$$\|T(x) - T(y)\| \leq \theta\|x - y\|, \quad \forall x, y \in X.$$

Definition 2.2. [19, 23] Let $\eta : X \times X \to X$ be a vector-valued operator. A multi-valued operator $M : X \rightrightarrows X$ is said to be

(i) monotone if,

$$\langle u - v, x - y\rangle \geq 0, \quad \forall (x,u), (y,v) \in \text{Graph}(M);$$

(ii) η-monotone if,

$$\langle u - v, \eta(x, y)\rangle \geq 0, \quad \forall (x, u), (y, v) \in \text{Graph}(M);$$

(iii) ς-strongly η-monotone if, there exists a constant $\varsigma > 0$ such that

$$\langle u - v, \eta(x, y)\rangle \geq \varsigma \|x - y\|^2, \quad \forall (x, u), (y, v) \in \text{Graph}(M);$$

(iii) maximal η-monotone if, M is η-monotone and $(I + \lambda M)(X) = X$, for every real constant $\lambda > 0$.

We note that M is a maximal η-monotone operator if and only if M is η-monotone and there is no other η-monotone operator whose graph contains strictly Graph(M). The maximal η-monotonicity is to be understood in terms of inclusion of graphs. If $M : X \rightrightarrows X$ is a maximal η-monotone operator, then adding anything to its graph so as to obtain the graph of a new multi-valued operator, destroys the η-monotonicity. In fact, the extended operator is no longer η-monotone. In other words, for every pair $(x, u) \in X \times X \setminus \text{Graph}(M)$ there exists $(y, v) \in \text{Graph}(M)$ such that $\langle u - v, \eta(x, y)\rangle < 0$. Owing to the above-mentioned arguments, a necessary and sufficient condition for a multi-valued operator $M : X \rightrightarrows X$ to be η-monotone is that the property

$$\langle u - v, \eta(x, y)\rangle \geq 0, \quad \forall (y, v) \in \text{Graph}(M)$$

is equivalent to $u \in M(x)$. The above characterization of maximal η-monotone operators provides a useful and manageable way for recognizing that an element u belongs to $M(x)$.

Fang et al. [19] introduced and studied the class of (H, η)-monotone operators as a unifying framework for the classes maximal monotone operators [38], maximal η-monotone operators [23] and H-monotone operators [17] as follows.

Definition 2.3. [19] For given vector-valued operators $\eta : X \times X \to X$ and $H : X \to X$, a multi-valued operator $M : X \rightrightarrows X$ is said to be (H, η)-monotone if M is η-monotone and $(H + \lambda M)(X) = X$ holds, for every real constant $\lambda > 0$.

It should be pointed out that for the case when $\eta(x, y) = x - y$ for all $x, y \in X$, then Definition 2.3 reduces to the definition of H-monotonicity of the multi-valued operator M which was introduced and studied by Fang and Huang [17].

It is worthwhile to stress that for given operators $\eta : X \times X \to X$ and $H : X \to X$, an (H, η)-monotone operator may be neither H-monotone nor maximal η-monotone. For illustration of this fact, the following example is given.

Example 2.4. Let $\phi : \mathbb{Z} \to (0, +\infty)$ and consider the complex linear space $l^2_\phi(\mathbb{Z})$, the weighted $l^2(\mathbb{Z})$ space, consisting of all bi-infinite complex sequences $(z_n)_{n=-\infty}^{\infty}$ such that $\sum\limits_{n=-\infty}^{\infty} |z_n|^2 \phi(n) < \infty$. It is a well known that

$$l^2_\phi(\mathbb{Z}) = \{z = (z_n)_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} |z_n|^2 \phi(n) < \infty, z_n \in \mathbb{C}\}$$

with respect to the inner product $\langle .,. \rangle : l^2_\phi(\mathbb{Z}) \times l^2_\phi(\mathbb{Z}) \to \mathbb{C}$ defined by

$$\langle z, w\rangle = \sum_{n=-\infty}^{\infty} z_n \overline{w_n}\phi(n), \quad \forall z = (z_n)_{n=-\infty}^{\infty}, w = (w_n)_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z}),$$

is a Hilbert space. The inner product defined above induces a norm on $l^2_\phi(\mathbb{Z})$ as follows:

$$\|z\|_{l^2_\phi(\mathbb{Z})} = \sqrt{\langle z, z\rangle} = (\sum_{n=-\infty}^{\infty} |z_n|^2\phi(n))^{\frac{1}{2}}, \quad \forall z = (z_n)_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z}).$$

Let

$$\delta_{j,(2k+1)\omega-j+1} = (\dots, 0, 0, \dots, 0, \frac{1}{\sqrt{2\phi(j)}} i_j, 0, \frac{1}{\sqrt{2\phi((2k+1)\omega - j + 1)}} i_{(2k+1)\omega-j+1}, 0, 0, \dots)$$

and

$$\delta'_{j,(2s+1)\omega-j+1} = (\dots, 0, 0, \dots, 0, \frac{1}{\sqrt{2\phi(j)}} i_j, 0, -\frac{1}{\sqrt{2\phi((2k+1)\omega-j+1)}} i_{(2k+1)\omega-j+1}, 0, 0, \dots).$$

Define the operators $M : l^2_\phi(\mathbb{Z}) \rightrightarrows l^2_\phi(\mathbb{Z})$, $\eta : l^2_\phi(\mathbb{Z}) \times l^2_\phi(\mathbb{Z}) \to l^2_\phi(\mathbb{Z})$ and $H : l^2_\phi(\mathbb{Z}) \to l^2_\phi(\mathbb{Z})$, respectively, by

$$M(z) = \begin{cases} \Phi, & z = \delta_{\alpha,(2\beta+1)\omega-\alpha+1}, \\ -z + \Big(\sqrt{\frac{\cos\gamma\pi n}{2(n^p+\theta)\phi(n)}} + i\sqrt{\frac{\cos\gamma\pi n}{2(n^p+\theta)\phi(n)}}\Big)_{n=-\infty}^{\infty}, & z \neq \delta_{\alpha,(2\beta+1)\omega-\alpha+1}, \end{cases}$$

$$\eta(z, w) = \begin{cases} \varsigma(w - z), & z, w \neq \delta_{\alpha,(2\beta+1)\omega-\alpha+1}, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

and $H(z) = \mu z + \zeta\Big(\sqrt{\frac{\cos\gamma\pi n}{2(n^p+\theta)\phi(n)}} + i\sqrt{\frac{\cos\gamma\pi n}{2(n^p+\theta)\phi(n)}}\Big)_{n=-\infty}^{\infty}$, for all $z, w \in l^2_\phi(\mathbb{Z})$, where

$$\Phi = \Big\{\delta_{j,(2k+1)\omega-j+1} - \delta_{\alpha,(2\beta+1)\omega-\alpha+1}, \delta'_{j,(2k+1)\omega-j+1} - \delta_{\alpha,(2\beta+1)\omega-\alpha+1} : \\ k \in \mathbb{Z}; j = k\omega + 1, k\omega + 2, \dots, \frac{(2k+1)\omega}{2}\Big\},$$

$\varsigma, \mu, \zeta \in \mathbb{R}$ are arbitrary constants such that $\mu < 0 < \varsigma$; $p \geq 2$ is an arbitrary even natural number, $\theta > 0$ and γ are arbitrary real constants, $\beta \in \mathbb{Z}$ and $\alpha \in \{\beta\omega + 1, \beta\omega + 2, \dots, \frac{(2\beta+1)\omega}{2}\}$ are chosen arbitrarily but fixed, k is an arbitrary but fixed natural number, and $\mathbf{0}$ is the zero vector of the space $l^2_\phi(\mathbb{Z})$. Owing to the fact that

$$\sum_{n=-\infty}^{\infty} \frac{\cos\gamma\pi n}{n^p + \theta} = \frac{1}{\theta} + 2\sum_{n=1}^{\infty} \frac{\cos\gamma\pi n}{n^p + \theta}$$

and $\sum\limits_{n=1}^{\infty} \frac{\cos\gamma\pi n}{n^p+\theta}$ is convergent, it follows that $\sum\limits_{n=-\infty}^{\infty} \frac{\cos\gamma\pi n}{n^p+\theta} < \infty$, and so

$$\Big(\sqrt{\frac{\cos\gamma\pi n}{2(n^p + \theta)\phi(n)}} + i\sqrt{\frac{\cos\gamma\pi n}{2(n^p + \theta)\phi(n)}}\Big)_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z}).$$

Then, for all $z, w \in l^2_\phi(\mathbb{Z})$, $z \neq w \neq \delta_{\alpha,(2\beta+1)\omega-\alpha+1}$, we have

$$\begin{aligned}\langle M(z)-M(w), z-w\rangle &= \langle w-z, z-w\rangle = -\|z-w\|^2_{l^2_\phi(\mathbb{Z})}\\ &= -\sum_{n=-\infty}^{\infty}|z_n-w_n|^2\phi(n) < 0,\end{aligned}$$

which means that M is not monotone and so it is not an H-monotone operator. For any given $z, w \in l^2_\phi(\mathbb{Z})$, $z \neq w \neq \delta_{\alpha,(2\beta+1)\omega-\alpha+1}$, yields

$$\begin{aligned}\langle M(z)-M(w), \eta(z,w)\rangle &= \alpha\langle w-z, w-z\rangle = \alpha\|w-z\|^2_{l^2_\phi(\mathbb{Z})}\\ &= \alpha\sum_{n=-\infty}^{\infty}|z_n-w_n|^2\phi(n) > 0.\end{aligned}$$

For each of the cases when $z \neq w = \delta'_{\alpha,(2\beta+1)\omega-\alpha+1}$, $w \neq z = \delta'_{\alpha,(2\beta+1)\omega-\alpha+1}$ and $z = w = \delta_{\alpha,(2\beta+1)\omega-\alpha+1}$, taking into account that $\eta(z,w) = \mathbf{0}$, we deduce that

$$\langle u-v, \eta(z,w)\rangle = 0, \quad \forall u \in M(z), v \in M(w).$$

Therefore, M is an η-monotone operator. By virtue of the fact that for any $\delta_{\alpha,(2\beta+1)\omega-\alpha+1} \neq z \in l^2_\phi(\mathbb{Z})$,

$$\|(I+M)(z)\|^2_{l^2_\phi(\mathbb{Z})} = \sum_{n=1}^{\infty}\frac{\cos\gamma\pi n}{n^p+\theta} > 0$$

and

$$\begin{aligned}(I+M)(\delta'_{\alpha,(2\beta+1)\omega-\alpha+1}) = \Big\{&\delta_{j,(2k+1)\omega-j+1}, \delta'_{j,(2k+1)\omega-j+1} :\\ &k \in \mathbb{Z}; j = k\omega+1, k\omega+2, \ldots, \frac{(2k+1)\omega}{2}\Big\},\end{aligned}$$

where I is the identity mapping on $X = l^2_\phi(\mathbb{Z})$, it follows that $\mathbf{0} \notin (I+M)(l^2_\phi(\mathbb{Z}))$. Hence, $I+M$ is not surjective and so M is not a maximal η-monotone operator. For any $\lambda > 0$ and $z \in l^2_\phi(\mathbb{Z})$, by taking

$$w = \frac{1}{\mu-\lambda}z + \frac{\zeta+\lambda}{\lambda-\mu}\Big(\sqrt{\frac{\cos\gamma\pi n}{2(n^p+\theta)\phi(n)}} + i\sqrt{\frac{\cos\gamma\pi n}{2(n^p+\theta)\phi(n)}}\Big)_{n=-\infty}^{\infty}$$

($\lambda \neq \mu$, because $\mu < 0$), we obtain

$$\begin{aligned}(H+\lambda M)(w) &= (H+\lambda M)\left(\frac{1}{\mu-\lambda}z + \frac{\zeta+\lambda}{\lambda-\mu}\left(\sqrt{\frac{\cos\gamma\pi n}{2(n^p+\theta)\phi(n)}} + i\sqrt{\frac{\cos\gamma\pi n}{2(n^p+\theta)\phi(n)}}\right)\right)_{n=-\infty}^{\infty}\\ &= z.\end{aligned}$$

Consequently, for any real constant $\lambda > 0$, the mapping $H+\lambda M$ is surjective and so M is an (H,η)-monotone operator.

It is significant to mention that for given operators $H : X \to X$ and $\eta : X\times X \to X$, a maximal η-monotone operator need not be (H,η)-monotone. In support of this fact, we present the following example.

Example 2.5. Let $m, n \in \mathbb{N}$ and $M_{m\times n}(\mathbb{F})$ be the space of all $m \times n$ matrices with real or complex entries. Then

$$M_{m\times n}(\mathbb{F}) = \{A = \left(a_{ij} \right) | a_{ij} \in \mathbb{F}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\}$$

is a Hilbert space with respect to the Hilbert-Schmidt norm

$$\|A\| = \Big(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\Big)^{\frac{1}{2}}, \quad \forall A \in M_{m\times n}(\mathbb{F})$$

induced by the Hilbert-Schmidt inner product

$$\langle A, B\rangle = tr(A^*B) = \sum_{i=1}^{m}\sum_{j=1}^{n}\bar{a}_{ij}b_{ij}, \quad \forall A, B \in M_{m\times n}(\mathbb{F}),$$

where tr denotes the trace, that is, the sum of the diagonal entries, and A^* denotes the Hermitian conjugate (or adjoint) of the matrix A, that is, $A^* = \overline{A^t}$, the complex conjugate of the transpose A. Denote by $D_n(\mathbb{R})$ the space of all diagonal $n \times n$ matrices with real entries. Then $D_n(\mathbb{R})$ is a subspace of $M_{n\times n}(\mathbb{R}) = M_n(\mathbb{R})$. Define the operators $H, M : D_n(\mathbb{R}) \to D_n(\mathbb{R})$ and $\eta : D_n(\mathbb{R})\times D_n(\mathbb{R}) \to D_n(\mathbb{R})$, respectively, as $H(A) = H(\left(a_{ij} \right)) = \left(\widetilde{a}_{ij} \right)$, $M(A) = M(\left(a_{ij} \right)) = \left(\widehat{a}_{ij} \right)$ and $\eta(A, B) = \eta(\left(a_{ij} \right), \left(b_{ij} \right)) = \left(c_{ij} \right)$ for all $A = \left(a_{ij} \right), B = \left(b_{ij} \right) \in D_n(\mathbb{R})$, where for each $i, j \in \{1, 2, \ldots, n\}$,

$$\widetilde{a}_{ij} = \begin{cases} \alpha \sin\gamma a_{ii}, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$\widehat{a}_{ij} = \begin{cases} \beta \cos\gamma a_{ii}, & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$c_{ij} = \begin{cases} \beta(\cos\gamma a_{ii} - \cos\gamma b_{ii}), & i = j, \\ 0, & i \neq j, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}\backslash\{0\}$ are arbitrary real constants.
Then, for any $A = \left(a_{ij} \right), B = \left(b_{ij} \right) \in D_n(\mathbb{R})$, yields

$$\langle M(A) - M(B), \eta(A, B)\rangle = \beta^2\sum_{i=1}^{n}(\cos\gamma a_{ii} - \cos\gamma b_{ii})^2 \geq 0,$$

which means that M is an η-monotone operator. Let us now define the function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) := \alpha \sin\gamma x + \beta \cos\gamma x$, for all $x \in \mathbb{R}$. Then, for any $A \in D_n(\mathbb{R})$, we get

$$(P + M)(A) = (P + M)(\left(a_{ij} \right)) = \left(\widetilde{a}_{ij} + \widehat{a}_{ij} \right) = \left(d_{ij} \right),$$

where for each $i, j \in \{1, 2, \ldots, n\}$,

$$d_{ij} = \begin{cases} f(a_{ii}) = \alpha \sin\gamma a_{ii} + \beta \cos\gamma a_{ii}, & i = j, \\ 0, & i \neq j. \end{cases}$$

It can be easily observed that $\mathrm{R}(f) = [-\sqrt{\alpha^2 + \beta^2}, \sqrt{\alpha^2 + \beta^2}]$. Owing to the fact that $f(\mathbb{R}) \neq \mathbb{R}$, it follows that $(H + M)(D_n(\mathbb{R})) \neq D_n(\mathbb{R})$, which ensures that $H + M$ is not surjective, and so M is not an (H, η)-monotone operator. Now, assume that

$\lambda > 0$ is an arbitrary real constant and let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x + \lambda\beta \cos\gamma x$, for all $x \in \mathbb{R}$. Then, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, we yield

$$(I + \lambda M)(A) = (I + \lambda M)(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a_{ij} + \lambda \widehat{a}_{ij} \end{pmatrix} = \begin{pmatrix} a'_{ij} \end{pmatrix},$$

where I is the identity mapping on $D_n(\mathbb{R})$ and for each $i, j \in \{1, 2, \dots, n\}$,

$$a'_{ij} = \begin{cases} g(a_{ii}) = a_{ii} + \lambda\beta \cos\gamma a_{ii}, & i = j, \\ 0, & i \neq j. \end{cases}$$

Considering the fact that $g(\mathbb{R}) = \mathbb{R}$, it follows that $(I + \lambda M)(D_n(\mathbb{R})) = D_n(\mathbb{R})$, that is, $I + \lambda M$ is surjective. Since $\lambda > 0$ was arbitrary, we deduce that M is a maximal η-monotone operator.

In the light of Example 2.4, for given operators $H : X \to X$ and $\eta : X \times X \to X$, an (H, η)-monotone operator is not maximal η-monotone in general. As a consequent of Theorem 2.1 in [25], the following assertion, in which the sufficient conditions for an (H, η)-monotone operator M to be maximal η-monotone are stated, can be derived.

Lemma 2.6. *Let $\eta : X \times X \to X$ be a vector-valued operator, $H : X \to X$ be a strictly η-monotone operator, $M : X \rightrightarrows X$ be an (H, η)-monotone operator, and let $x, u \in X$ be two given points. If $\langle u - v, \eta(x, y)\rangle \geq 0$ holds, for all $(v, y) \in \mathrm{Graph}(M)$, then $(u, x) \in \mathrm{Graph}(M)$, that is, M is a maximal η-monotone operator.*

According to Example 2.5, for given operators $H : X \to X$ and $\eta : X \times X \to X$, a maximal η-monotone operator need not be (H, η)-monotone. In the next conclusion, the sufficient conditions for a maximal η-monotone operator to be (H, η)-monotone are provided. Before proceeding to it, let us recall the following definitions.

Definition 2.7. [17, Definition 2.2] An operator $H : X \to X$ is said to be coercive if

$$\lim_{\|x\| \to +\infty} \frac{\langle H(x), x\rangle}{\|x\|} = +\infty.$$

Definition 2.8. [17, Definition 2.3] An operator $A : X \to X$ is said to be bounded if $A(B)$ is bounded for every bounded subset B of X. A is said to be hemi-continuous if for any fixed $x, y, z \in X$, the function $t \mapsto \langle A(x + ty), z\rangle$ is continuous at 0^+.

Lemma 2.9. *Let $\eta : X \times X \to X$ be a vector-valued operator and $H : X \to X$ be a bounded, coercive, hemi-continuous and η-monotone operator. If $M : X \rightrightarrows X$ is a maximal η-monotone operator, then M is (H, η)-monotone.*

Proof. Owing to the fact that H is bounded, coercive, hemi-continuous and η-monotone, invoking Corollary 32.26 of [38], we conclude that the operator $H + \lambda M$ is surjective for every $\lambda > 0$, that is, the range of $H + \lambda M$ is precisely X for all $\lambda > 0$. Thereby, M is an (H, η)-monotone operator. The proof is completed. □

Theorem 2.10. *Let $\eta : X \times X \to X$ be a vector-valued operator, $H : X \to X$ be a strictly η-monotone operator and $M : X \rightrightarrows X$ be an η-monotone operator. Then, for every real constant $\lambda > 0$, the operator $(H + \lambda M)^{-1}$ from $\mathrm{Range}(H + \lambda M)$ to X is single-valued.*

Proof. Suppose, on the contrary, that there exists $z \in \mathrm{Range}(H + \lambda M)$ such that $x, y \in (H + \lambda M)^{-1}(z)$ and $x \neq y$. Then, we have $z \in (H + \lambda M)(x)$ and $z \in (H + \lambda M)(y)$, and so there exist $u \in M(x)$ and $v \in M(y)$ such that

$$H(x) + \lambda u = H(y) + \lambda v. \tag{2.1}$$

Since M and H are η-monotone, with the help of (2.1), we derive that

$$0 \leq \langle \lambda(u - v), \eta(x, y)\rangle = -\langle H(x) - H(y), \eta(x, y)\rangle \leq 0,$$

which guarantees that $\langle H(x)-H(y), \eta(x, y)\rangle = 0$. In view of the fact that H is strictly η-monotone, it follows that $x = y$ which is in contradiction to our assumption. □

In order to define the resolvent operator associated with an (H, η)-monotone operator, Fang et al. [19] presented the following statement which is an immediate consequence of the previous theorem.

Lemma 2.11. [19, Lemma 2.1] *Let $\eta : X \times X \to X$ be a vector-valued operator, $H : X \to X$ be a strictly η-monotone and $M : X \rightrightarrows X$ be a (H, η)-monotone operator. Then, for every real constant $\lambda > 0$, the operator $(H + \lambda M)^{-1}$ is single-valued.*

Based on Lemma 2.11, for an arbitrary real constant $\lambda > 0$, Fang et al. [19] defined the resolvent operator $R^{H,\eta}_{M,\lambda}$ associated with an (H, η)-monotone operator M as follows.

Definition 2.12. [19, Definition 2.4] Let $\eta : X \times X \to X$ be a vector-valued operator, $H : X \to X$ be a strictly η-monotone operator and $M : X \rightrightarrows X$ be an (H, η)-monotone operator. The resolvent operator $R^{H,\eta}_{M,\lambda} : X \to X$ is defined by

$$R^{H,\eta}_{M,\lambda}(u) = (H + \lambda M)^{-1}(u), \quad \forall u \in X,$$

where $\lambda > 0$ is an arbitrary real constant.

Let us emphasize that in the rest of the paper, we say that M is an (H, η)-γ-strongly monotone operator, means that M is a γ-strongly η-monotone operator and $(H + \lambda M)(X) = X$, for every real constant $\lambda > 0$.

We now prove the Lipschitz continuity of the resolvent operator $R^{H,\eta}_{M,\lambda}$ associated with an (H, η)-monotone operator M and an arbitrary real constant $\lambda > 0$ under some appropriate conditions and compute an estimate of its Lipschitz constant. For this end, we need to recall the following definition.

Definition 2.13. A vector-valued operator $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that $\|\eta(x, y)\| \leq \tau\|x - y\|$ for all $x, y \in X$.

Theorem 2.14. *Let $\eta : X\times X \to X$ be a τ-Lipschitz continuous operator, $H : X \to X$ be a ϱ-strongly η-monotone operator and let $M : X \rightrightarrows X$ be an (H, η)-γ-strongly monotone operator. Then, the resolvent operator $R^{H,\eta}_{M,\lambda} : X \to X$ is $\frac{\tau}{\lambda\gamma+\varrho}$-Lipschitz continuous, i.e.,*

$$\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\| \leq \frac{\tau}{\lambda\gamma + \varrho}\|u - v\|, \quad \forall u, v \in X.$$

Proof. Owing to the fact that M is an (H,η)-monotone operator, for any given points $u, v \in X$ with $\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\| \neq 0$, yields

$$R^{H,\eta}_{M,\lambda}(u) = (H + \lambda M)^{-1}(u) \text{ and } R^{H,\eta}_{M,\lambda}(v) = (H + \lambda M)^{-1}(v),$$

which implies that

$$\lambda^{-1}(u - H(R^{H,\eta}_{M,\lambda}(u))) \in M(R^{H,\eta}_{M,\lambda}(u)) \text{ and } \lambda^{-1}(v - H(R^{H,\eta}_{M,\lambda}(v))) \in M(R^{H,\eta}_{M,\lambda}(v)).$$

Since M is γ-strongly η-monotone, we deduce that

$$\begin{aligned}\lambda^{-1}\langle u - H(R^{H,\eta}_{M,\lambda}(u)) - (v - H(R^{H,\eta}_{M,\lambda}(v))), \eta(R^{H,\eta}_{M,\lambda}(u), R^{H,\eta}_{M,\lambda}(v)))\rangle \\ \geq \gamma\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\|^2.\end{aligned}$$

In virtue of the fact that $\lambda^{-1} > 0$, the last inequality ensures that

$$\begin{aligned}\langle u - v, \eta(R^{H,\eta}_{M,\lambda}(u), R^{H,\eta}_{M,\lambda}(v))\rangle &\geq \lambda\gamma\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\|^2 \\ &\quad + \langle H(R^{H,\eta}_{M,\lambda}(u)) - H(R^{H,\eta}_{M,\lambda}(v)), \eta(R^{H,\eta}_{M,\lambda}(u), R^{H,\eta}_{M,\lambda}(v))\rangle.\end{aligned}$$

Considering the facts that the operator η is τ-Lipschitz continuous and the operator H is ϱ-strongly η-monotone, the preceding inequality guarantees that

$$\begin{aligned}\|u - v\|\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\| &\geq \|u - v\|\|\eta(R^{H,\eta}_{M,\lambda}(u), R^{H,\eta}_{M,\lambda}(v))\| \\ &\geq \lambda\gamma\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\|^2 \\ &\quad + \langle H(R^{H,\eta}_{M,\lambda}(u)) - H(R^{H,\eta}_{M,\lambda}(v)), \eta(R^{H,\eta}_{M,\lambda}(u), R^{H,\eta}_{M,\lambda}(v))\rangle \\ &\geq \lambda\gamma\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\|^2 + \varrho\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\|^2 \\ &= (\lambda\gamma + \varrho)\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\|^2. \qquad (2.2)\end{aligned}$$

Since $\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\| \neq 0$, by (2.2), it follows that

$$\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\| \leq \frac{\tau}{\lambda\gamma + \varrho}\|u - v\|.$$

The proof is finished. □

3. Formulation of the problem and iterative algorithms

Recall that a mapping $T : X \to X$ which has Lipschitz's constant equal to 1, that is, $\|T(x) - T(y)\| \leq \|x - y\|$ for every $x, y \in X$, is said to be nonexpansive. In the last forty years, there has been a major activity in the study of nonexpansive mappings under appropriate conditions and there is an extensive literature on the iterative methods to approximate fixed points of them, see, for example, [21, 33, 37]. Goebel and Kirk [20] defined in 1972 a class of generalized nonexpansive mappings, the so-called asymptotically nonexpansive mappings as an extension of the class of nonexpansive mappings as follows.

Definition 3.1. [20] A mapping $T : X \to X$ is said to be asymptotically nonexpansive if, there exists a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x, y \in X$,

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\|, \quad \forall n \in \mathbb{N}.$$

In 2005, Sahu [29] made an effort to unify some generalizations of nonexpansive mappings. He introduced the class of nearly asymptotically nonexpansive mappings as an extension of nonexpansive and asymptotically nonexpansive mappings as follows.

Definition 3.2. [29] A mapping $T : X \to X$ is said to be nearly asymptotically nonexpansive if, there exist real sequences $\{k_n\} \subseteq [1, +\infty)$ and $\{\mu_n\} \subseteq [0, +\infty)$ with $k_n \to 1$ and $\mu_n \to 0$ as $n \to \infty$, such that for all $x, y \in X$,

$$\|T^n(x) - T^n(y)\| \leq k_n(\|x - y\| + \mu_n), \quad \forall n \in \mathbb{N}.$$

Remark 3.3. It should be pointed out that if $k_n = L$ for all $n \in \mathbb{N}$, then the class of nearly asymptotically nonexpansive mappings coincides with the class of nearly uniformly L-Lipschitzian mappings [29]. In the meanwhile, for the case when $k_n = 1$ for all $n \in \mathbb{N}$, then the class of nearly asymptotically nonexpansive mappings becomes actually the same class of nearly nonexpansive mappings [29].

Recently, Alber et al. [1] introduced the notion of total asymptotically nonexpansive mappings as a unifying framework for some classes of generalized nonexpansive mappings available in the literature as follows.

Definition 3.4. [1] A mapping $T : X \to X$ is said to be total asymptotically nonexpansive (also referred to as $(\{a_n\}, \{b_n\}, \phi)$-total asymptotically nonexpansive) if, there exist nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in X$,

$$\|T^n(x) - T^n(y)\| \leq \|x - y\| + a_n\phi(\|x - y\|) + b_n, \quad \forall n \in \mathbb{N}.$$

They further studied the iterative approximation of fixed point of total asymptotically nonexpansive mappings using a modified Mann iterative process.

It should be remarked that from the definitions, it is obvious that every nonexpansive mapping is asymptotically nonexpansive with $k_n = 1$ for all $n \in \mathbb{N}$, every asymptotically nonexpansive mapping is total asymptotically nonexpansive with $b_n = 0$, $a_n = k_n - 1$ for all $n \in \mathbb{N}$ and $\phi(t) = t$ for all $t \geq 0$, and every nearly asymptotically nonexpansive mapping is also total asymptotically nonexpansive with $b_n = \mu_n k_n$, $a_n = k_n - 1$ for all $n \in \mathbb{N}$, and $\phi(t) = t$ for all $t \geq 0$.

The following example illustrates that the class of total asymptotically nonexpansive mappings is more general than the class of asymptotically nonexpansive mappings.

Example 3.5. For $1 \leq p < \infty$, consider the classical space

$$l^p = \big\{x = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty, x_n \in \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\big\},$$

consisting of all p-power summable sequences, with the p-norm $\|.\|_p$ defined on it by

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}, \quad \forall x = (x_n)_{n\in\mathbb{N}} \in l^p.$$

Let B be the closed unit ball in the Banach space l^p and consider the subset

$$X := [\sigma, \delta] \times B$$

of $\mathbb{R} \times l^p$ with the norm $\|.\|_X = |.|_{\mathbb{R}} + \|.\|_p$ defined on $\mathbb{R} \times l^p$, where $\sigma < 0$ and $\delta \geq 1$ are arbitrary real constants. Furthermore, let the self-mapping T of X be defined by

$$T(u,x) = \begin{cases} (u, \widetilde{x}), & \text{if } u \in [\sigma, 0), \\ (\alpha, \widetilde{x}), & \text{if } u = 0, \\ (\alpha u, \widetilde{x}), & \text{if } u \in (0, \delta], \end{cases}$$

where

$$\widetilde{x} = (\underbrace{0, 0, \ldots, 0}_{k\ times}, \beta \sin|x_m|^{q_{\frac{m+2}{3}}}, 0, \frac{\beta}{\sqrt[p]{2^{p+1}}}(\sin|x_{m+1}|^{s_{\frac{m+2}{3}}} - |x_{m+1}|^{t_{\frac{m+2}{3}}}), 0,$$

$$\frac{\beta}{\sqrt[p]{2^{p+1}}}(|x_{m+2}|^{l_{\frac{m+2}{3}}} - \sin^{k_{\frac{m+2}{3}}}|x_{m+2}|), 0, \beta x_{m+3}, 0, \beta x_{m+4}, \ldots),$$

$\alpha, \beta \in (0,1)$ are arbitrary real constants; $m \in \{3j-2 | j \in \mathbb{N}\}$, $k \geq m+2$ and $k_i, l_i, q_i, s_i, t_i \in \mathbb{N}\backslash\{1\}$ $(i = 1, 2, \ldots, \frac{m+2}{3})$ are arbitrary but fixed natural numbers.

It can be easily seen that the mapping T is discontinuous at the points $(0, x)$ for all $x \in B$. Taking into account that every asymptotically nonexpansive mapping is Lipschitzian and every Lipschitzian mapping is continuous, it follows that T is not Lipschitzian and so it is not an asymptotically nonexpansive mapping.

It is easy to prove that for all $(u,x), (v,y) \in [\sigma, 0) \times B$,

$$\begin{aligned} &\|T(u,x) - T(v,y)\|_X = \|(u - v, \widetilde{x} - \widetilde{y})\|_X \\ &\leq |u - v| + \beta \max\Big\{ \sum_{r''=1}^{q_i} |x_{3i-2}|^{q_i - r''} |y_{3i-2}|^{r''-1}, \\ &\sum_{s'=1}^{s_i} |x_{3i-1}|^{s_i - s'} |y_{3i-1}|^{s'-1}, \sum_{j=1}^{t_i} |x_{3i-1}|^{t_i - j} |y_{3i-1}|^{j-1}, \\ &\sum_{r=1}^{l_i} |x_{3i}|^{l_i - r} |y_{3i}|^{r-1}, \sum_{r'=1}^{k_i} |x_{3i}|^{k_i - r'} |y_{3i}|^{r'-1}, 1 : i = 1, 2, \ldots, \frac{m+2}{3} \Big\} \|x - y\|_p. \end{aligned} \tag{3.1}$$

Considering the fact that $x, y \in B$, and making use of (3.1) it follows that for all $(u,x), (v,y) \in [\sigma, 0) \times B$,

$$\|T(u,x) - T(v,y)\|_X \leq |u - v| + \beta\xi\|x - y\|_p, \tag{3.2}$$

where $\xi = \max\{k_i, l_i, q_i, s_i, t_i : i = 1, 2, \ldots, \frac{m+2}{3}\}$. By following the same arguments as above, on can prove that

(i) for all $(u,x), (v,y) \in \{0\} \times B$,

$$\|T(u,x) - T(v,y)\|_X \leq \beta\xi\|x - y\|_p \leq |u - v| + \beta\xi\|x - y\|_p; \tag{3.3}$$

(ii) for all $(u,x), (v,y) \in (0, \delta] \times B$,

$$\|T(u,x) - T(v,y)\|_X \leq |u - v| + \beta\xi\|x - y\|_p + \alpha\delta; \tag{3.4}$$

(iii) for all $(u,x) \in [\sigma,0) \times B$ and $(v,y) \in (0,\delta] \times B$,

$$\|T(u,x) - T(v,y)\|_X \leq |u-v| + \beta\xi\|x-y\|_p + \alpha\delta; \tag{3.5}$$

(iv) for all $(u,x) \in [\sigma,0) \times B$ and $(v,y) \in \{0\} \times B$,

$$\|T(u,x) - T(v,y)\|_X \leq |u-v| + \beta\xi\|x-y\|_p + \alpha; \tag{3.6}$$

(v) for all $(u,x) \in \{0\} \times B$ and $(v,y) \in (0,\delta] \times B$,

$$\|T(u,x) - T(v,y)\|_X \leq |u-v| + \beta\xi\|x-y\|_p + \alpha. \tag{3.7}$$

Using (3.2)–(3.7) and in virtue of the fact that $\alpha > 0$ and $\delta \geq 1$, we conclude that for all $(u,x),(v,y) \in X$,

$$\begin{aligned} \|T(u,x) - T(v,y)\|_X &\leq |u-v| + \beta\xi\|x-y\|_p + \alpha\delta \\ &\leq |u-v| + \|x-y\|_p + \beta\xi(|u-v| + \|x-y\|_p) + \alpha\delta. \end{aligned} \tag{3.8}$$

For all $(u,x) \in [\sigma,0) \times B$ and $n \geq 2$, we have $T^n(u,x) = (u,\widehat{x})$, where

$$\begin{aligned} \widehat{x} = \Big(&\underbrace{0,\dots,0}_{(2^n-1)k\ times}, \beta^n \sin|x_m|^{q^{\frac{m+2}{3}}}, \underbrace{0,\dots,0}_{(2^n-1)\ times}, \\ &(\frac{\beta}{\sqrt[p]{2^{p+1}}})^n (\sin|x_{m+1}|^{s^{\frac{m+2}{3}}} - |x_{m+1}|^{t^{\frac{m+2}{3}}}), \underbrace{0,\dots,0}_{(2^n-1)\ times}, \\ &(\frac{\beta}{\sqrt[p]{2^{p+1}}})^n (|x_{m+2}|^{l^{\frac{m+2}{3}}} - \sin^{k^{\frac{m+2}{3}}}|x_{m+2}|), \\ &\underbrace{0,\dots,0}_{(2^n-1)\ times}, \beta^n x_{m+3}, \underbrace{0,\dots,0}_{(2^n-1)\ times}, \beta^n x_{m+4}, \dots \Big). \end{aligned}$$

At the same time, for each $n \in \mathbb{N}$, $T^n(u,x) = (\alpha^n, \widehat{x})$ and $T^n(u,x) = (\alpha^n u, \widehat{x})$ for all $(u,x) \in \{0\} \times B$ and $(u,x) \in (0,\delta] \times B$, respectively. Then, using the same arguments as for (3.1)–(3.8), one can show that for all $(u,x),(v,y) \in X$ and $n \geq 2$,

$$\begin{aligned} \|T^n(u,x) - T^n(v,y)\|_X &\leq |u-v| + \beta^n\xi\|x-y\|_p + \alpha^n\delta \\ &\leq |u-v| + \|x-y\|_p + \beta^n\xi(|u-v| + \|x-y\|_p) + \alpha^n\delta. \end{aligned} \tag{3.9}$$

Thereby, employing (3.8) and (3.9), it follows that for all $(u,x),(v,y) \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned} \|T^n(u,x) - T^n(v,y)\|_X &\leq |u-v| + \|x-y\|_p + \beta^n\xi(|u-v| + \|x-y\|_p) + \alpha^n\delta \\ &= \|(u,x)-(v,y)\|_X + \beta^n\xi\|(u,x)-(v,y)\|_X + \alpha^n\delta. \end{aligned} \tag{3.10}$$

Taking $a_n = \beta^n$ and $b_n = \alpha^n\delta$ for each $n \in \mathbb{N}$, since $\alpha, \beta \in (0,1)$, we deduce that $a_n, b_n \to 0$, as $n \to \infty$. Defining the mapping $\phi : [0,+\infty) \to [0,+\infty)$ as $\phi(w) = \xi w$ for all $w \in [0,+\infty)$, making use of (3.10), for all $(u,x),(v,y) \in X$ and $n \in \mathbb{N}$, yields

$$\|T^n(u,x) - T^n(v,y)\|_X \leq \|(u,x)-(v,y)\|_X + a_n\phi(\|(u,x)-(v,y)\|_X) + b_n,$$

that is, T is an $(\{a_n\},\{b_n\},\phi)$-total asymptotically nonexpansive mapping.

Let $P : X \to X$ be a single-valued operator and $M : X \rightrightarrows X$ be a multi-valued operator. We now consider the problem of finding $x \in X$ such that

$$0 \in P(x) + M(x), \tag{3.11}$$

which is called the *variational inclusion problem* (VIP).

The VIP (3.11) has been studied by many authors in the setting of Hilbert spaces when M is maximal monotone and P is strongly monotone, see, for example, [22]. For the case when M is an H-monotone operator, where $H : X \to X$ is a single-valued operator, and P is strongly monotone with respect to H and Lipschitz continuous, the VIP (3.11) is the same variational inclusion problem involving H-monotone operator considered by Fang and Huang [17, 39]. Note, in particular, that the VIP (3.11) has been studied by Bi et al. [15] and Fang and Huang [18] in the setting of Banach spaces.

The following conclusion tells the VIP (3.11) is equivalent to a fixed point problem.

Lemma 3.6. *Let $\eta : X \times X \to X$ and $P : X \to X$ be vector-valued operators, $H : X \to X$ be a strictly η-monotone operator and $M : X \rightrightarrows X$ be an (H,η)-monotone operator. Then $x \in X$ is a solution of the* VIP (3.11) *if and only if*

$$x = R^{H,\eta}_{M,\lambda}[H(x) - \lambda P(x)],$$

where $\lambda > 0$ is a real constant.

Proof. It follows directly from Definition 2.12 and some simple arguments. □

Let $S : X \to X$ be a total asymptotically nonexpansive mapping and let M, H, P and η be the same as in Lemma 3.6. We denote by $\mathrm{Fix}(S)$ and $\mathrm{VIP}(X, M, H, P, \eta)$, respectively, the set of all the fixed points of S and the set of the solutions of the VIP (3.11). We now characterize the problem. If $x^* \in \mathrm{Fix}(S) \cap \mathrm{VIP}(X, M, H, P, \eta)$, then from Lemma 3.6 it follows that for all $n \geq 0$,

$$x^* = S^n x^* = R^{H,\eta}_{M,\lambda}[H(x^*) - \lambda P(x^*)] = S^n R^{H,\eta}_{M,\lambda}[H(x^*) - \lambda P(x^*)]. \tag{3.12}$$

The fixed point formulation (3.12) allows us to construct the following resolvent iterative algorithm for approximating a common element of the two sets $\mathrm{Fix}(S)$ and $\mathrm{VIP}(X, M, H, P, \eta)$.

Algorithm 3.7. Let $P : X \to X$ and $\eta_n : X \times X \to X$ $(n \geq 0)$ be vector-valued operators, $H_n : X \to X$ be a strictly η_n-monotone operator, and $M_n : X \rightrightarrows X$ be an (H_n, η_n)-monotone operator. Suppose further that $S : X \to X$ is a total asymptotically nonexpansive mapping. For an arbitrary chosen initial point $x_0 \in X$, compute the iterative sequence $\{x_n\}_{n=0}^{\infty}$ in X in the following way:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n R^{H_n,\eta_n}_{M_n,\lambda_n}[H_n(x_n) - \lambda_n P(x_n)], \tag{3.13}$$

where $n = 0, 1, 2, \ldots$; $\lambda_n > 0$ is a real constant for each $n \geq 0$; and $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in the interval $[0, 1]$ with $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.

If $S \equiv I$, the identity mapping on X, Algorithm 3.7 collapses to the following iterative algorithm.

Algorithm 3.8. Suppose that P, η_n, H_n, M_n $(n \geq 0)$ are the same as in Algorithm 3.7. For an arbitrary chosen initial point $x_0 \in X$, define the iterative sequence $\{x_n\}_{n=0}^{\infty}$ in X by the iterative scheme

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n R^{H_n,\eta_n}_{M_n,\lambda_n}[H_n(x_n) - \lambda_n P(x_n)],$$

where $n = 0, 1, 2, \ldots$; and $\lambda_n > 0$, $\{\alpha_n\}_{n=0}^{\infty}$ are the same as in Algorithm 3.7.

4. Graph Convergence and an Application

Definition 4.1. Given multi-valued operators $M_n, M : X \rightrightarrows X$ $(n \geq 0)$, the sequence $\{M_n\}_{n=0}^{\infty}$ is said to be graph-convergent to M, denoted by $M_n \xrightarrow{G} M$, if for every point $(x, u) \in \text{Graph}(M)$, there exists a sequence of points $(x_n, u_n) \in \text{Graph}(M_n)$ such that $x_n \to x$ and $u_n \to u$ as $n \to \infty$.

In the next theorem, the equivalence between the graph convergence of a sequence of (H, η)-strongly monotone operators and their associated resolvent operators, respectively, to a given (H, η)-strongly monotone operator and its associated resolvent operator is proved.

Theorem 4.2. *Let $\eta : X \times X \to X$ be a vector-valued operator, $H : X \to X$ be a strictly η-monotone operator, and let $M : X \rightrightarrows X$ be an (H, η)-monotone operator. Suppose that for each $n \geq 0$, $\eta_n : X \times X \to X$ is a τ_n-Lipschitz continuous operator, $H_n : X \to X$ is a ϱ_n-strongly η_n-monotone and σ_n-Lipschitz continuous operator such that $\{\sigma_n\}_{n=0}^{\infty}$ is a bounded sequence, and $M_n : X \rightrightarrows X$ is an (H_n, η_n)-γ_n-strongly monotone operator. Moreover, assume that $\lim\limits_{n\to\infty} H_n(x) = H(x)$ for any $x \in X$, $\{\lambda_n\}_{n=0}^{\infty}$ is a sequence of positive real constants convergent to a positive real constant λ such that the sequence $\{\frac{\tau_n}{\lambda_n\gamma_n+\varrho_n}\}_{n=0}^{\infty}$ is bounded. Then, $M_n \xrightarrow{G} M$ if and only if $\lim\limits_{n\to\infty} R^{H_n,\eta_n}_{M_n,\lambda_n}(z) = R^{H,\eta}_{M,\lambda}(z)$, for all $z \in X$.*

Proof. Suppose first that $\lim\limits_{n\to\infty} R^{H_n,\eta_n}_{M_n,\lambda_n}(z) = R^{H,\eta}_{M,\lambda}(z)$, for all $z \in X$. Then, for any $(x, u) \in \text{Graph}(M)$, we have $x = R^{H,\eta}_{M,\lambda}[H(x)+\lambda u]$ and so $\lim\limits_{n\to\infty} R^{H_n,\eta_n}_{M_n,\lambda_n}[H(x)+\lambda u] = x$. Letting $x_n = R^{H_n,\eta_n}_{M_n,\lambda_n}[H(x) + \lambda u]$ for each $n \geq 0$, we deduce that $H(x) + \lambda u \in (H_n + \lambda_n M_n)(x_n)$. Thereby, for each $n \geq 0$, we can choose $u_n \in M_n(x_n)$ such that $H(x) + \lambda u = H_n(x_n) + \lambda_n u_n$. Then, we derive for all $n \geq 0$,

$$\begin{aligned}
\|\lambda_n u_n - \lambda u\| = \|H_n(x_n) - H(x)\| &\leq \|H_n(x_n) - H_n(x)\| + \|H_n(x) - H(x)\| \\
&\leq \sigma_n\|x_n - x\| + \|H_n(x) - H(x)\|.
\end{aligned}$$

Since the sequence $\{\sigma_n\}_{n=0}^{\infty}$ is bounded, $x_n \to x$ and $\lim\limits_{n\to\infty} H_n(x) = H(x)$, it follows that $\lim\limits_{n\to\infty} \lambda_n u_n = \lambda u$. At the same time, for all $n \geq 0$, yields

$$\begin{aligned}
\lambda\|u_n - u\| = \|\lambda u_n - \lambda u\| &\leq \|\lambda_n u_n - \lambda u_n\| + \|\lambda_n u_n - \lambda u\| \\
&= |\lambda_n - \lambda|\|u_n\| + \|\lambda_n u_n - \lambda u\|.
\end{aligned}$$

The facts that $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lim_{n\to\infty} \lambda_n u_n = \lambda u$ imply that the right-hand side of the last inequality tends to zero, as $n \to \infty$. Accordingly, $u_n \to u$, as $n \to \infty$. Now, invoking Definition 4.1, we conclude that $M_n \xrightarrow{G} M$.

Converse, assume that $M_n \xrightarrow{G} M$, and let $z \in X$ be chosen arbitrarily but fixed. Taking into account that M is an (H, η)-monotone operator, it follows that the range of $H + \lambda M$ is precisely X and so there exists $(x, u) \in \text{Graph}(M)$ such that $z = H(x) + \lambda u$. Thanks to Definition 4.1, there exists a sequence $\{(x_n, u_n)\}_{n=0}^{\infty} \subset \text{Graph}(M_n)$ such that $x_n \to x$ and $u_n \to u$, as $n \to \infty$. Relying on the facts that $(x, u) \in \text{Graph}(M)$ and $(x_n, u_n) \in \text{Graph}(M_n)$ $(n \geq 0)$ we get

$$x = R^{H,\eta}_{M,\lambda}[H(x) + \lambda u] \quad \text{and} \quad x_n = R^{H_n,\eta_n}_{M_n,\lambda_n}[H_n(x_n) + \lambda_n u_n], \quad \forall n \geq 0. \tag{4.1}$$

Putting $z_n = H_n(x_n) + \lambda_n u_n$ for all $n \geq 0$, utilizing Theorem 2.14, (4.1) and the assumptions, yields

$$\begin{aligned}
&\|R^{H_n,\eta_n}_{M_n,\lambda_n}(z) - R^{H,\eta}_{M,\lambda}(z)\| \\
&\leq \|R^{H_n,\eta_n}_{M_n,\lambda_n}(z) - R^{H_n,\eta_n}_{M_n,\lambda_n}(z_n)\| + \|R^{H_n,\eta_n}_{M_n,\lambda_n}(z_n) - R^{H,\eta}_{M,\lambda}(z)\| \\
&\leq \frac{\tau_n}{\lambda_n\gamma_n + \varrho_n}\|z_n - z\| + \|R^{H_n,\eta_n}_{M_n,\lambda_n}[H_n(x_n) + \lambda_n u_n] - R^{H,\eta}_{M,\lambda}[H(x) + \lambda u]\| \\
&\leq \frac{\tau_n}{\lambda_n\gamma_n + \varrho_n}\|z_n - z\| + \|x_n - x\| \\
&= \frac{\tau_n}{\lambda_n\gamma_n + \varrho_n}\|H_n(x_n) + \lambda_n u_n - H(x) - \lambda u\| + \|x_n - x\| \\
&\leq \frac{\tau_n}{\lambda_n\gamma_n + \varrho_n}(\|H_n(x_n) - H(x)\| + \|\lambda_n u_n - \lambda u\|) + \|x_n - x\| \\
&\leq \frac{\tau_n}{\lambda_n\gamma_n + \varrho_n}(\|H_n(x_n) - H_n(x)\| + \|H_n(x) - H(x)\| \\
&\quad + \|\lambda_n u_n - \lambda_n u\| + \|\lambda_n u - \lambda u\|) + \|x_n - x\| \\
&\leq (1 + \frac{\sigma_n\tau_n}{\lambda_n\gamma_n + \varrho_n})\|x_n - x\| + \frac{\tau_n}{\lambda_n\gamma_n + \varrho_n}\|H_n(x) - H(x)\| \\
&\quad + \frac{\lambda_n\tau_n}{\lambda_n\gamma_n + \varrho_n}\|u_n - u\| + \frac{|\lambda_n - \lambda|\tau_n}{\lambda_n\gamma_n + \varrho_n}\|u\|.
\end{aligned}$$

Since the sequences $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\frac{\tau_n}{\lambda_n\gamma_n+\varrho_n}\}_{n=0}^{\infty}$ are bounded and $\lim_{n\to\infty} \lambda_n = \lambda$, we conclude that the sequences $\{\frac{\sigma_n\tau_n}{\lambda_n\gamma_n+\varrho_n}\}_{n=0}^{\infty}$ and $\{\frac{\lambda_n\tau_n}{\lambda_n\gamma_n+\varrho_n}\}_{n=0}^{\infty}$ are also bounded. Using the assumptions, it is easy to see that the right-hand side of the preceding inequality approaches zero, as $n \to \infty$, which ensures $\lim_{n\to\infty} R^{H_n,\eta_n}_{M_n,\lambda_n}(z) = R^{H,\eta}_{M,\lambda}(z)$. The proof is now complete. $\square$

Before turning to the convergence analysis of the suggested iterative algorithm for computation of a common element of the two sets $\text{Fix}(S)$ and $\text{VIP}(X, M, H, P, \eta)$, let us give the following lemma which will be used efficiently in the proof of our main result in this section.

Lemma 4.3. [36] *Let $\{a_n\}_{n=0}^{\infty}$ be a nonnegative real sequence and $\{b_n\}_{n=0}^{\infty}$ be a real sequence in $[0,1]$ such that $\sum\limits_{n=0}^{\infty} b_n = \infty$. If there exists a positive integer n_0 such that*

$$a_{n+1} \leq (1-b_n)a_n + b_n c_n, \quad \forall n \geq n_0,$$

where $c_n \geq 0$ for all $n \geq 0$ and $\lim\limits_{n\to\infty} c_n = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Theorem 4.4. *Let $\eta : X \times X \to X$ be a τ-Lipschitz continuous operator, $H : X \to X$ be a ϱ-strongly η-monotone and σ-Lipschitz continuous operator, $M : X \rightrightarrows X$ be an (H,η)-γ-strongly monotone operator, and $P : X \to X$ be a ω-Lipschitz continuous operator. Assume that for each $n \geq 0$, $\eta_n : X \times X \to X$ is a τ_n-Lipschitz continuous operator, the operator $H_n : X \to X$ is ϱ_n-strongly η_n-monotone, κ_n-strongly monotone with respect to P, and σ_n-Lipschitz continuous. Suppose that for each $n \geq 0$, $M_n : X \rightrightarrows X$ is an (H_n,η_n)-γ_n-strongly monotone operator and $\lim\limits_{n\to\infty} H_n(z) = H(z)$ for any $z \in X$. Let $\kappa_n \to \kappa$, $\tau_n \to \tau$, $\gamma_n \to \gamma$, $\varrho_n \to \varrho$, $\sigma_n \to \sigma$ and $M_n \xrightarrow{G} M$, as $n \to \infty$. Furthermore, let $S : X \to X$ be a $(\{\nu_n\}, \{\mu_n\}, \phi)$-total asymptotically nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \mathrm{VIP}(X,M,H,P,\eta) \neq \emptyset$. *If there exist real constants $\lambda_n > 0$ $(n \geq 0)$ satisfying* (3.13) *and a real constant $\lambda > 0$ such that $\lambda_n \to \lambda$ as $n \to \infty$, and*

$$\sqrt{\sigma^2 - 2\lambda\kappa + \lambda^2\omega^2} < \frac{\lambda\gamma + \varrho}{\tau}, \quad 2\lambda\kappa < \sigma^2 + \lambda^2\omega^2, \tag{4.2}$$

then

(i) $\mathrm{VIP}(X,M,H,P,\eta)$ *is a singleton set;*
(ii) *the iterative sequence $\{x_n\}_{n=0}^{\infty}$ generated by Algorithm 3.7 converges strongly to the only element x^* of* $\mathrm{Fix}(S) \cap \mathrm{VIP}(X,M,H,P,\eta)$.

Proof. Let us define a mapping $Q : X \to X$ by

$$Q(x) = R_{M,\lambda}^{H,\eta}[H(x) - \lambda P(x)], \quad \forall x \in X. \tag{4.3}$$

Then, making use of (4.3) and Theorem 2.14, for all $x, y \in X$, yields

$$\begin{aligned} \|Q(x) - Q(y)\| &= \|R_{M,\lambda}^{H,\eta}[H(x) - \lambda P(x)] - R_{M,\lambda}^{H,\eta}[H(y) - \lambda P(y)]\| \\ &\leq \frac{\tau}{\lambda\gamma + \varrho}\|H(x) - H(y) - \lambda(P(x) - P(y))\|. \end{aligned} \tag{4.4}$$

Since H is κ-strongly monotone with respect to P and t-Lipschitz continuous, and P is ω-Lipschitz continuous, utilizing well known property of the norm arising from inner product in Hilbert space X, it follows that for all $x, y \in X$,

$$\begin{aligned} &\|H(x) - H(y) - \lambda(P(x) - P(y))\|^2 \\ &= \|H(x) - H(y)\|^2 - 2\lambda\langle H(x) - H(y), P(x) - P(y)\rangle \\ &\quad + \lambda^2\|P(x) - P(y)\|^2 \\ &\leq (\sigma^2 - 2\lambda\kappa + \lambda^2\omega^2)\|x - y\|^2, \end{aligned}$$

which implies that

$$\|H(x) - H(y) - \lambda(P(x) - P(y))\| \leq \sqrt{\sigma^2 - 2\lambda\kappa + \lambda^2\omega^2}\|x - y\|. \tag{4.5}$$

Substituting (4.5) into (4.4), we obtain

$$\|Q(x)-Q(y)\| \leq \theta\|x-y\|, \tag{4.6}$$

where $\theta=\frac{\tau}{\lambda\gamma+\varrho}\sqrt{\sigma^2-2\lambda\kappa+\lambda^2\omega^2}$. Taking into account (4.2), we know that $\theta\in[0,1)$ and so (4.6) ensures that the mapping Q is contraction. Invoking Banach fixed point theorem, Q has a unique fixed point in X, that is, there exists a unique point $x^*\in X$ such that $Q(x^*)=x^*$. In virtue of (4.3), it follows that $x^*=R^{H,\eta}_{M,\lambda}[H(x^*)-\lambda P(x^*)]$. According to Lemma 3.6, $x^*\in X$ is a unique solution of the VIP (4.4). Thereby, VIP(X,M,H,P,η) is a singleton set.
Now, we prove the conclusion (ii). Owing to the fact that VIP$(X,M,H,P,\eta)=\{x^*\}$, from Fix(S) VIP$(X,M,H,P,\eta)\neq\emptyset$, we conclude that $x^*\in$ Fix(S). Hence, for all $n\geq 0$, one can write

$$x^*=(1-\alpha_n)x^*+\alpha_n S^n R^{H,\eta}_{M,\lambda}[H(x^*)-\lambda P(x^*)], \tag{4.7}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is the same as in Algorithm 3.7. Applying (4.6), (4.7) and the fact that S is a $(\{\nu_n\},\{\mu_n\},\varrho)$-total asymptotically nonexpansive mapping, we get

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|S^nR^{H_n,\eta_n}_{M_n,\lambda_n}[H_n(x_n)-\lambda_nP(x_n)]\\
&\quad -S^nR^{H,\eta}_{M,\lambda}[H(x^*)-\lambda P(x^*)]\|\\
&\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\big(\|R^{H_n,\eta_n}_{M_n,\lambda_n}[H_n(x_n)-\lambda_nP(x_n)]\\
&\quad -R^{H,\eta}_{M,\lambda}[H(x^*)-\lambda P(x^*)]\|+\nu_n\phi(\|R^{H_n,\eta_n}_{M_n,\lambda_n}[H_n(x_n)-\lambda_nP(x_n)]\\
&\quad -R^{H,\eta}_{M,\lambda}[H(x^*)-\lambda P(x^*)]\|)+\mu_n\big).\end{aligned} \tag{4.8}$$

Using Theorem 2.14 and the assumptions, for each $n\geq 0$, we obtain

$$\begin{aligned}&\|R^{H_n,\eta_n}_{M_n,\lambda_n}[H_n(x_n)-\lambda_nP(x_n)]-R^{H,\eta}_{M,\lambda}[H(x^*)-\lambda P(x^*)]\|\\
&\leq \|R^{H_n,\eta_n}_{M_n,\lambda_n}[H_n(x_n)-\lambda_nP(x_n)]-R^{H_n,\eta_n}_{M_n,\lambda_n}[H(x^*)-\lambda P(x^*)]\|\\
&\quad +\|R^{H_n,\eta_n}_{M_n,\lambda_n}[H(x^*)-\lambda P(x^*)]-R^{H,\eta}_{M,\lambda}[H(x^*)-\lambda P(x^*)]\|\\
&\leq \frac{\tau_n}{\lambda_n\gamma_n+\varrho_n}\|H_n(x_n)-\lambda_nP(x_n)-(H(x^*)-\lambda P(x^*))\|\\
&\leq \frac{\tau_n}{\lambda_n\gamma_n+\varrho_n}\big(\|H_n(x_n)-H(x^*)-\lambda_n(P(x_n)-P(x^*))\|\\
&\quad +|\lambda_n-\lambda|\|P(x^*)\|\big)+\|\Gamma_n\|\\
&\leq \frac{\tau_n}{\lambda_n\gamma_n+\varrho_n}\big(\sqrt{\sigma_n^2-2\lambda_n\kappa_n+\lambda_n^2\omega^2}\|x_n-x^*\|\\
&\quad +|\lambda_n-\lambda|\|P(x^*)\|\big)+\|\Gamma_n\|\\
&=\theta_n\|x_n-x^*\|+\frac{\tau_n|\lambda_n-\lambda|}{\lambda_n\gamma_n+\varrho_n}\|P(x^*)\|+\|\Gamma_n\|,\end{aligned} \tag{4.9}$$

where for each $n\geq 0$,

$$\Gamma_n=R^{H_n,\eta_n}_{M_n,\lambda_n}[H(x^*)-\lambda P(x^*)]-R^{H,\eta}_{M,\lambda}[H(x^*)-\lambda P(x^*)]$$

and

$$\theta_n = \frac{\tau_n}{\lambda_n\gamma_n + \varrho_n}\sqrt{\sigma_n^2 - 2\lambda_n\kappa_n + \lambda_n^2\omega^2}.$$

Substituting (4.9) into (4.8) and considering the fact that ϕ is an increasing function, yields

$$\begin{aligned}\|x_{n+1} - x^*\| \leq{} & (1-\alpha_n)\|x_n - x^*\| + \alpha_n\big(\theta_n\|x_n - x^*\| \\ & + \frac{\tau_n|\lambda_n - \lambda|}{\lambda_n\gamma_n + \varrho_n}\|P(x^*)\| + \|\Gamma_n\|\big) \\ & + \nu_n\phi(\theta_n\|x_n - x^*\| + \frac{\tau_n|\lambda_n - \lambda|}{\lambda_n\gamma_n + \varrho_n}\|P(x^*)\| + \|\Gamma_n\|) + \mu_n\big) \\ ={} & (1 - \alpha_n(1-\theta_n))\|x_n - x^*\| + \alpha_n\big(\frac{\tau_n|\lambda_n - \lambda|}{\lambda_n\gamma_n + \varrho_n}\|P(x^*)\| + \|\Gamma_n\| \\ & + \nu_n\phi(\theta_n\|x_n - x^*\| + \frac{\tau_n|\lambda_n - \lambda|}{\lambda_n\gamma_n + \varrho_n}\|P(x^*)\| + \|\Gamma_n\|) + \mu_n\big).\end{aligned} \tag{4.10}$$

Taking into account of the facts that $\tau_n \to \tau$, $\lambda_n \to \lambda$, $\gamma_n \to \gamma$, $\varrho_n \to \varrho$, $\sigma_n \to \sigma$ and $\kappa_n \to \kappa$, as $n \to \infty$, we deduce that $\theta_n \to \theta$, as $n \to \infty$, where θ is the same as in (4.6). Since $\theta \in (0,1)$ there exist $\hat{\theta} \in (0,1)$ (take $\hat{\theta} = \frac{\theta+1}{2} \in (\theta, 1)$) and $n_0 \in \mathbb{N}$ such that $\theta_n \leq \hat{\theta}$ for all $n \geq n_0$. Consequently, for all $n \geq n_0$, we have $1 - \alpha_n(1-\theta_n) \leq 1 - \alpha_n(1-\hat{\theta})$. Then, by (4.10), for all $n \geq n_0$, it follows that

$$\begin{aligned}\|x_{n+1} - x^*\| \leq{} & (1 - \alpha_n(1-\hat{\theta}))\|x_n - x^*\| \\ & + \alpha_n(1-\hat{\theta})\frac{\Psi_n + \nu_n\phi(\hat{\theta}\|x_n - x^*\| + \Psi_n) + \mu_n}{1-\hat{\theta}},\end{aligned}$$

where $\Psi_n = \frac{\tau_n|\lambda_n-\lambda|}{\lambda_n\gamma_n+\varrho_n}\|P(x^*)\| + \|\Gamma_n\|$. Let us now take for each $n \geq n_0$, $a_n = \|x_n - x^*\|$, $b_n = \alpha_n(1-\hat{\theta})$ and

$$c_n = \frac{\frac{\tau_n|\lambda_n-\lambda|}{\lambda_n\gamma_n+\varrho_n}\|P(x^*)\| + \|\Gamma_n\| + \nu_n\phi(\hat{\theta}\|x_n - x^*\| + \frac{\tau_n|\lambda_n-\lambda|}{\lambda_n\gamma_n+\varrho_n}\|P(x^*)\| + \|\Gamma_n\|) + \mu_n}{1-\hat{\theta}}.$$

It is obvious that $\sum\limits_{n=0}^{\infty} b_n = \infty$, because of $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$. Thanks to the fact that $M_n \xrightarrow{G} M$, in the light of Theorem 4.2, we deduce that $R^{H_n,\eta_n}_{M_n,\lambda_n}[H(x^*) - \lambda P(x^*)] \to R^{H,\eta}_{M,\lambda}[H(x^*) - \lambda P(x^*)]$, as $n \to \infty$ and so $\lim\limits_{n\to\infty} \Gamma_n = 0$. Since $\lambda_n \to \lambda$, $\nu_n \to 0$ and $\mu_n \to 0$, as $n \to \infty$, we conclude that $c_n \to 0$, as $n \to \infty$. Owing to the fact that all the conditions of Lemma 4.3 are satisfied, Lemma 4.3 guarantees that $\lim\limits_{n\to\infty} a_n = 0$, and so $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

As an immediate consequence of the above theorem we obtain the following assertion.

Corollary 4.5. *Let $P, \eta, \eta_n, H, H_n, M, M_n$ $(n \geq 0)$ be the same as in Theorem 4.4 and let all the conditions of Theorem 4.4 hold. If there exist real constants $\lambda_n > 0$ $(n \geq 0)$ satisfying* (3.13) *and a real constant $\lambda > 0$ such that $\lambda_n \to \lambda$ as $n \to \infty$, and*

(4.2) *holds, then the iterative sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by Algorithm 3.8 converges strongly to the unique solution of the* VIP (3.11).

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