Fixed Point Theory, 24(2023), No. 1, 49-78
DOI: 10.24193/fpt-ro.2023.1.03
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ITERATIVE ALGORITHMS FOR VARIATIONAL INCLUSIONS IN BANACH SPACES

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Abstract. The present paper is in two folds. In the first fold, we prove the Lipschitz continuity of the proximal mapping associated with a general strongly H-monotone mapping and compute an estimate of its Lipschitz constant under some mild assumptions imposed on the mapping H involved in the proximal mapping. We provide two examples to show that a maximal monotone mapping need not be a general H-monotone for a single-valued mapping H from a Banach space to its dual space. A class of multi-valued nonlinear variational inclusion problems is considered, and by using the notion of proximal mapping and Nadler's technique, an iterative algorithm with mixed errors is suggested to compute its solutions. Under some appropriate hypotheses imposed on the mappings and parameters involved in the multi-valued nonlinear variational inclusion problem, the strong convergence of the sequences generated by the proposed algorithm to a solution of the aforesaid problem is verified. The second fold of this paper investigates and analyzes the notion of C_n-monotone mappings defined and studied in [S.Z. Nazemi, A new class of monotone mappings and a new class of variational inclusions in Banach spaces, J. Optim. Theory Appl. 155(3)(2012) 785–795]. Several comments related to the results and algorithm appeared in the above mentioned paper are given.
Key Words and Phrases: Variational inclusion problems, general H-monotone operators, proximal mapping, iterative algorithm, C_n-monotone mapping, convergence analysis.
2010 Mathematics Subject Classification: 47H05, 47J22, 47J25, 49J40, 90C33.

1. Introduction

It is well established that the variational inclusion $0 \in \partial f(x)$ provides the necessary and sufficient condition for a solution of an optimization problem where the objective function f is convex but not necessarily differentiable, see, for example, [5]. In the last two decades, several solution methods for computing the approximate solutions of variational inclusions are studied. The proximal point method [21] is one them which is a generalization of projection method. It has been widely used to study the existence of solution and to develop iterative algorithms for variational inclusions, see, for example, [**?**, 7, 11, 12, 14, 15, 17, 19, **?**] and the references therein. Fang and Huang [12, 11] introduced the notions of H-monotone operators and H-accretive mappings, and by defining the resolvent operators associated with these notions, they studied a class of variational inclusions in the framework of Hilbert / Banach spaces. Subsequently, Xia and Huang [24] introduced the concept of general H-monotone operators as an extension of J-proximal mapping [10] and H-monotone operator [11], and defined proximal mapping associated with general H-monotone operator, which is different from the resolvent operator associated with the H-accretive mapping [12]. At the same time, by using the proximal mapping, they proposed an iterative algorithm for solving a class of variational inclusions involving general H-monotone operators. Luo and Huang [17] introduced the concept of B-monotone operators as a generalization of general H-monotone mapping and by using the notion of the proximal mapping, they constructed an iterative algorithm for solving a class of variational inclusions involving the B-monotone operators in the setting of Banach spaces. Recently, Nazemi [19] introduced and studied the notion of C_n-monotone mappings as a generalization of general H-monotone and B-monotone operators. She considered a class of variational inclusions involving the C_n-monotone mappings in the setting of Banach spaces and suggested an iterative algorithm for solving this class of variational inclusions by using the technique of proximal mapping. She also discussed the convergence of sequences generated by the proposed iterative algorithm under some suitable conditions.

The main motivation of this paper is to discuss general H-monotone multi-valued operators and their properties, and consider a general class of variational inclusions and to develop a new iterative algorithm for computing their solutions. We also investigate and analyze the notion of C_n-monotone mappings defined and studied in [19], and give several comments related to the results and algorithm appeared in [19].

The rest of this paper is organized as follows. In Section 2, we recall some preliminary notions, basic definitions and their properties which will be used in the sequel. We also provide two examples to show that a maximal monotone mapping need not be a general H-monotone for a single-valued mapping H from a Banach space to its dual space. In Section 3, we consider a class of multi-valued nonlinear variational inclusion problems (in short, MNVIP) and by using the notion of proximal mapping and Nadler's technique, we suggest an iterative algorithm with mixed errors for this class of inclusion problems. Imposing some appropriate assumptions on the mappings and parameters involved in the MNVIP, we prove the strong convergence of the sequences generated by the proposed iterative algorithm to a solution of the MNVIP. Finally, in Section 4, we investigate and analyze the concept of C_n-monotone mapping introduced and studied by Nazemi [19]. We point out that under the assumptions imposed in [19] on the C_n-monotone mapping, every C_n-monotone mapping is actually a general H-monotone mapping. We also derive the results related to the C_n-monotone mappings [19] as a special case of our main results presented in Section 3.

2. Preliminary materials and basic results

Throughout the paper, unless otherwise specified, we use the following notations, terminology and assumptions. Let E be a real Banach space whose topological dual is denoted by E^*. The pairing between E and E^* is designated by $\langle .,. \rangle$, and for the sake of simplicity, the norms of E and E^* are denoted by the symbol $\|\cdot\|$. We further denote the family of all nonempty closed and bounded subsets of E by $CB(E)$.

For a given multi-valued mapping $\widehat{M} : E \rightrightarrows E^*$,

(a) the set $\text{Range}(\widehat{M}) = \bigcup_{u\in E} \widehat{M}(u)$ is the range of $\widehat{M}$;

(b) the set $\text{Gph}(\widehat{M}) := \{(u,v) \in E \times E^* : v \in \widehat{M}(u)\}$ is the graph of $\widehat{M}$.

We denote by B_E and S_E, respectively, the unit ball and the unit sphere in E.

Recall that a normed space E is called strictly convex if S_E is strictly convex, that is, the inequality $\|x + y\| < 2$ holds for all distinct unit vectors x and y in E. It is said to be smooth if for every vector x in B_E, there exists a unique vector $x^* \in E^*$ such that $\|x^*\| = x^*(x) = 1$. It is known that E is smooth if E^* is strictly convex, and that E is strictly convex if E^* is smooth.

Definition 2.1. A normed space E is said to be uniformly convex if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in B_E$ with $\|x - y\| \geq \varepsilon$ the inequality $\|x + y\| \leq 2(1 - \delta)$ holds.

The modulus of convexity of E is a function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in B_E, \|x - y\| \geq \varepsilon\right\}.$$

It is known (see, e.g. [8]) that this modulus function can be defined equivalently as

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_E, \|x - y\| = \varepsilon\right\}.$$

The function δ_E is continuous and increasing on the interval $[0, 2]$ and $\delta_E(0) = 0$. Clearly, in the light of the definition of the function δ_E, a normed space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$.

For any Banach space E, its modulus of convexity is bounded from above by the modulus of convexity of a Hilbert space $\mathcal{H}$, that is, $\delta_E(\varepsilon) \leq \delta_{\mathcal{H}}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$. This means that Hilbert spaces are the most convex among all Banach spaces.

Definition 2.2. A normed space E is called uniformly smooth if for any given $\varepsilon > 0$, there exists $\tau > 0$ such that for all $x \in B_E$ and $y \in \tau B_E$, the following inequality holds $\|x + y\| + \|x - y\| \leq 2 + \varepsilon\|y\|$.

The function $\rho_E : [0, +\infty) \to [0, +\infty)$ defined by

$$\begin{aligned}\rho_E(\tau) &= \sup\left\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : x, y \in B_E\right\}\\ &= \sup\left\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : x, y \in S_E\right\}\end{aligned}$$

is called the modulus of smoothness of E.

Note that the function ρ_E is convex, continuous and increasing on the interval $[0,+\infty)$ and $\rho_E(0)=0$. In addition, $\rho_E(\tau)\leq\tau$ for all $\tau\geq 0$. By the definition of the function ρ_E, a normed space E is uniformly smooth if and only if $\lim\limits_{\tau\to 0}\frac{\rho_E(\tau)}{\tau}=0$.

Any uniformly convex and any uniformly smooth Banach space is reflexive. A Banach space E is uniformly convex (respectively, uniformly smooth) if and only if E^* is uniformly smooth (respectively, uniformly convex). The spaces l^p, L^p and W_m^p, $1<p<\infty$, $m\in\mathbb{N}$, are uniformly convex as well as uniformly smooth, see, for example, [9, 13, 16].
Furthermore, more details on the modulus of convexity and smoothness of a Hilbert space and the spaces l^p, L^p and W_m^p, $1<p<\infty$, $m\in\mathbb{N}$ can be found in [9, 13, 16].

Definition 2.3. A mapping $A:E\to E^*$ is said to be

(a) monotone if $\langle A(x)-A(y),x-y\rangle\geq 0,\quad\forall x,y\in E$;
(b) strictly monotone if A is monotone and equality holds if and only if $x=y$;
(c) k-strongly monotone if there exists a constant $k>0$ such that
$$\langle A(x)-A(y),x-y\rangle\geq k\|x-y\|^2,\quad\forall x,y\in E;$$
(d) γ-Lipschitz continuous if there exists a constant $\gamma>0$ such that
$$\|A(x)-A(y)\|\leq\gamma\|x-y\|,\quad\forall x,y\in E.$$

Definition 2.4. A multi-valued mapping $\widehat{M}:E\rightrightarrows E^*$ is said to be

(a) monotone if
$$\langle u-v,x-y\rangle\geq 0,\quad\forall(x,u),(y,v)\in\mathrm{Gph}(\widehat{M});$$
(b) r-strongly monotone if there exists a constant $r>0$ such that
$$\langle u-v,x-y\rangle\geq r\|x-y\|^2,\quad\forall(x,u),(y,v)\in\mathrm{Gph}(\widehat{M});$$
(c) maximal monotone if $\widehat{M}$ is monotone and $(J+\lambda\widehat{M})(E)=E^*$ for every $\lambda>0$, where $J:E\rightrightarrows E^*$ is the normalized duality mapping defined by
$$J(x):=\{x^*\in E^*:\langle x^*,x\rangle=\|x^*\|\|x\|,\|x^*\|=\|x\|\},\quad\forall x\in E.$$

We observe immediately that if $E=\mathcal{H}$, a Hilbert space, then J is the identity mapping on $\mathcal{H}$. In the meanwhile, it is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x\in E$.

We note that $\widehat{M}$ is a maximal monotone multi-valued mapping if and only if $\widehat{M}$ is monotone and there is no other monotone multi-valued mapping whose graph contains strictly $\mathrm{Gph}(\widehat{M})$. The maximality is to be understood in terms of bifunctions of graphs. If $\widehat{M}:E\rightrightarrows E^*$ is a maximal monotone multi-valued mapping, then adding anything to its graph so as to obtain the graph of a new multi-valued mapping, destroys the monotonicity. In fact, the extended mapping is no longer monotone. In other words, for every pair $(x,x^*)\in E\times E^*\setminus\mathrm{Gph}(\widehat{M})$, there exists $(y,y^*)\in\mathrm{Gph}(\widehat{M})$ such that $\langle x^*-y^*,x-y\rangle<0$. In the light of the above-mentioned arguments, a necessary and sufficient condition for a multi-valued mapping $\widehat{M}:E\rightrightarrows E^*$ to be maximal monotone if that the property $\langle x^*-y^*,x-y\rangle\geq 0,\ \forall(y,y^*)\in\mathrm{Gph}(\widehat{M})$ is equivalent to $x^*\in\widehat{M}(x)$.

Fang and Huang [11] introduced the concept of an H-monotone operator which is defined as follows.

Definition 2.5. [11] Let $\mathcal{H}$ be a Hilbert space and $H : \mathcal{H} \to \mathcal{H}$ be a single-valued operator. A multi-valued operator $\widehat{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ is said to be H-monotone if $\widehat{M}$ is monotone and $\mathrm{Range}(H + \lambda\widehat{M}) = \mathcal{H}$, for all $\lambda > 0$.

We remark that if $H \equiv I$, the identity mapping on $\mathcal{H}$, then the definition of I-monotone operators is that of maximal monotone operators.

Subsequently, Xia and Huang [24], by replacing the Hilbert space $\mathcal{H}$ by a Banach space, introduced a more general class of monotone multi-valued operators, so-called general H-monotone multi-valued operators, which is an extension of the class of H-monotone multi-valued operators.

Definition 2.6. [24] Let E be a Banach space and $H : E \to E^*$ be a single-valued operator. A multi-valued operator $\widehat{M} : E \rightrightarrows E^*$ is said to be general H-monotone if $\widehat{M}$ is monotone and $\mathrm{Range}(H + \lambda\widehat{M}) = E^*$, for all $\lambda > 0$.

It is significant to emphasize that if $E = \mathcal{H}$ is a Hilbert space, then the class of general H-monotone operators coincides exactly with the class of H-monotone operators. The following example shows that for a given single-valued mapping $H : E \to E^*$, a maximal monotone mapping need not be general H-monotone.

Example 2.7. Let $\phi : \mathbb{Z} \to (0, +\infty)$ and consider the complex linear space $l^2_\phi(\mathbb{Z})$, the weighted $l^2(\mathbb{Z})$ space, consisting of all bi-infinite complex sequences $\{z_n\}_{n=-\infty}^{\infty}$ such that $\sum\limits_{n=-\infty}^{\infty} |z_n|^2\phi(n) < \infty$. It is a well known that

$$l^2_\phi(\mathbb{Z}) = \left\{ z = \{z_n\}_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} |z_n|^2\phi(n) < \infty, z_n \in \mathbb{C} \right\}$$

with respect to the inner product $\langle .,. \rangle : l^2_\phi(\mathbb{Z}) \times l^2_\phi(\mathbb{Z}) \to \mathbb{C}$ defined by

$$\langle z, w \rangle = \sum_{n=-\infty}^{\infty} z_n \overline{w_n} \phi(n), \quad \forall z = \{z_n\}_{n=-\infty}^{\infty},\ w = \{w_n\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z}),$$

is a Hilbert space. The above inner product induces a norm as

$$\|z\|_{l^2_\phi(\mathbb{Z})} = \sqrt{\langle z, z \rangle} = \left(\sum_{n=-\infty}^{\infty} |z_n|^2\phi(n) \right)^{\frac{1}{2}}, \quad \forall z = \{z_n\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z}).$$

An arbitrary element $z = \{z_n\}_{n=-\infty}^{\infty} = \{x_n + iy_n\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z})$ can be written as

$$z = \sum_{t=-\infty}^{\infty} (\dots, 0, \dots, 0, x_{tq+1} + iy_{tq+1}, x_{tq+2} + iy_{tq+2}, \dots, x_{(t+1)q} + iy_{(t+1)q}, 0, \dots),$$

where $q \geq 2$ is an arbitrary but fixed natural number. For each $t \in \mathbb{Z}$, we obtain

$$\begin{aligned}&(\dots, 0, 0, \dots, 0, x_{tq+1} + iy_{tq+1}, x_{tq+2} + iy_{tq+2}, \dots, x_{(t+1)q} + iy_{(t+1)q}, 0, 0, \dots)\\ &= (\dots, 0, 0, \dots, 0, x_{tq+1} + iy_{tq+1}, 0, 0, \dots, 0, x_{(t+1)q} + iy_{(t+1)q}, 0, 0, \dots)\end{aligned}$$

$$+(\dots,0,0,\dots,0,x_{tq+2}+iy_{tq+2},0,0,\dots,0,x_{(t+1)q-1}+iy_{(t+1)q-1},0,0,\dots)$$
$$+\cdots+(\dots,0,0,\dots,0,x_{\frac{(2t+1)q}{2}}+iy_{\frac{(2t+1)q}{2}},x_{\frac{(2t+1)q}{2}+1}+iy_{\frac{(2t+1)q}{2}+1},0,0,\dots)$$
$$=\sum_{j=tq+1}^{\frac{(2t+1)q}{2}}(\dots,0,0,\dots,0,x_j+iy_j,0,0,\dots,0,x_{(2t+1)q-j+1}+iy_{(2t+1)q-j+1},0,0,\dots).$$

Therefore, for any $z=\{z_n\}_{n=-\infty}^{\infty}=\{x_n+iy_n\}_{n=-\infty}^{\infty}\in l^2_\phi(\mathbb{Z})$, we have

$$\begin{aligned}z&=\sum_{t=-\infty}^{\infty}(\dots,0,\dots,0,x_{tq+1}+iy_{tq+1},x_{tq+2}+iy_{tq+2},\dots,x_{(t+1)q}+iy_{(t+1)q},0,\dots)\\
&=\sum_{t=-\infty}^{\infty}\sum_{j=tq+1}^{\frac{(2t+1)q}{2}}(\dots,0,\dots,0,x_j+iy_j,0,\dots,0,x_{(2t+1)q-j+1}+iy_{(2t+1)q-j+1},0,\dots)\\
&=\sum_{t=-\infty}^{\infty}\sum_{j=tq+1}^{\frac{(2t+1)q}{2}}\Big[\frac{y_j+y_{(2t+1)q-j+1}-i(x_j+x_{(2t+1)q-j+1})}{2}\sigma_{j,(2t+1)q-j+1}\\
&\quad+\frac{y_j-y_{(2t+1)q-j+1}-i(x_j-x_{(2t+1)q-j+1})}{2}\sigma'_{j,(2t+1)q-j+1}\Big],\end{aligned}$$

where for each $t\in\mathbb{Z}$ and $j\in\Big\{tq+1,tq+2,\dots,\frac{(2t+1)q}{2}\Big\}$,

$$\sigma_{j,(2t+1)q-j+1}=(\dots,0,0,\dots,0,i_j,0,0,\dots,0,i_{(2t+1)q-j+1},0,0,\dots),$$

i at the jth and $((2t+1)q-j+1)$th coordinates and all other coordinates are zero, and

$$\sigma'_{j,(2t+1)q-j+1}=(\dots,0,0,\dots,0,i_j,0,0,\dots,0,-i_{(2t+1)q-j+1},0,0,\dots),$$

i and $-i$ at the jth and $((2t+1)q-j+1)$th coordinates, respectively, and all other coordinates are zero. Thus, the set

$$\mathfrak{B}=\Big\{\sigma_{j,(2t+1)q-j+1},\sigma'_{j,(2t+1)q-j+1}:t\in\mathbb{Z};j=tq+1,tq+2,\dots,\frac{(2t+1)q}{2}\Big\}$$

spans the Banach space $l^2_\phi(\mathbb{Z})$. It can be easily seen that the set $\mathfrak{B}$ is linearly independent, and so it is a basis for the Hilbert space $l^2_\phi(\mathbb{Z})$. For each $t\in\mathbb{Z}$ and $j\in\Big\{tq+1,tq+2,\dots,\frac{(2t+1)q}{2}\Big\}$, let us take

$$\omega_{j,(2t+1)q-j+1}=\left(\dots,0,\frac{1}{\sqrt{2\phi(j)}}i_j,0,\dots,0,\frac{1}{\sqrt{2\phi((2t+1)q-j+1)}}i_{(2t+1)q-j+1},0,\dots\right),$$

and

$$\omega'_{j,(2t+1)q-j+1}=\left(\dots,0,\frac{1}{\sqrt{2\phi(j)}}i_j,0,\dots,0,-\frac{1}{\sqrt{2\phi((2t+1)q-j+1)}}i_{(2t+1)q-j+1},0,\dots\right).$$

Obviously, the set

$$\Big\{\omega_{j,(2t+1)q-j+1},\omega'_{j,(2t+1)q-j+1}:t\in\mathbb{Z};j=tq+1,tq+2,\dots,\frac{(2t+1)q}{2}\Big\}$$

is linearly independent and orthonormal. Let the mappings $\widehat{M}, H : l^2_\phi(\mathbb{Z}) \to l^2_\phi(\mathbb{Z})$ be defined, respectively, by

$$\widehat{M}(z) = \alpha z + \beta\omega_{s,(2k+1)q-s+1} + \gamma\omega'_{s,(2k+1)q-s+1} \quad \text{and} \quad H(z) = -\alpha z,$$

for all $z = \{z_n\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z})$, where $\alpha > 0$ is an arbitrary real constant, β and γ are two arbitrary nonzero real constants, and $k \in \mathbb{Z}$ and $s \in \left\{kq+1, kq+2, \dots, \frac{(2k+1)q}{2}\right\}$ are two arbitrary but fixed integers. Then, for all $z, w \in l^2_\phi(\mathbb{Z})$, we have

$$\begin{aligned}\langle \widehat{M}(z) - \widehat{M}(w), z - w\rangle &= \langle \alpha z + \beta\omega_{s,(2k+1)q-s+1} + \gamma\omega'_{s,(2k+1)q-s+1} \\ &\qquad - \alpha w - \beta\omega_{s,(2k+1)q-s+1} - \gamma\omega'_{s,(2k+1)q-s+1}, z - w\rangle \\ &= \alpha\langle z - w, z - w\rangle = \alpha\|z - w\|^2 \\ &= \alpha\left(\sum_{n=-\infty}^{\infty} |z_n - w_n|^2\phi(n)\right)^{\frac{1}{2}} \geq 0,\end{aligned}$$

which means that $\widehat{M}$ is monotone. Relying on the fact that for any $z \in l^2_\phi(\mathbb{Z})$, we have

$$\begin{aligned}&(H + \widehat{M})(z) = \beta\omega_{s,(2k+1)q-s+1} + \gamma\omega'_{s,(2k+1)q-s+1} \\ &= \left(\dots, 0, \frac{\beta}{\sqrt{2\phi(s)}}i_s, 0, \dots, 0, \frac{\beta}{\sqrt{2\phi((2k+1)q-s+1)}}i_{(2k+1)q-s+1}, 0, \dots\right) \\ &\quad + \left(\dots, 0, \dots, 0, \frac{\gamma}{\sqrt{2\phi(s)}}i_s, 0, \dots, 0, -\frac{\gamma}{\sqrt{2\phi((2k+1)q-s+1)}}i_{(2k+1)q-s+1}, 0, \dots\right) \\ &= \left(\dots, 0, \dots, 0, \frac{\beta+\gamma}{\sqrt{2\phi(s)}}i_s, 0, \dots, 0, \frac{\beta-\gamma}{\sqrt{2\phi((2k+1)q-s+1)}}i_{(2k+1)q-s+1}, 0, \dots\right),\end{aligned}$$

it follows that for any $z \in l^2_\phi(\mathbb{Z})$,

$$\|(H + \widehat{M})(z)\|^2_{l^2_\phi(\mathbb{Z})} = \frac{(\beta+\gamma)^2}{2} + \frac{(\beta-\gamma)^2}{2} = \beta^2 + \gamma^2 > 0.$$

This fact guarantees that $0 \notin (H + \widehat{M})(l^2_\phi(\mathbb{Z}))$, i.e., $H + \widehat{M}$ is not surjective and so the operator $\widehat{M}$ is not general H-monotone.

Let $\lambda > 0$ be an arbitrary real constant. Since for any $z = \{z_n\}_{n=-\infty}^{\infty} \in l^2_\phi(\mathbb{Z})$,

$$(J + \lambda\widehat{M})(z) = (I + \lambda\widehat{M})(z) = (1 + \alpha\lambda)z + \lambda\beta\omega_{s,(2k+1)q-s+1} + \lambda\gamma\omega'_{s,(2k+1)q-s+1},$$

where J is the identity mapping on $l^2_\phi(\mathbb{Z})$, we deduce that $(J + \lambda\widehat{M})(l^2_\phi(\mathbb{Z})) = l^2_\phi(\mathbb{Z})$, that is, the mapping $J + \lambda\widehat{M}$ is surjective. Since $\lambda > 0$ was arbitrary, it follows that $\widehat{M}$ is a maximal monotone multi-valued operator.

Denoting the set of all functions $\phi : \mathbb{Z} \to (0, 1]$ by Φ and $l^2_\Phi(\mathbb{Z}) = \{l^2_\phi(\mathbb{Z}) : \phi \in \Phi\}$, it can be easily seen that $l^2(\mathbb{Z}) \subseteq l^2_\phi(\mathbb{Z})$ for each $\phi \in \Phi$ so that for some $\phi_0 \in \Phi$, we have $l^2(\mathbb{Z}) \subset l^2_{\phi_0}(\mathbb{Z})$, that is, $l^2(\mathbb{Z})$ is strictly contained within $l^2_{\phi_0}(\mathbb{Z})$. Recall that

$$l^2(\mathbb{Z}) = \left\{ x = \{x_n\}_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} |x_n|^2 < \infty, x_n \in \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C} \right\}$$

denotes the real or complex linear space consisting of all bi-infinite real or complex sequences $x = \{x_n\}_{n=-\infty}^{\infty}$, for which $\|x\|_{l^2(\mathbb{Z})} < \infty$. Evidently, if $\phi(n) = 1$ for all $n \in \mathbb{Z}$, then the weight space $l^2_\phi(\mathbb{Z})$ coincides exactly with the linear space $l^2(\mathbb{Z})$. It is to be noted that the two Hilbert spaces $l^2(\mathbb{Z})$ and $l^2_\phi(\mathbb{Z})$ need not be the same for all $\phi \in \Phi$. To show this claim, we consider the two cases when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If $\mathbb{F} = \mathbb{R}$, letting $x_n = e^{\frac{n^2}{2}} \sqrt{\alpha n^p + m}$ for all $n \in \mathbb{Z}$, where $\alpha > 1$ is an arbitrary real constant, m is an arbitrary positive real constant, and p is an arbitrary but fixed even natural number. Then, we have

$$\sum_{n=-\infty}^{\infty} |x_n|^2 = \sum_{n=-\infty}^{\infty} e^{n^2}(\alpha n^p + m) = \infty,$$

that is, $x = \{x_n\}_{n=-\infty}^{\infty} \notin l^2(\mathbb{Z})$. Let $\phi_1 : \mathbb{Z} \to (0, +\infty)$ be a function defined as $\phi_1(n) = \frac{Ln(n^k+l)}{e^{n^2}(\alpha n^p+m)^2}$, for all $n \in \mathbb{Z}$, where k is an arbitrary but fixed even natural number and l is an arbitrary positive real constant such that $k < p$ and $l < m$. In view of the fact that $Ln(n^k + l) < n^k + l < \alpha n^p + m$, for all $n \in \mathbb{Z}$, it follows that $\phi_1 \in \Phi$ and we have

$$\sum_{n=-\infty}^{\infty} |x_n|^2 \phi_1(n) = \sum_{n=-\infty}^{\infty} \frac{Ln(n^k + l)}{\alpha n^p + m} = 2 \sum_{n=1}^{\infty} \frac{Ln(n^k + l)}{\alpha n^p + m}.$$

Taking into consideration the facts that $\frac{Ln(n^k+l)}{\alpha n^p+m} \leq \frac{n^k+l}{n^p}$ for all $n \in \mathbb{Z}$, and $\lim_{n\to\infty} n^{p-k}\left(\frac{n^k+l}{n^p}\right) = 1$, we conclude that $\sum_{n=1}^{\infty} \frac{Ln(n^k+l)}{\alpha n^p+m}$ is convergent, which implies that $x \in l^2_{\phi_1}(\mathbb{Z})$. For the case when $\mathbb{F} = \mathbb{C}$, taking $z_n = \sqrt{\frac{n^k+\sqrt{2}}{2}} + i\sqrt{\frac{n^k+\sqrt{2}}{2}}$, for all $n \in \mathbb{Z}$, where k is an arbitrary but fixed even natural number, we have

$$\sum_{n=-\infty}^{\infty} |z_n|^2 = \sum_{n=-\infty}^{\infty} (n^k + \sqrt{2}) = \infty,$$

that is, $z = \{z_n\}_{n=-\infty}^{\infty} \notin l^2(\mathbb{Z})$. Let the function $\phi_2 : \mathbb{Z} \to (0, +\infty)$ be defined by $\phi_2(n) = \frac{1+cosn}{(n^k+\sqrt{2})^2}$ for all $n \in \mathbb{Z}$. Then, we have $\phi_2 \in \Phi$ and

$$\sum_{n=-\infty}^{\infty} |z_n|^2 \phi_2(n) = \sum_{n=-\infty}^{\infty} \frac{1 + \cos n}{n^k + \sqrt{2}} = 2\left(\frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \frac{1 + \cos n}{n^k + \sqrt{2}}\right).$$

Since $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^k+\sqrt{2}}$ is absolutely convergent, it follows that $z = \{z_n\}_{n=-\infty}^{\infty} \in l^2_{\phi_2}(\mathbb{Z})$. Consequently, for some $\phi \in \Phi$, $l^2_\phi(\mathbb{Z})$ is a proper superset of the Hilbert space $l^2(\mathbb{Z})$.

Example 2.8. Let $m, n \in \mathbb{N}$ and $M_{m\times n}(\mathbb{F})$ be the space of all $m \times n$ matrices with real or complex entries. Then

$$M_{m\times n}(\mathbb{F}) = \{A = [a_{ij}] : a_{ij} \in \mathbb{F}, i = 1, 2, \dots, m;\ j = 1, 2, \dots, n;\ \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\}$$

is a Hilbert space with respect to the Hilbert-Schmidt norm

$$\|A\| = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}, \quad \forall A \in M_{m\times n}(\mathbb{F})$$

induced by the Hilbert-Schmidt inner product

$$\langle A, B\rangle = tr(A^*B) = \sum_{i=1}^{m}\sum_{j=1}^{n}\bar{a}_{ij}b_{ij}, \quad \forall A, B \in M_{m\times n}(\mathbb{F}),$$

where tr denotes the trace, that is, the sum of the diagonal entries, and A^* denotes the Hermitian conjugate (or adjoint) of the matrix A, that is, $A^* = \overline{A^\top}$, the complex conjugate of the transpose A, and the bar denotes the transpose of the entries. Let us denote by $D_n(\mathbb{R})$ the space of all diagonal $n \times n$ matrices with real entries, that is, the (i, j)-entry is an arbitrary real number if $i = j$, and is zero if $i \neq j$. Then

$$D_n(\mathbb{R}) = \{A = [a_{ij}] : a_{ij} \in \mathbb{R},\ a_{ij} = 0 \text{ if } i \neq j;\ i, j = 1, 2, \dots, n\}$$

is a subspace of $M_{n\times n}(\mathbb{R}) = M_n(\mathbb{R})$ with respect to the operations of addition and scalar multiplication defined on $M_n(\mathbb{R})$, and the Hilbert-Schmidt inner product on $D_n(\mathbb{R})$, and the Hilbert-Schmidt norm induced by it are $\langle A, B\rangle = tr(A^*B) = tr(AB)$ and $\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{tr(AA)} = \left(\sum\limits_{i=1}^{n} a_{ii}^2\right)^{\frac{1}{2}}$, respectively.

Let the mappings $H_1, H_2, \widehat{M} : D_n(\mathbb{R}) \to D_n(\mathbb{R})$ be defined, respectively, by

$$H_1(A) = H_1([a_{ij}]) = [a'_{ij}], \quad H_2(A) = H_2([a_{ij}]) = [a''_{ij}] \text{ and } \widehat{M}(A) = \widehat{M}([a_{ij}]) = [a'''_{ij}],$$

for all $A = [a_{ij}] \in D_n(\mathbb{R})$, where for each $i, j \in \{1, 2, \dots, n\}$,

$$a'_{ij} = \begin{cases} |\sin a_{ii} - \sqrt{3}\cos a_{ii}| - \beta a_{ii}^k, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$a''_{ij} = \begin{cases} \gamma a_{ii}^l, & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$a'''_{ij} = \begin{cases} \beta a_{ii}^k, & i = j, \\ 0, & i \neq j, \end{cases}$$

where β is an arbitrary positive real constant, γ is an arbitrary real constant, and k, l are arbitrary but fixed odd and even natural numbers, respectively, such that $k > n$.

Then, for any $A = [a_{ij}], B = [b_{ij}] \in D_n(\mathbb{R})$, yields

$$\begin{aligned}\Big\langle \widehat{M}(A) - \widehat{M}(B), A - B \Big\rangle &= tr\big([a'''_{ij} - b'''_{ij}][a_{ij} - b_{ij}]\big) \\ &= \beta \sum_{i=1}^{n} (a^k_{ii} - b^k_{ii})(a_{ii} - b_{ii}) \\ &= \beta \sum_{i=1}^{n} (a_{ii} - b_{ii})^2 \sum_{s=1}^{k} a^{k-s}_{ii} b^{s-1}_{ii}.\end{aligned}$$

Thanks to the fact that k is an odd natural number, it is easy to see that for each $i \in \{1, 2, \dots, n\}$, $\sum\limits_{s=1}^{k} a^{k-s}_{ii} b^{s-1}_{ii} \geq 0$. Since $\beta > 0$, this fact implies that

$$\Big\langle \widehat{M}(A) - \widehat{M}(B), A - B \Big\rangle \geq 0, \quad \forall A, B \in D_n(\mathbb{R}),$$

that is, $\widehat{M}$ is a monotone mapping.

Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := \big|\sin x - \sqrt{3}\cos x\big|$ for all $x \in \mathbb{R}$. Then, for any $A \in D_n(\mathbb{R})$, we have

$$(H_1 + \widehat{M})(A) = (H_1 + \widehat{M})([a_{ij}]) = [a'_{ij} + a'''_{ij}] = [\hat{a}_{ij}],$$

where for each $i, j \in \{1, 2, \dots, n\}$,

$$\hat{a}_{ij} = \begin{cases} \big|\sin a_{ii} - \sqrt{3}\cos a_{ii}\big|, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$= \begin{cases} f(a_{ii}), & i = j, \\ 0, & i \neq j. \end{cases}$$

Owing to the fact that $f(\mathbb{R}) = [0, 2]$, it follows that $(H_1 + \widehat{M})(D_n(\mathbb{R})) \neq D_n(\mathbb{R})$, i.e., $H_1 + \widehat{M}$ is not surjective, and so $\widehat{M}$ is not general H_1-monotone. Let us now assume that $\lambda > 0$ is an arbitrary real constant and let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := \gamma x^l + \lambda\beta x^k$, for all $x \in \mathbb{R}$. Then, for any $A = [a_{ij}] \in D_n(\mathbb{R})$, we obtain

$$(H_2 + \lambda\widehat{M})(A) = (H_2 + \lambda\widehat{M})([a_{ij}]) = [a''_{ij} + \lambda a'''_{ij}] = [\tilde{a}_{ij}],$$

where for each $i, j \in \{1, 2, \dots, n\}$,

$$\tilde{a}_{ij} = \begin{cases} \gamma a^l_{ii} + \lambda\beta a^k_{ii}, & i = j, \\ 0, & i \neq j, \end{cases} = \begin{cases} g(a_{ii}), & i = j, \\ 0, & i \neq j. \end{cases}$$

Taking into consideration the fact that l is an even natural number and k is an odd natural number such that $k > l$, it is easy to observe that $g(\mathbb{R}) = \mathbb{R}$ and so $(H_2 + \lambda\widehat{M})(D_n(\mathbb{R})) = D_n(\mathbb{R})$, which means that $H_2 + \lambda\widehat{M}$ is surjective. Since $\lambda > 0$ was arbitrary, we conclude that $\widehat{M}$ is a general H_2-monotone operator.

Rest of the paper, unless otherwise specified, we assume that E is a reflexive Banach space with the dual space E^*, and $\langle ., . \rangle$ is the dual pair between E and E^*.

Theorem 2.9. *Let $H : E \to E^*$ be a monotone mapping and $\widehat{M} : E \rightrightarrows E^*$ be a γ-strongly monotone mapping. Then, the mapping $(H+\lambda\widehat{M})^{-1} : \mathrm{Range}(H+\lambda\widehat{M}) \to E$ is single-valued for every $\lambda > 0$.*

Proof. Let $\lambda > 0$ be chosen arbitrarily but fixed and take $x, y \in (H + \lambda\widehat{M})^{-1}(z^*)$ for any given $z^* \in \mathrm{Range}(H + \lambda\widehat{M})$. Then, we have

$$z^* = (H + \lambda\widehat{M})(x) = (H + \lambda\widehat{M})(y),$$

which implies that $\lambda^{-1}(z^* - H(x)) \in \widehat{M}(x)$ and $\lambda^{-1}(z^* - H(y)) \in \widehat{M}(y)$. Since H is monotone and $\widehat{M}$ is γ-strongly monotone, it follows that

$$\begin{aligned}\lambda\gamma\|x - y\|^2 &\leq \lambda\langle\lambda^{-1}(z^* - H(x)) - \lambda^{-1}(z^* - H(y)), x - y\rangle \\ &\quad + \langle H(x) - H(y), x - y\rangle = 0.\end{aligned}$$

Since $\lambda, \gamma > 0$, from the last inequality we conclude that $x = y$, which ensures that the mapping $H + \lambda\widehat{M}$ from $\mathrm{Range}(H + \lambda\widehat{M})$ into E is single-valued. □

Rest of the paper, we say that the multi-valued mapping $\widehat{M} : E \rightrightarrows E^*$ is a general strongly H-monotone with constant γ if $\widehat{M}$ is a γ-strongly monotone mapping and $\mathrm{Range}(H + \lambda\widehat{M}) = E^*$ for all $\lambda > 0$.

As a consequence of the last result, one can deduce immediately that the mapping $(H + \lambda\widehat{M})^{-1} : E^* \to E$ is single-valued.

Corollary 2.10. *Suppose that $H : E \to E^*$ is a monotone mapping and $\widehat{M} : E \rightrightarrows E^*$ is a general strongly H-monotone mapping with constant γ. Then, the mapping $(H + \lambda\widehat{M})^{-1} : E^* \to E$ is single-valued for all $\lambda > 0$.*

Based on Corollary 2.10, we are able to define the proximal mapping $R^H_{\widehat{M},\lambda}$ associated with the mappings $H, \widehat{M}$ and an arbitrary positive real constant λ as follows.

Definition 2.11. Let $H : E \to E^*$ be a monotone mapping and $\widehat{M} : E \rightrightarrows E^*$ be a general strongly H-monotone mapping with constant γ. For every real constant $\lambda > 0$, the proximal mapping $R^H_{\widehat{M},\lambda} : E^* \to E$ is defined by

$$R^H_{\widehat{M},\lambda}(x^*) = (H + \lambda\widehat{M})^{-1}(x^*), \quad \forall x^* \in E^*.$$

This section is concluded with the following assertion in which the required conditions to prove the Lipschitz continuity of the proximal mapping $R^H_{\widehat{M},\lambda}$ and to compute an estimate of its Lipschitz constant are stated.

Theorem 2.12. *Let $H : E \to E^*$ be a monotone mapping and $\widehat{M} : E \rightrightarrows E^*$ be a general strongly H-monotone mapping with constant γ. Then, for any real constant $\lambda > 0$, the proximal mapping $R^H_{\widehat{M},\lambda} : E^* \to E$ is $\frac{1}{\lambda\gamma}$-Lipschitz continuous.*

Proof. Taking into account that $\widehat{M}$ is a general H-monotone mapping, for any given $x^*, y^* \in E^*$ with $\left\|R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\| \neq 0$, we have

$$R^H_{\widehat{M},\lambda}(x^*) = (H + \lambda\widehat{M})^{-1}(x^*) \text{ and } R^H_{\widehat{M},\lambda}(y^*) = (H + \lambda\widehat{M})^{-1}(y^*),$$

whence we conclude that

$$\lambda^{-1}\left(x^* - H\left(R^H_{\widehat{M},\lambda}(x^*)\right)\right) \in \widehat{M}\left(R^H_{\widehat{M},\lambda}(x^*)\right)$$

and

$$\lambda^{-1}\left(y^* - H\left(R^H_{\widehat{M},\lambda}(y^*)\right)\right) \in \widehat{M}\left(R^H_{\widehat{M},\lambda}(y^*)\right).$$

Since $\widehat{M}$ is γ-strongly monotone, we yield

$$\begin{aligned}\lambda^{-1}\left\langle x^* - H\left(R^H_{\widehat{M},\lambda}(x^*)\right) - \left(y^* - H\left(R^H_{\widehat{M},\lambda}(y^*)\right)\right), R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\rangle& \\ \geq \gamma\left\|R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\|^2&.\end{aligned}$$

By virtue of the fact that $\lambda^{-1} > 0$, the preceding inequality implies that

$$\begin{aligned}&\left\langle x^* - y^*, R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\rangle \\ &\geq \left\langle H\left(R^H_{\widehat{M},\lambda}(x^*)\right) - H\left(R^H_{\widehat{M},\lambda}(y^*)\right), R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\rangle \\ &\quad + \lambda\gamma\left\|R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\|^2.\end{aligned}$$

By using the later inequality and monotonicity of H, it follows that

$$\begin{aligned}&\|x^* - y^*\|\left\|R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\| \\ &\quad\geq \left\langle x^* - y^*, R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\rangle \\ &\quad\geq \left\langle H\left(R^H_{\widehat{M},\lambda}(x^*)\right) - H\left(R^H_{\widehat{M},\lambda}(y^*)\right), R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\rangle \\ &\qquad + \lambda\gamma\left\|R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\|^2 \\ &\quad\geq \lambda\gamma\left\|R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\|^2.\end{aligned}$$

Thanks to the fact that $\left\|R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\| \neq 0$, dividing both the sides of the last inequality by $\left\|R^H_{\widehat{M},\lambda}(x^*) - R^H_{\widehat{M},\lambda}(y^*)\right\| \neq 0$, we obtain that the mapping $R^H_{\widehat{M},\lambda}$ is $\frac{1}{\lambda\gamma}$-Lipschitz continuous. The proof is complete. □

Remark 2.13. The Lipschitz continuity of the proximal mapping $R^H_{\widehat{M},\lambda}$ and its Lipschitz constant is proved and computed under the strict monotonicity assumption of the mapping $H : E \to E^*$ in [24]. But, Theorem 2.12 tells us that the same results can be derived without imposing the strict monotonicity condition on the mapping $H : E \to E^*$. Indeed, one can prove the Lipschitz continuity of the proximal mapping $R^H_{\widehat{M},\lambda}$ and approximate its Lipschitz constant under the mild condition of monotonicity of H instead of strict monotonicity. Thereby, Theorem 2.12 improves [24, Theorem 3.2 (ii)].

3. Formulation of the problem, iterative algorithms and convergence analysis

Assume that $p : E \to E$, $A : E \to E^*$ and $F : \underbrace{E \times E \times \cdots \times E}_{k-terms} = E^k \to E^*$ are single-valued mappings. Suppose further that for $i = 1, 2, \dots, k$, $T_i : E \to CB(E)$ and $\widehat{M} : E \rightrightarrows E^*$ are multi-valued mappings. For any given $a \in E^*$, we consider a multi-valued nonlinear variational inclusion problem (for short, MNVIP) as follows: Find $x \in E$ and $(t_1, t_2, \dots, t_k) \in \prod_{i=1}^{k} T_i(x)$ such that

$$a \in A(x - p(x)) + \widehat{M}(x) - F(t_1, t_2, \dots, t_k). \tag{3.1}$$

For suitable and appropriate choices of the mappings $T, A, F, \widehat{M}, p_i$ $(i = 1, 2, \dots, k)$ and the underlying space E, the MNVIP (3.1) reduces to various classes of variational inclusions and variational inequalities, see for example, [6, 11, 15] and the references therein.

We now derive the equivalence between the MNVIP (3.1) and a fixed point problem, which will be used in the sequel.

Theorem 3.1. *Let $E, E^*, A, F, T_i (i = 1, 2, \dots, k), \widehat{M}, p$ and a be the same as in the* MNVIP (3.1). *Suppose further that $H : E \to E^*$ is a monotone mapping and $\widehat{M} : E \rightrightarrows E^*$ is a general strongly H-monotone multi-valued mapping. Then, $x \in E$ and $(t_1, t_2, \dots, t_k) \in \prod_{i=1}^{k} T_i(x)$ are a solution of the* MNVIP (3.1) *if and only if*

$$x = R^H_{\widehat{M},\lambda}[H(x) - \lambda(A(x - p(x)) - a - F(t_1, t_2, \dots, t_k))], \tag{3.2}$$

where $\lambda > 0$ is a real constant and $R^H_{\widehat{M},\lambda}(x^*) = (H + \lambda \widehat{M})^{-1}(x^*)$ for all $x^* \in E^*$.

Proof. The conclusion follows directly from Definition 2.11. □

Lemma 3.2. [18] *Let (E, d) be a complete metric space and $T : E \to CB(E)$ be a multi-valued mapping. Then for any $\epsilon > 0$ and for any given $x, y \in E$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u, v) \le (1 + \epsilon)\hat{H}(T(x), T(y)),$$

where $\hat{H}(.,.)$ denotes the Hausdorff metric on $CB(E)$ defined by

$$\hat{H}(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B} \|x - y\|, \sup_{y\in B}\inf_{x\in A} \|x - y\|\right\}, \quad \forall A, B \in CB(E).$$

By using the fixed point formulation (3.3) and Nadler's technique [18], we are able to construct the folloiwng iterative algorithm for solving the MNVIP (3.1).

Algorithm 3.3. Let $E, E^*, A, F, T_i (i = 1, 2, \dots, k), H, \widehat{M}, p$ and a be the same as in Theorem 3.1. For any given $x^0, t_1^0, t_2^0, \dots, t_k^0 \in E$, compute the iterative sequences

$\{x^n\}_{n=0}^{\infty}, \{t_1^n\}_{n=0}^{\infty}, \{t_2^n\}_{n=0}^{\infty}, \dots, \{t_k^n\}_{n=0}^{\infty}$ in E in the following way:

$$\begin{aligned} x^{n+1} &= (1-\alpha)x^n + \alpha R^H_{M,\lambda}[H(x^n) - \lambda(A(x^n - p_i(x^n)) - a \\ &\quad - F_i(t_1^n, t_2^n, \dots, t_k^n))] + \alpha e^n + r^n, \end{aligned} \tag{3.3}$$

$$t_i^n \in T_i(x^n) : \|t_i^{n+1} - t_i^n\| \le (1 + (1+n)^{-1})\hat{H}(T_i(x^{n+1}), T_i(x^n)), \tag{3.4}$$

where $i = 1, 2, \dots, k$; $n = 0, 1, 2, \dots$; $\lambda > 0$ is a real constant, $\alpha \in (0, 1]$ is a parameter, and $\{e^n\}_{n=0}^{\infty}$ and $\{r^n\}_{n=0}^{\infty}$ are two sequences in E to into account a possible computation of the resolvent operator point satisfying the following conditions:

$$\begin{cases} \lim\limits_{n\to\infty} \|e^n\| = \lim\limits_{n\to\infty} \|r^n\| = 0, \\ \sum\limits_{n=0}^{\infty} \|e^n - e^{n-1}\| < \infty, \sum\limits_{n=0}^{\infty} \|r^n - r^{n-1}\| < \infty. \end{cases} \tag{3.5}$$

If $\alpha = 1$ and $e^n = r^n = 0$ for all $n \ge 0$, then Algorithm 3.3 reduces to the following algorithm.

Algorithm 3.4. Suppose that $E, E^*, A, F, T_i (i = 1, 2, \dots, k), H, \widehat{M}, p$ and a are the same as in Theorem 3.1. For any given $x^0, t_1^0, t_2^0, \dots, t_k^0 \in E$, define the iterative sequences $\{x^n\}_{n=0}^{\infty}, \{t_1^n\}_{n=0}^{\infty}, \{t_2^n\}_{n=0}^{\infty}, \dots, \{t_k^n\}_{n=0}^{\infty}$ in E by the iterative schemes

$$x^{n+1} = R^H_{\widehat{M},\lambda}[H(x^n) - \lambda(A(x^n - p(x^n)) - a - F(t_1^n, t_2^n, \dots, t_k^n))],$$

$$t_i^n \in T_i(x^n) : \|t_i^{n+1} - t_i^n\| \le (1 + (1+n)^{-1})\hat{H}(T_i(x^{n+1}), T_i(x^n)),$$

where $i = 1, 2, \dots, k$; $n = 0, 1, 2, \dots$ and $\lambda > 0$ is a real constant.

Before proceeding to our results, we give the following lemma and definitions which will be used efficiently in the proof of main results.

Definition 3.5. A multi-valued mapping $T : E \rightrightarrows CB(E)$ is said to be $\hat{H}$-Lipschitz continuous with constant ϱ (or ϱ-$\hat{H}$-Lipschitz continuous) if and only if there exists a constant $\varrho > 0$ such that

$$\hat{H}(T(x), T(y)) \le \varrho\|x - y\|, \quad \forall x, y \in E.$$

Definition 3.6. Let J be the normalized duality mapping from E into E^*. A single-valued mapping $g : E \to E$ is said to be (γ, μ)-relaxed cocoercive if there exist two constants $\gamma, \mu > 0$ such that

$$\langle J(x - y), g(x) - g(y)\rangle \ge -\gamma\|g(x) - g(y)\|^2 + \mu\|x - y\|^2, \quad \forall x, y \in E.$$

Definition 3.7. For each $i = 1, 2, \dots, k$, let $T_i : E \rightrightarrows CB(E)$ be a multi-valued mapping. A single-valued mapping $F : E^k \to E^*$ is said to be λ_{F_i}-Lipschitz continuous in the ith argument with respect to T_i $(i = 1, 2, \dots, k)$ if there exists a constant $\lambda_{F_i} > 0$ such that

$$\begin{aligned} &\|F(x_1, x_2, \dots, x_{i-1}, u_{i,1}, x_{i+1}, \dots, x_k) - F(x_1, x_2, \dots, x_{i-1}, u_{i,2}, x_{i+1}, \dots, x_k)\| \\ &\le \lambda_{F_i}\|u_{i,1} - u_{i,2}\|, \quad \forall x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_k \in E, u_{i,1} \in T_i(y_1), u_{i,2} \in T_i(y_2). \end{aligned}$$

Lemma 3.8. [20] *Let J be the normalized duality mapping from E into E^*. Then for all $x, y \in E$, we have*

(a) $\|x + y\|^2 \le \|x\|^2 + 2\langle J(x + y), y\rangle$;

(b) $\langle J(x)-J(y),x-y\rangle \leq 2d^2(x,y)\rho_E\left(\frac{4\|x-y\|}{d(x,y)}\right)$, *where* $d(x,y)=\sqrt{\frac{\|x\|^2+\|y\|^2}{2}}$.

In the next theorem, under the appropriate conditions, the existence of a solution to the SGMNVI (3.1) is proved and the convergence analysis of the sequences generated by Algorithm 3.3 is studied.

Theorem 3.9. *Let E be a real uniformly smooth Banach space and $\rho_E(t)\leq Ct^2$ for some $C>0$. Let $H:E\to E^*$ be a monotone and δ-Lipschitz continuous mapping, $p:E\to E$ be a (γ,μ)-relaxed cocoercive and λ_p-Lipschitz continuous mapping, and $\widehat{M}:E\rightrightarrows E^*$ be a general strongly H-monotone multi-valued mapping with constant θ. Suppose that $A:E\to E^*$ is a τ-Lipschitz continuous mapping, $T_i:E\rightrightarrows CB(E)$ is a λ_{T_i}-$\hat{H}$-Lipschitz continuous multi-valued mapping for each $i\in\{1,2,\dots,k\}$, and $F:E^k\to E^*$ is λ_{F_i}-Lipschitz continuous in the ith argument with respect to T_i $(i=1,2,\dots,k)$. If there exists a constant $\lambda>0$ satisfying*

$$\begin{cases} \frac{1}{\lambda\theta}\left(\delta+\lambda\tau\sqrt{1+2\gamma\lambda_p^2-2\mu+64C\lambda_p^2}+\lambda\sum\limits_{i=1}^{k}\lambda_{F_i}\lambda_{T_i}\right)<1,\\ 2\mu<1+2\gamma\lambda_p^2+64C\lambda_p^2, \end{cases} \tag{3.6}$$

then, the iterative sequences $\{x^n\}_{n=0}^\infty,\{t_1^n\}_{n=0}^\infty,\{t_2^n\}_{n=0}^\infty,\ \dots,\ \{t_k^n\}_{n=0}^\infty$ generated by Algorithm 3.3 converge strongly to $x,t_1,t_2,\dots,t_k$, respectively, and $(x,t_1,t_2,\dots,t_k)$ is a solution of the MNVIP (3.1).

Proof. From (3.3), Theorem 2.12 and using the assumptions, it follows that for each $n\in\mathbb{N}$

$$\begin{aligned}
\|x^{n+1}-x^n\| &= \|(1-\alpha)x^n+\alpha R^H_{M,\lambda}[H(x^n)-\lambda(A(x^n-p(x^n))-a\\
&\quad -F(t_1^n,t_2^n,\dots,t_k^n))]+\alpha e^n+r^n-\big((1-\alpha)x^{n-1}\\
&\quad +\alpha R^H_{M,\lambda}[H(x^{n-1})-\lambda(A(x^{n-1}-p(x^{n-1}))-a\\
&\quad -F(t_1^{n-1},t_2^{n-1},\dots,t_k^{n-1}))]+\alpha e^{n-1}+r^{n-1}\big)\|\\
&\leq (1-\alpha)\|x^n-x^{n-1}\|+\alpha\|R^H_{M,\lambda}[H(x^n)-\lambda(A(x^n-p(x^n))\\
&\quad -a-F(t_1^n,t_2^n,\dots,t_k^n))]-R^H_{M,\lambda}[H(x^{n-1})-\lambda(A(x^{n-1}-p(x^{n-1}))\\
&\quad -a-F(t_1^{n-1},t_2^{n-1},\dots,t_k^{n-1}))]\|+\alpha\|e^n-e^{n-1}\|+\|r^n-r^{n-1}\|\\
&\leq (1-\alpha)\|x^n-x^{n-1}\|+\frac{\alpha}{\lambda\theta}\|H(x^n)-\lambda(A(x^n-p(x^n))\\
&\quad -a-F(t_1^n,t_2^n,\dots,t_k^n))-(H(x^{n-1})-\lambda(A(x^{n-1}-p(x^{n-1}))\\
&\quad -a-F(t_1^{n-1},t_2^{n-1},\dots,t_k^{n-1})))\|+\alpha\|e^n-e^{n-1}\|+\|r^n-r^{n-1}\|\\
&\leq (1-\alpha)\|x^n-x^{n-1}\|+\frac{\alpha}{\lambda\theta}\big(\|H(x^n)-H(x^{n-1})\|\\
&\quad +\lambda\|A(x^n-p(x^n))-A(x^{n-1}-p(x^{n-1}))\|\\
&\quad +\lambda\|F(t_1^n,t_2^n,\dots,t_k^n)-F(t_1^{n-1},t_2^{n-1},\dots,t_k^{n-1})\|\big)+\alpha\|e^n-e^{n-1}\|\\
&\quad +\|r^n-r^{n-1}\|.
\end{aligned} \tag{3.7}$$

Since the mapping H is δ-Lipschitz continuous, for each $n \in \mathbb{N}$, we yield

$$\|H(x^n) - H(x^{n-1})\| \leq \delta \|x^n - x^{n-1}\|. \tag{3.8}$$

Relying on the fact that the mapping A is τ-Lipschitz continuous, we derive that for each $n \in \mathbb{N}$,

$$\|A(x^n - p(x^n)) - A(x^{n-1} - p(x^{n-1}))\| \leq \tau \|x^n - x^{n-1} - (p(x^n) - p(x^{n-1}))\|. \tag{3.9}$$

Taking into account that p is a (γ, μ)-relaxed cocoercive and λ_p-Lipschitz continuous, and E is a real uniformly smooth Banach space with $\rho_E(t) \leq Ct^2$ for some $C > 0$, utilizing Lemma 3.8, we get, for each $n \in \mathbb{N}$, that

$$\begin{aligned}
&\|x^n - x^{n-1} - (p(x^n) - p(x^{n-1}))\|^2 \\
&\leq \|x^n - x^{n-1}\|^2 + 2\langle J(x^n - x^{n-1} - (p(x^n) - p(x^{n-1}))), -(p(x^n) - p(x^{n-1}))\rangle \\
&= \|x^n - x^{n-1}\|^2 - 2\langle J(x^n - x^{n-1}), p(x^n) - p(x^{n-1})\rangle \\
&\quad + 2\langle J(x^n - x^{n-1} - (p(x^n) - p(x^{n-1}))) - J(x^n - x^{n-1}), -(p(x^n) - p(x^{n-1}))\rangle \\
&\leq \|x^n - x^{n-1}\|^2 - 2(-\gamma \|p(x^n) - p(x^{n-1})\|^2 + \mu \|x^n - x^{n-1}\|^2) \\
&\quad + 4d^2(x^n - x^{n-1} - (p(x^n) - p(x^{n-1})), x^n - x^{n-1}) \\
&\quad \times \rho_E\left(\frac{4\|p(x^n) - p(x^{n-1})\|}{d(x^n - x^{n-1} - (p(x^n) - p(x^{n-1})), x^n - x^{n-1})}\right) \\
&\leq (1 + 2\gamma\lambda_p^2 - 2\mu + 64C\lambda_p^2)\|x^n - x^{n-1}\|^2,
\end{aligned} \tag{3.10}$$

where J is the normalized duality mapping from E into E^*.

By virtue of the facts that F is λ_{F^j}-Lipschitz continuous in the jth argument $(j = 1, 2, \ldots, k)$ with respect to T_i $(i = 1, 2, \ldots, k)$, the mapping T_i is λ_{T_i}-$\hat{H}$-Lipschitz continuous, making use of (3.5), for each $n \in \mathbb{N}$, we get

$$\begin{aligned}
&\|F(t_1^n, t_2^n, \ldots, t_k^n) - F(t_1^{n-1}, t_2^{n-1}, \ldots, t_k^{n-1})\| \\
&\leq \|F(t_1^n, t_2^n, \ldots, t_k^n) - F(t_1^{n-1}, t_2^n, \ldots, t_k^n)\| \\
&\quad + \|F(t_1^{n-1}, t_2^n, t_3^n, \ldots, t_k^n) - F(t_1^{n-1}, t_2^{n-1}, t_3^n, \ldots, t_k^n)\| + \ldots \\
&\quad + \|F(t_1^{n-1}, t_2^{n-1}, \ldots, t_{k-1}^{n-1}, t_k^n) - F(t_1^{n-1}, t_2^{n-1}, \ldots, t_{k-1}^{n-1}, t_k^{n-1})\| \\
&\leq \sum_{j=1}^{k} \lambda_{F^j} \|t_j^n - t_j^{n-1}\|_j \\
&\leq \sum_{j=1}^{k} \lambda_{F^j} (1 + n^{-1}) \hat{H}_j(T_j(x^n), T_j(x^{n-1})) \leq \sum_{j=1}^{k} \lambda_{F^j} \lambda_{T_j} (1 + n^{-1}) \|x^n - x^{n-1}\|.
\end{aligned} \tag{3.11}$$

Combining (3.7)–(3.11), for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\|x^{n+1} - x^n\| \\
&\le (1-\alpha)\|x^n - x^{n-1}\| + \alpha\Big(\frac{\delta + \lambda\tau\sqrt{1 + 2\gamma\lambda_p^2 - 2\mu + 64C\lambda_p^2}}{\lambda\theta}\|x^n - x^{n-1}\| \\
&\quad + \frac{1}{\theta}\sum_{j=1}^{k}\lambda_{F^j}\lambda_{T_j}(1+n^{-1})\|x^n - x^{n-1}\|\Big) + \alpha\|e^n - e^{n-1}\| + \|r^n - r^{n-1}\| \\
&= (1-\alpha)\|x^n - x^{n-1}\| + \alpha\Phi(n)\|x^n - x^{n-1}\| + \alpha\|e^n - e^{n-1}\| + \|r^n - r^{n-1}\|,
\end{aligned} \tag{3.12}$$

where for each $n \in \mathbb{N}$,

$$\Phi(n) = \frac{\delta + \lambda\tau\sqrt{1 + 2\gamma\lambda_p^2 - 2\mu + 64C\lambda_p^2}}{\lambda\theta} + \frac{1}{\theta}\sum_{j=1}^{k}\lambda_{F^j}\lambda_{T_j}(1+n^{-1}).$$

Clearly, $\Phi(n) \to \Phi$, as $n \to \infty$, where

$$\Phi = \frac{\delta + \lambda\tau\sqrt{1 + 2\gamma\lambda_p^2 - 2\mu + 64C\lambda_p^2}}{\lambda\theta} + \frac{1}{\theta}\sum_{j=1}^{k}\lambda_{F^j}\lambda_{T_j}.$$

Now, letting $\varphi(n) = 1 - \alpha + \alpha\Phi(n)$, for each $n \in \mathbb{N}$, we know that $\varphi(n) \to \varphi$, as $n \to \infty$, where $\varphi = 1 - \alpha + \alpha\Phi$. Obviously, (3.6) implies that $\Phi \in (0,1)$, and so $\varphi \in (0,1)$. Therefore, there exists $\hat{\varphi} \in (0,1)$ (take $\hat{\varphi} = \frac{\varphi+1}{2} \in (\varphi, 1)$) and $n_0 \in \mathbb{N}$ such that $\varphi(n) \le \hat{\varphi}$, for all $n \ge n_0$. Accordingly, for all $n > n_0$, by means of (3.12), for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\|x^{n+1} - x^n\| &\le \hat{\varphi}\|x^n - x^{n-1}\| + \alpha\|e^n - e^{n-1}\| + \|r^n - r^{n-1}\| \\
&\le \hat{\varphi}[\hat{\varphi}\|x^{n-1} - x^{n-2}\| + \alpha\|e^{n-1} - e^{n-2}\| + \|r^{n-1} - r^{n-2}\|] \\
&\quad + \alpha\|e^n - e^{n-1}\| + \|r^n - r^{n-1}\| \\
&= \hat{\varphi}^2\|x^{n-1} - x^{n-2}\| + \alpha(\hat{\varphi}\|e^{n-1} - e^{n-2}\| + \|e^n - e^{n-1}\|) \\
&\quad + \hat{\varphi}\|r^{n-1} - r^{n-2}\| + \|r^n - r^{n-1}\| \\
&\le \cdots \\
&\le \hat{\varphi}^{n-n_0}\|x^{n_0+1} - x^{n_0}\| + \alpha\sum_{j=1}^{n-n_0}\hat{\varphi}^{j-1}\|e^{n-(j-1)} - e^{n-j}\| \\
&\quad + \sum_{j=1}^{n-n_0}\hat{\varphi}^{j-1}\|r^{n-(j-1)} - r^{n-j}\|.
\end{aligned} \tag{3.13}$$

The inequality (3.13) implies that for any $m \geq n > n_0$,

$$\begin{aligned}\|x^m - x^n\| \leq \sum_{l=n}^{m-1} \|x^{l+1} - x^l\| \leq \sum_{l=n}^{m-1} \hat{\varphi}^{l-n_0} \|x^{n_0+1} - x^{n_0}\| \\ + \alpha \sum_{l=n}^{m-1} \sum_{j=1}^{l-n_0} \hat{\varphi}^{j-1} \|e^{n-(j-1)} - e^{n-j}\| \\ + \sum_{l=n}^{m-1} \sum_{j=1}^{l-n_0} \hat{\varphi}^{j-1} \|r^{n-(j-1)} - r^{n-j}\|.\end{aligned} \tag{3.14}$$

Since $\hat{\varphi} < 1$, it follows from (3.5) and (3.14) that $\|x^m - x^n\| \to \infty$, as $n \to \infty$, that is, $\{x^n\}_{n=0}^{\infty}$ is a Cauchy sequence in E. In view of the completeness of E, there exists $x \in E$ such that $x^n \to x$, as $n \to \infty$. Thanks to the fact that for each $i \in \{1, 2, \dots, k\}$, T_i is a λ_{T_i}-$\hat{H}$-Lipschitz continuous mapping, by using (3.4), for each $n \geq 0$ and $i \in \{1, 2, \dots, k\}$, we have

$$\begin{aligned}\|t_i^{n+1} - t_i^n\| &\leq (1 + (1+n)^{-1})\hat{H}(T_i(x^{n+1}), T_i(x^n)) \\ &\leq (1 + (1+n)^{-1})\lambda_{T_i}\|x^{n+1} - x^n\|.\end{aligned} \tag{3.15}$$

Owing to the fact that $\|x^{n+1} - x^n\| \to 0$, as $n \to \infty$, (3.15) implies that for each $i \in \{1, 2, \dots, k\}$, $\|t_i^{n+1} - t_i^n\| \to 0$, as $n \to \infty$. Consequently, for each $i \in \{1, 2, \dots, k\}$, $\{t_i^n\}_{n=0}^{\infty}$ is also a Cauchy sequence in E. Thereby, for each $i \in \{1, 2, \dots, k\}$, there exists $t_i \in E$ such that $t_i^n \to t_i$, as $n \to \infty$. At the same time, in the light of λ_{T_i}-$\hat{H}$-Lipschitz continuity of the mapping T_i for each $i \in \{1, 2, \dots, k\}$, we have

$$\begin{aligned}d(t_i, T_i(x)) &= \inf\{\|t_i - z\| : z \in T_i(x)\} \\ &\leq \|t_i - t_i^n\| + d(t_i^n, T_i(x)) \\ &\leq \|t_i - t_i^n\| + \hat{H}(T_i(x^n), T_i(x)) \\ &\leq \|t_i - t_i^n\| + \lambda_{T_i}\|x^n - x\|.\end{aligned} \tag{3.16}$$

The right-hand side of (3.16) tends to zero, as $n \to \infty$. Hence, $t_i \in T_i(x)$ for each $i \in \{1, 2, \dots, k\}$. Since the mappings $R_{M,\lambda}^H, H, A, T_i$ $(i = 1, 2, \dots, k), F, p$ are continuous, $x^n \to x$ and $t_i^n \to t_i$, as $n \to \infty$, passing to the limit in (3.3) as $n \to \infty$, it follows that $x, t_1, t_2, \dots, t_k$ satisfy (3.2). Now, Theorem 3.1 guarantees that $(x, t_1, t_2, \dots, t_k)$ is a solution of the MNVIP (3.1). □

As a direct consequence of the above theorem we obtain the following corollary by taking $\alpha = 1$ and $e^n = r^n = 0$ $(n \geq 0)$ in Algorithm 3.3 immediately.

Corollary 3.10. *Let E be a real uniformly smooth Banach space and $\rho_E(t) \leq Ct^2$ for some $C > 0$. Let $H : E \to E^*$ be a monotone and δ-Lipschitz continuous mapping, $p : E \to E$ be a (γ, μ)-relaxed cocoercive and λ_p-Lipschitz continuous mapping, and $\widehat{M} : E \rightrightarrows E^*$ be a general strongly H-monotone multi-valued mapping with constant θ. Suppose that $A : E \to E^*$ is a τ-Lipschitz continuous mapping, $T_i : E \rightrightarrows CB(E)$ is a λ_{T_i}-$\hat{H}$-Lipschitz continuous multi-valued mapping for each $i \in \{1, 2, \dots, k\}$, and*

$F : E^k \to E^*$ is λ_{F_i}-Lipschitz continuous in the ith argument with respect to T_i $(i = 1, 2, \ldots, k)$. If there exists a constant $\lambda > 0$ satisfying

$$\begin{cases} \frac{1}{\lambda\theta}\left(\delta + \lambda\tau\sqrt{1 + 2\gamma\lambda_p^2 - 2\mu + 64C\lambda_p^2} + \lambda\sum\limits_{i=1}^{k}\lambda_{F_i}\lambda_{T_i}\right) < 1, \\ 2\mu < 1 + 2\gamma\lambda_p^2 + 64C\lambda_p^2, \end{cases} \tag{3.17}$$

then, the iterative sequences $\{x^n\}_{n=0}^{\infty}, \{t_1^n\}_{n=0}^{\infty}, \{t_2^n\}_{n=0}^{\infty}, \ldots, \{t_k^n\}_{n=0}^{\infty}$ generated by Algorithm 3.4 converge strongly to $x, t_1, t_2, \ldots, t_k$, respectively, and $(x, t_1, t_2, \ldots, t_k)$ is a solution of the MNVIP (3.1).

4. Comments on C_n-monotone mappings

This section investigates and analyzes the notion of C_n-monotone mapping introduced in [19] and pointing out some important facts related to it. We also derive the results of [19] by using the results of previous section of this paper.

Definition 4.1. [19, Definition 2.2] Let $n \geq 3$ and $M : E^n \rightrightarrows E^*$ be a multi-valued mapping and $f_1, f_2, \ldots, f_n : E \to E$ be single-valued mappings.

(a) For each $1 \leq i \leq n$, $M(\ldots, f_i, \ldots)$ is said to be α_i-strongly monotone with respect to f_i (in the ith argument) if there exists a constant $\alpha_i > 0$ such that

$$\begin{aligned} \langle w_i - w_i', x - y\rangle \geq \alpha_i\|x - y\|^2, \quad &\forall x, y, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n \in E, \\ &w_i \in M(u_1, \ldots, u_{i-1}, f_i(x), u_{i+1}, \ldots, u_n), \\ &w_i' \in M(u_1, \ldots, u_{i-1}, f_i(y), u_{i+1}, \ldots, u_n). \end{aligned}$$

(b) For each $1 \leq i \leq n$, $M(\ldots, f_i, \ldots)$ is said to be β_i-relaxed monotone with respect to f_i (in the ith argument) if there exists a constant $\beta_i > 0$ such that

$$\begin{aligned} \langle w_i - w_i', x - y\rangle \geq -\beta_i\|x - y\|^2, \quad &\forall x, y, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n \in E, \\ &w_i \in M(u_1, \ldots, u_{i-1}, f_i(x), u_{i+1}, \ldots, u_n), \\ &w_i' \in M(u_1, \ldots, u_{i-1}, f_i(y), u_{i+1}, \ldots, u_n). \end{aligned}$$

(c) Let n be an even natural number. The set-valued mapping M is said to be $\alpha_1\beta_2\alpha_3\beta_4\ldots\alpha_{n-1}\beta_n$-symmetric monotone with respect to $f_1, f_2, \ldots, f_n$ if for each $i \in \{1, 3, \ldots, n-1\}$, $M(\ldots, f_i, \ldots)$ is α_i-strongly monotone with respect to f_i (in the ith argument) and for each $j \in \{2, 4, \ldots, n\}$, $M(\ldots, f_j, \ldots)$ is β_j-relaxed monotone with respect to f_j (in the jth argument) with

$$\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} \geq \beta_2 + \beta_4 + \cdots + \beta_n,$$

and $\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} = \beta_2 + \beta_4 + \cdots + \beta_n$ if and only if $x = y$.

(d) Let n be an odd natural number. The set-valued mapping M is said to be $\alpha_1\beta_2\alpha_3\beta_4\ldots\beta_{n-1}\alpha_n$-symmetric monotone with respect to $f_1, f_2, \ldots, f_n$ if for each $i \in \{1, 3, \ldots, n\}$, $M(\ldots, f_i, \ldots)$ is α_i-strongly monotone with respect to f_i (in the ith argument) and for each $j \in \{2, 4, \ldots, n-1\}$, $M(\ldots, f_j, \ldots)$ is β_j-relaxed monotone with respect to f_j (in the jth argument) with

$$\alpha_1 + \alpha_3 + \cdots + \alpha_n \geq \beta_2 + \beta_4 + \cdots + \beta_{n-1},$$

and $\alpha_1 + \alpha_3 + \cdots + \alpha_n = \beta_2 + \beta_4 + \cdots + \beta_{n-1}$ if and only if $x = y$.

Proposition 4.2. *Let $f_1, f_2, \ldots, f_n : E \to E$ $(n \geq 3)$ be single-valued mappings and $M : E^n \rightrightarrows E^*$ be a multi-valued mapping. Suppose further that the mapping $\widehat{M} : E \rightrightarrows E^*$ is defined by $\widehat{M}(x) = M(f_1(x), f_2(x), \ldots, f_n(x))$, for all $x \in E$. Then the following statements hold:*

(a) *If n is an even natural number and M is an $\alpha_1\beta_2\alpha_3\beta_4\ldots\alpha_{n-1}\beta_n$-symmetric monotone mapping with respect to $f_1, f_2, \ldots, f_n$, then $\widehat{M}$ is a $\sum\limits_{i=1}^{\frac{n}{2}}(\alpha_{2i-1} - \beta_{2i})$-strongly monotone mapping.*

(b) *If n is an odd natural number and M is an $\alpha_1\beta_2\alpha_3\beta_4\ldots\beta_{n-1}\alpha_n$-symmetric monotone mapping with respect to $f_1, f_2, \ldots, f_n$, then $\widehat{M}$ is a $\Big(\sum\limits_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} - \sum\limits_{i=1}^{\frac{n-1}{2}} \beta_{2i}\Big)$-strongly monotone mapping.*

Proof. Let n be an even natural number. Taking into account that M is an $\alpha_1\beta_2\alpha_3\beta_4\ldots\alpha_{n-1}\beta_n$-symmetric monotone mapping with respect to $f_1, f_2, \ldots, f_n$, for all $x, y \in E$,

$$u \in \widehat{M}(x) = M(f_1(x), f_2(x), \ldots, f_n(x))$$

and

$$v \in \widehat{M}(y) = M(f_1(y), f_2(y), \ldots, f_n(y)),$$

we have

$$\begin{aligned}
\langle u - v, x - y\rangle &= \left\langle u + \sum_{i=1}^{n-1}(-w_i + w_i) - v, x - y \right\rangle \\
&= \langle u - w_1, x - y\rangle + \sum_{i=1}^{n-2}\langle w_i - w_{i+1}, x - y\rangle + \langle w_{n-1} - v, x - y\rangle \\
&= \langle u - w_1, x - y\rangle + \sum_{i=1}^{\frac{n-2}{2}}\langle w_{2i-1} - w_{2i}, x - y\rangle \\
&\quad + \sum_{i=1}^{\frac{n-2}{2}}\langle w_{2i} - w_{2i+1}, x - y\rangle + \langle w_{n-1} - v, x - y\rangle
\end{aligned}$$

$$\begin{aligned}
&\geq \alpha_1\|x - y\|^2 - \sum_{i=1}^{\frac{n-2}{2}}\beta_{2i}\|x - y\|^2 + \sum_{i=1}^{\frac{n-2}{2}}\alpha_{2i+1}\|x - y\|^2 - \beta_n\|x - y\|^2 \\
&= \sum_{i=1}^{\frac{n}{2}}\alpha_{2i-1}\|x - y\|^2 - \sum_{i=1}^{\frac{n}{2}}\beta_{2i}\|x - y\|^2 \qquad (4.1) \\
&= \sum_{i=1}^{\frac{n}{2}}(\alpha_{2i-1} - \beta_{2i})\|x - y\|^2,
\end{aligned}$$

where for each $i \in \{1, 2, \dots, n-1\}$, $w_i \in M(f_1(y), f_2(y), \dots, f_i(y), f_{i+1}(x), \dots, f_n(x))$. Since

$$\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} = \sum_{i=1}^{\frac{n}{2}} \alpha_{2i-1} \geq \beta_2 + \beta_4 + \cdots + \beta_n = \sum_{i=1}^{\frac{n}{2}} \beta_{2i}$$

and

$$\sum_{i=1}^{\frac{n}{2}} \alpha_{2i-1} = \sum_{i=1}^{\frac{n}{2}} \beta_{2i}$$

if and only if $x = y$, it follows from (4.1) that $\widehat{M}$ is a $\sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i})$-strongly monotone mapping.

We now prove conclusion (b). Suppose that n is an odd natural number. Thanks to the fact that M is an $\alpha_1\beta_2\alpha_3\beta_4 \dots \beta_{n-1}\alpha_n$-symmetric monotone mapping with respect to $f_1, f_2, \dots, f_n$, for all $x, y \in E$, $u \in \widehat{M}(x) = M(f_1(x), f_2(x), \dots, f_n(x))$ and $v \in \widehat{M}(y) = M(f_1(y), f_2(y), \dots, f_n(y))$, yields

$$\begin{aligned}
\langle u - v, x - y \rangle &= \left\langle u + \sum_{i=1}^{n-1} (-w_i + w_i) - v, x - y \right\rangle \\
&= \langle u - w_1, x - y \rangle + \sum_{i=1}^{n-2} \langle w_i - w_{i+1}, x - y \rangle + \langle w_{n-1} - v, x - y \rangle \\
&= \langle u - w_1, x - y \rangle + \sum_{i=1}^{\frac{n-1}{2}} \langle w_{2i-1} - w_{2i}, x - y \rangle \\
&\quad + \sum_{i=1}^{\frac{n-3}{2}} \langle w_{2i} - w_{2i+1}, x - y \rangle + \langle w_{n-1} - v, x - y \rangle \qquad (4.2) \\
&\geq \alpha_1 \|x - y\|^2 - \sum_{i=1}^{\frac{n-1}{2}} \beta_{2i} \|x - y\|^2 + \sum_{i=1}^{\frac{n-3}{2}} \alpha_{2i+1} \|x - y\|^2 + \alpha_n \|x - y\|^2 \\
&= \sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} \|x - y\|^2 - \sum_{i=1}^{\frac{n-1}{2}} \beta_{2i} \|x - y\|^2 \\
&= \left(\sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} - \sum_{i=1}^{\frac{n-1}{2}} \beta_{2i} \right) \|x - y\|^2,
\end{aligned}$$

where for each $i \in \{1, 2, \dots, n-1\}$, $w_i \in M(f_1(y), f_2(y), \dots, f_i(y), f_{i+1}(x), \dots, f_n(x))$. In virtue of the facts that

$$\alpha_1 + \alpha_3 + \cdots + \alpha_n = \sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} \geq \beta_2 + \beta_4 + \cdots + \beta_{n-1} = \sum_{i=1}^{\frac{n-1}{2}} \beta_{2i}$$

and

$$\sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} = \sum_{i=1}^{\frac{n-1}{2}} \beta_{2i}$$

if and only if $x = y$, making use of (4.2) we deduce that $\widehat{M}$ is a $\left(\sum\limits_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} - \sum\limits_{i=1}^{\frac{n-1}{2}} \beta_{2i}\right)$-strongly monotone mapping. □

Remark 4.3. In the light of Proposition 4.2 and the arguments mentioned above, we observe that the notions of $\alpha_1\beta_2\alpha_3\beta_4\ldots\alpha_{n-1}\beta_n$-symmetric monotonicity and $\alpha_1\beta_2\alpha_3\beta_4\ldots\beta_{n-1}\alpha_n$-symmetric monotonicity of the multi-valued mapping $M : E^n \rightrightarrows E^*$ with respect to the mappings $f_1, f_2, \ldots, f_n : E \to E$, presented in Definition 4.1 (c) and (d), are actually r-strong monotonicity of the mapping $\widehat{M} = M(f_1, f_2, \ldots, f_n)$ given in Definition 2.4 (c), where $r = \sum\limits_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i})$ for the case when n is an even natural number, and $r = \sum\limits_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} - \sum\limits_{i=1}^{\frac{n-1}{2}} \beta_{2i}$ when n is an odd natural number.

Nazemi[19] introduced the following concept of so-called C_n-monotone mappings.

Definition 4.4. [19, Definition 3.1] Let $f_1, f_2, \ldots, f_n : E \to E$ and $C_n : E \to E^*$ $(n \geq 3)$ be single-valued mappings and $M : E^n \rightrightarrows E^*$ be a multi-valued mapping.

(a) Let n be an even natural number. The multi-valued mapping M is said to be a C_n-monotone mapping if M is $\alpha_1\beta_2\alpha_3\beta_4\ldots\alpha_{n-1}\beta_n$-symmetric monotone with respect to $f_1, f_2, \ldots, f_n$ and $(C_n + \lambda M(f_1, f_2, \ldots, f_n))(E) = E^*$ for all $\lambda > 0$.

(b) Let n be an odd natural number. The multi-valued mapping M is said to be a C_n-monotone mapping if M is $\alpha_1\beta_2\alpha_3\beta_4\ldots\beta_{n-1}\alpha_n$-symmetric monotone with respect to $f_1, f_2, \ldots, f_n$ and $(C_n + \lambda M(f_1, f_2, \ldots, f_n))(E) = E^*$ for all $\lambda > 0$.

By providing an example in [19], it has been shown that the class of mappings defined in the above definition is nonempty. However, in view of Proposition 4.2, it is expected that this mapping to be actually a general strongly H-monotone mapping. This fact is illustrated in the following example.

Example 4.5. Let $E = l^2$ denote the space of all square-summable sequences, i.e., the space of all sequences $\{x_m\}_{m=1}^{\infty}$ for which $\sum\limits_{m=1}^{\infty} |x_m|^2$ converges, and $\|.\|_2$ be a norm defined on l^2 by $\|x\|_2 = \left(\sum\limits_{m=1}^{\infty} |x_m|^2\right)^{\frac{1}{2}}$, for all $x = \{x_m\}_{m=1}^{\infty} \in l^2$. It is well known that l^2 together with the inner product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i, \quad \forall x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in l^2,$$

is a Hilbert space and so $E^* = l^2$. Moreover, $\{e_m\}_{m=1}^{\infty}$, where for each $m \in \mathbb{N}$, e_m is the sequence with 1 in the mth position and 0's elsewhere, is a basis of $E = l^2$.

Let n be an even natural number and suppose that the mappings $f_1, f_2, \dots, f_n : E \to E$ and $C_n : E \to E^*$ are defined by

$$f_{2i-1}(x) = \begin{cases} e_k + \alpha_1 x, & \text{if } i = 1, \\ \alpha_{2i-1} x, & \text{if } i = 2, 3, \dots, \frac{n}{2}, \end{cases} \tag{4.3}$$

$f_{2i}(x) = -\beta_{2i} x$ $(i = 1, 2, \dots, \frac{n}{2})$ and $C_n(x) = x - e_k$, for all $x = \{x_m\}_{m=1}^{\infty} \in l^2$, where $e_k = (0, 0, \dots, 1, 0, \dots) \in l^2$ (1 in the kth place) and $\alpha_{2i-1}, \beta_{2i} > 0$ $(i = 1, 2, \dots, \frac{n}{2})$ are arbitrary real constants such that $\sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i}) = 1$. Further, let $M : E^n \rightrightarrows E^*$ be defined by $M(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} x_i$, for all $(x_1, x_2, \dots, x_n) \in E^n$. Taking into account that for all $x, y \in E$ and $u_j \in E_j$ $(j = 2, 3, \dots, n)$,

$$\begin{aligned} &\langle M(f_1(x), u_2, u_3, \dots, u_n) - M(f_1(y), u_2, u_3, \dots, u_n), x - y\rangle \\ &= \langle f_1(x) - f_1(y), x - y\rangle \\ &= \langle \alpha_1 x - \alpha_1 y, x - y\rangle = \alpha_1 \|x - y\|_2^2, \end{aligned}$$

and due to the facts that for all $j = 1, 2, \dots, 2(i-1), 2i, \dots, n$ and $i = 2, 3, \dots, \frac{n}{2}$,

$$\begin{aligned} &\langle M(u_1, u_2, \dots, u_{2i-2}, f_{2i-1}(x), u_{2i}, \dots, u_n) \\ &- M(u_1, u_2, \dots, u_{2i-2}, f_{2i-1}(y), u_{2i}, \dots, u_n), x - y\rangle \\ &= \langle f_{2i-1}(x) - f_{2i-1}(y), x - y\rangle \\ &= \langle \alpha_{2i-1} x - \alpha_{2i-1} y, x - y\rangle = \alpha_{2i-1} \|x - y\|_2^2 \end{aligned}$$

and

$$\begin{aligned} &\langle M(u_1, u_2, \dots, u_{2i-1}, f_{2i}(x), u_{2i+1}, \dots, u_n) \\ &- M(u_1, u_2, \dots, u_{2i-1}, f_{2i}(y), u_{2i+1}, \dots, u_n), x - y\rangle \\ &= \langle f_{2i}(x) - f_{2i}(y), x - y\rangle \\ &= \langle -\beta_{2i} x + \beta_{2i} y, x - y\rangle = -\beta_{2i} \|x - y\|_2^2, \end{aligned}$$

it follows that for each $i \in \{1, 2, \dots, \frac{n}{2}\}$, $M(\dots, f_{2i-1}, \dots)$ is α_{2i-1}-strongly monotone with respect to f_{2i-1} in the $(2i-1)$th argument, and $M(\dots, f_{2i}, \dots)$ is β_{2i}-relaxed monotone with respect to f_{2i} in the $(2i)$th argument. Since $M(\dots, f_i, \dots)$ is α_i-strongly monotone with respect to f_i in the ith argument for each $i \in \{1, 3, \dots, n-1\}$, and $M(\dots, f_j, \dots)$ is β_j-relaxed monotone with respect to f_j in the jth argument for each $j \in \{2, 4, \dots, n\}$, and $\sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i}) = 1 > 0$, it follows that M is an $\alpha_1\beta_2\alpha_3\beta_4 \dots \alpha_{n-1}\beta_n$-symmetric monotone mapping with respect to the mappings

$f_1, f_2, \dots, f_n$. At the same time, for all $x = \{x_m\}_{m=1}^{\infty} \in l^2$, we have

$$\begin{aligned} M(f_1(x), f_2(x), \dots, f_n(x)) &= \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{\frac{n}{2}} (f_{2i-1}(x) + f_{2i}(x)) \\ &= e_k + \sum_{i=1}^{\frac{n}{2}} (\alpha_{2i-1} - \beta_{2i})x = e_k + x. \end{aligned}$$

Suppose that the real constant $\lambda > 0$ is chosen arbitrarily but fixed. Thanks to the fact that for any $x \in l^2$, there is $\frac{x+(1-\lambda)e_k}{1+\lambda} \in l^2$ such that $(C_n + \lambda M)(\frac{x+(1-\lambda)e_k}{1+\lambda}) = x$, it follows that $(C_n + \lambda M)(E) = E^*$. In the light of the above-mentioned arguments, it is deduced in [19] that M is a C_n-monotone mapping.

Let us now define the mapping $\widehat{M} : E \rightrightarrows E^*$ as $\widehat{M}(x) = M(f_1(x), f_2(x), \dots, f_n(x))$, for all $x \in E = l^2$. Then, for all $x = \{x_m\}_{m=1}^{\infty} \in l^2$, we have $\widehat{M}(x) = \sum\limits_{i=1}^{n} f_i(x) = e_k + x$. Meanwhile, for all $x = \{x_m\}_{m=1}^{\infty}, y = \{y_m\}_{m=1}^{\infty} \in l^2$, we obtain

$$\langle \widehat{M}(x) - \widehat{M}(y), x - y\rangle = \langle e_k + x - (e_k + y), x - y\rangle = \|x - y\|_2^2,$$

that is, $\widehat{M}$ is a 1-strongly monotone mapping. By taking $H = C_n$, it can be easily seen that $\text{Range}(H + \lambda\widehat{M}) = E^*$ holds, for every real constant $\lambda > 0$. Hence, according to Definition 2.6, $\widehat{M}$ is a general H-monotone mapping.

Remark 4.6. In view of the above discussion, the C_n-monotone mapping given in [19, Example 3.1] is actually a general strongly H-monotone mapping with constant 1. In general, if E is a real Banach space with its dual E^*, $f_1, f_2, \dots, f_n : E \to E$ and $C_n : E \to E^*$ ($n \geq 3$) are single-valued mappings and $M : E^n \rightrightarrows E^*$ is a C_n-monotone multi-valued mapping, then in accordance with Definition 4.4, for the case when n is an even natural number, M is an $\alpha_1\beta_2\alpha_3\beta_4 \dots \alpha_{n-1}\beta_n$-symmetric monotone mapping with respect to $f_1, f_2, \dots, f_n$, and in the case where n is an odd natural number, M is an $\alpha_1\beta_2\alpha_3\beta_4 \dots \beta_{n-1}\alpha_n$-symmetric monotone mapping with respect to $f_1, f_2, \dots, f_n$. At the same time, in both the cases, we have $\text{Range}(C_n + \lambda M(f_1, f_2, \dots, f_n)) = E^*$ for every real constant $\lambda > 0$. Then, by defining $\widehat{M} : E \rightrightarrows E^*$ by $\widehat{M}(x) := M(f_1(x), f_2(x), \dots, f_n(x))$ for all $x \in E$, and by taking $H = C_n$, Proposition 4.2 implies that $\widehat{M}$ is a strongly monotone mapping. Accordingly, invoking Definition 2.6, $\widehat{M}$ is a general H-monotone mapping and so Definition 4.4 reduces to the definition of a general H-monotone mapping which has been introduced in [24].

Lemma 4.7. [19, Lemma 3.1] *Let E be a reflexive Banach space. Let $n \geq 3$ and $f_1, f_2, \dots, f_n : E \to E$ be single-valued mappings, $C_n : E \to E^*$ be a monotone mapping and $M : E^n \rightrightarrows E^*$ be a C_n-monotone mapping. Then the mapping*

$$(C_n + \lambda M(f_1, f_2, \dots, f_n))^{-1}$$

is single-valued for every $\lambda > 0$.

Proof. Let $\widehat{M} : E \rightrightarrows E^*$ be defined by $\widehat{M}(x) := M(f_1(x), f_2(x), \dots, f_n(x))$ for all $x \in E$. Proposition 4.2 implies that $\widehat{M}$ is a strongly monotone mapping. By taking

$H = C_n$, it follows that $\widehat{M}$ is a general H-monotone mapping. Then, all the conditions of Corollary 2.10 hold, and therefore, $(H + \lambda\widehat{M})^{-1} = (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1}$ is single-valued for every $\lambda > 0$. □

Based on Lemma 4.7, Nazemi [19] defined the proximal mapping $R^{C_n}_{M(.,.,\ldots,.),\lambda}$ associated with C_n, λ and the C_n-monotone mapping $M(.,.,\ldots,.)$ as follows.

Definition 4.8. [19, Definition 3.2] Let E be a reflexive Banach space. Let $n \geq 3$ and $f_1, f_2, \ldots, f_n : E \to E$ be single-valued mappings, $C_n : E \to E^*$ be a monotone mapping and $M : E^n \rightrightarrows E^*$ be a C_n-monotone mapping. A mapping $R^{C_n}_{M(.,.,\ldots,.),\lambda} : E^* \to E$ is defined by

$$R^{C_n}_{M(.,.,\ldots,.),\lambda}(x^*) = (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1}(x^*), \quad \forall x^* \in E^*,$$

is called proximal associated with C_n, λ and the C_n-monotone mapping $M(.,.,\ldots,.)$.

Remark 4.9. By defining $\widehat{M} : E \rightrightarrows E^*$ as $\widehat{M}(x) := M(f_1(x), f_2(x), \ldots, f_n(x))$ for all $x \in E$, and taking $H = C_n$, invoking Proposition 4.2, we conclude that $\widehat{M}$ is a general H-monotone mapping. In accordance with Definition 2.11, for any real constant $\lambda > 0$, the mapping $R^H_{\widehat{M},\lambda}$, that is, the proximal mapping associated with H, λ and $\widehat{M}$ is defined for any $x^* \in E^*$ as follows:

$$R^H_{\widehat{M},\lambda}(x^*) = R^{C_n}_{M(.,.,\ldots,.),\lambda}(x^*) = (H + \lambda\widehat{M})^{-1}(x^*) = (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1}(x^*).$$

In fact, the notion of the proximal mapping $R^{C_n}_{M(.,.,\ldots,.),\lambda}$ associated with a monotone mapping C_n, an arbitrary real constant $\lambda > 0$ and a C_n-monotone mapping $M(.,.,\ldots,.)$ is actually the same as the notion of the proximal mapping $R^{C_n}_{\widehat{M},\lambda}$ associated with C_n, λ and the general strongly $H = C_n$-monotone mapping $\widehat{M} = M(f_1, f_2, \ldots, f_n)$.

Nazemi [19] also proved the Lipschitz continuity of the proximal mapping $R^{C_n}_{M(.,.,\ldots,.),\lambda}$ associated with C_n, λ and the C_n-monotone mapping $M(.,.,\ldots,.)$, and computed an estimate of its Lipschitz constant under some appropriate conditions imposed on the parameters and mappings.

Theorem 4.10. [19, Theorem 3.1] *Let E be a reflexive Banach space. Let $n \geq 3$ and $f_1, f_2, \ldots, f_n : E \to E$ be single-valued mappings, $C_n : E \to E^*$ be a monotone mapping and $M : E^n \rightrightarrows E^*$ be a C_n-monotone mapping. Then, the proximal mapping $R^{C_n}_{M(.,.,\ldots,.),\lambda} : E^* \to E$ is*

(a) $\frac{1}{\lambda(\alpha_1+\alpha_3+\cdots+\alpha_{n-1}-(\beta_2+\beta_4+\cdots+\beta_n))}$*-Lipschitz continuous, when n is an even natural number;*

(b) $\frac{1}{\lambda(\alpha_1+\alpha_3+\cdots+\alpha_n-(\beta_2+\beta_4+\cdots+\beta_{n-1}))}$*-Lipschitz continuous, when n is an odd natural number.*

Proof. Let $\widehat{M} : E \rightrightarrows E^*$ be defined by $\widehat{M}(x) := M(f_1(x), f_2(x), \ldots, f_n(x))$ for all $x \in E$. When n is an even natural number, due to the fact that M is a C_n-monotone mapping, using Definition 4.4 (a), it follows that M is an $\alpha_1\beta_2\alpha_3\beta_4\ldots\alpha_{n-1}\beta_n$-symmetric monotone mapping with respect to $f_1, f_2, \ldots, f_n$. Now, Definition 4.1

(a) implies that $\widehat{M}$ is a $\sum\limits_{i=1}^{\frac{n}{2}}(\alpha_{2i-1}-\beta_{2i})$-strongly monotone mapping. Meanwhile, picking $H=C_n$, we note that $\widehat{M}$ is a general H-monotone mapping. Now, by taking $\gamma=\sum\limits_{i=1}^{\frac{n}{2}}(\alpha_{2i-1}-\beta_{2i})$ and utilizing Theorem 2.12, it follows that $R^H_{\widehat{M},\lambda}=R^{C_n}_{M(.,.,\dots,.),\lambda}: E^*\to E$ is $\frac{1}{\lambda\sum\limits_{i=1}^{\frac{n}{2}}(\alpha_{2i-1}-\beta_{2i})}$-Lipschitz continuous, that is, the statement (a) holds.

When n is an odd natural number, in view of the C_n-monotonicity of the mapping M and Definition 4.4 (b), we conclude that M is an $\alpha_1\beta_2\alpha_3\beta_4\dots\beta_{n-1}\alpha_n$-symmetric monotone with respect to $f_1,f_2,\dots,f_n$. Then, Proposition 4.2 (b) implies that $\widehat{M}$ is a $\left(\sum\limits_{i=1}^{\frac{n+1}{2}}\alpha_{2i-1}-\sum\limits_{i=1}^{\frac{n-1}{2}}\beta_{2i}\right)$-strongly monotone mapping. Similarly, letting $H=C_n$, $\widehat{M}$ is a general H-monotone mapping. Now, by taking $\gamma=\sum\limits_{i=1}^{\frac{n+1}{2}}\alpha_{2i-1}-\sum\limits_{i=1}^{\frac{n-1}{2}}\beta_{2i}$, Theorem 2.12 ensures that the proximal mapping $R^H_{\widehat{M},\lambda}=R^{C_n}_{M(.,.,\dots,.),\lambda}: E^*\to E$ is $\frac{1}{\lambda(\sum\limits_{i=1}^{\frac{n+1}{2}}\alpha_{2i-1}-\sum\limits_{i=1}^{\frac{n-1}{2}}\beta_{2i})}$-Lipschitz continuous, that is, the statement (b) holds. □

Let $k\geq 3$ and $A:E\to E^*$, $p,f_1,,f_2,\dots,f_k:E\to E$, $F:E^k\to E^*$ be single-valued mappings and let $T_1,T_2,\dots,T_k:E\rightrightarrows CB(E)$ and $M:E^k\rightrightarrows E^*$ be multi-valued mappings. For any given $a\in E^*$, Nazemi [19] considered and studied the variational inclusion problem of finding $x\in E$, $t_1\in T_1(x),t_2\in T_2(x),\dots,t_k\in T_k(x)$ such that

$$a\in A(x-p(x))+M(f_1(x),f_2(x),\dots,f_k(x))-F(t_1,t_2,\dots,t_k). \tag{4.4}$$

In order to identify a solution of the problem (4.4), she gave the following characterization for the solution of the problem (4.4) by utilizing the proximal mapping $R^{C_k}_{M(.,.,\dots,.),\lambda}$.

Theorem 4.11. [19, Theorem 4.1] *Suppose that $k\geq 3$ and $A:E\to E^*$, $p,f_1,f_2,\dots,f_k:E\to E$, $F:E^k\to E^*$ are single-valued mappings and $T_1,T_2,\dots,T_k:E\rightrightarrows CB(E)$ are multi-valued mappings. Let $C_k:E\to E^*$ be a monotone mapping and $M:E^k\rightrightarrows E^*$ be a C_k-monotone mapping. Then $(x,t_1,t_2,\dots,t_k)$ is a solution of the problem* (4.4) *if and only if*

$$x=R^{C_k}_{M(.,.,\dots,.),\lambda}[C_k(x)-\lambda A(x-p(x))+\lambda a+\lambda F(t_1,t_2,\dots,t_k)],$$

where $t_1\in T_1(x),t_2\in T_2(x),\dots,t_k\in T_k(x)$ and $\lambda>0$ is a real constant.

Proof. Let $\widehat{M}:E\rightrightarrows E^*$ be defined by $\widehat{M}(x):=M(f_1(x),f_2(x),\dots,f_k(x))$ for all $x\in E$. By Proposition 4.2, we have that $\widehat{M}$ is a strongly monotone mapping. By taking $H=C_k$ and using the assumption, we conclude that $\widehat{M}$ is a general strongly H-monotone mapping. Then, the conclusion follows from Theorem 3.1. □

In view of the above-mentioned argument, it is clear that the characterization, presented in Theorem 4.10, for the solution of the problem (4.4) is exactly the same characterization of the solution for the problem (3.3) given in Theorem 3.1.

Based on Theorem 4.10, Nazemi[19] suggested the following iterative algorithm for solving the problem (4.4).

Algorithm 4.12. [19, Algorithm 4.1] Let E be a reflexive Banach space. For any given $x^0 \in E$, we choose $t_1^0 \in T_1(x^0), t_2^0 \in T_2(x^0), \dots, t_k^0 \in T_k(x^0)$ and compute $\{x^n\}, \{t_1^n\}, \{t_2^n\}, \dots, \{t_k^n\}$ by iterative schemes

$$
\begin{aligned}
&x^{n+1} = R^{C_k}_{M(.,.,\dots,.),\lambda}[C_k(x^n) - \lambda A(x^n - p(x^n)) + \lambda a + \lambda F(t_1^n, t_2^n, \dots, t_k^n)],\\
&t_1^n \in T_1(x^n); \|t_1^{n+1} - t_1^n\| \leq (1 + \frac{1}{n+1})\hat{H}(T_1(x^{n+1}), T_1(x^n)),\\
&t_2^n \in T_2(x^n); \|t_2^{n+1} - t_2^n\| \leq (1 + \frac{1}{n+1})\hat{H}(T_2(x^{n+1}), T_2(x^n)),\\
&\vdots\\
&t_k^n \in T_k(x^n); \|t_k^{n+1} - t_k^n\| \leq (1 + \frac{1}{n+1})\hat{H}(T_k(x^{n+1}), T_k(x^n)),
\end{aligned}
$$

for all $n = 0, 1, 2, \dots$.

Remark 4.13. (a) By defining $\widehat{M} : E \rightrightarrows E^*$ as $\widehat{M}(x) := M(f_1(x), f_2(x), \dots, f_k(x))$ for all $x \in E$, and by taking $H = C_k$, it follows that $\widehat{M}$ is a general H-monotone mapping and we observe that Algorithm 4.12 coincides with Algorithm 3.4.

(b) In Lemma 3.1, Theorem 4.1 and Algorithm 4.1 of [19], the Banach space should be reflexive as we have considered in Lemma 4.7, Theorem 4.10 and Algorithm 4.12.

Theorem 4.14. [19, Theorem 4.2] *Let E be a uniformly smooth Banach space with $\rho_E(t) \leq Ct^2$ for some $C > 0$. Let $k \geq 3$ and $f_1, f_2, \dots, f_k : E \to E$ be single-valued mappings, $C_k : E \to E^*$ be a monotone and δ-Lipschitz continuous mapping, $p : E \to E$ be a (γ, μ)-relaxed cocoercive and λ_p-Lipschitz continuous mapping and $M : E^k \rightrightarrows E^*$ be a C_k-monotone mapping. Let $A : E \to E^*$ be a τ-Lipschitz continuous mapping and for each $i \in \{1, 2, \dots, k\}$, $T_i : E \rightrightarrows CB(E)$ be $\hat{H}$-Lipschitz continuous with constant λ_{T_i}. Suppose that $F : E^k \to E^*$ is λ_{F_i}-Lipschitz continuous in the ith argument with respect to T_i $(i = 1, 2, \dots, k)$ and there exists a constant $\lambda > 0$ such that the following conditions are satisfied:*

$$
\begin{cases}
\frac{1}{\lambda(\alpha_1+\alpha_3+\cdots+\alpha_{k-1}-(\beta_2+\beta_4+\cdots+\beta_k))}\big(\delta + \lambda\tau\sqrt{1 + 2\gamma\lambda_p^2 - 2\mu + 64C\lambda_p^2}\\
+\lambda \sum\limits_{i=1}^{k} \lambda_{F_i}\lambda_{T_i}\big) < 1, \ 2\mu < 1 + 2\gamma\lambda_p^2 + 64C\lambda_p^2.
\end{cases}
\tag{4.5}
$$

Then the iterative sequences $\{x^n\}, \{t_1^n\}, \{t_2^n\}, \dots, \{t_k^n\}$ generated by Algorithm 4.12 converge strongly to $x, t_1, t_2, \dots, t_k$, respectively, and $(x, t_1, t_2, \dots, t_k)$ is a solution of the problem (4.4).

Proof. Define $\widehat{M} : E \rightrightarrows E^*$ by $\widehat{M}(x) := M(f_1(x), f_2(x), \dots, f_k(x))$ for all $x \in E$. Since M is a C_k-monotone mapping, and $k > 3$ is an even natural number, it follows that M is $\alpha_1\beta_2\alpha_3\beta_4\ldots\alpha_{k-1}\beta_k$-symmetric monotone mapping with respect to $f_1, f_2, \ldots, f_k$, and Proposition 4.2 (a) implies that $\widehat{M}$ is a $\sum_{i=1}^{\frac{k}{2}}(\alpha_{2i-1}-\beta_{2i})$-strongly monotone mapping. At the same time, by taking $H = C_k$, we conclude that $\widehat{M}$ is a general strongly H-monotone mapping with constant $\sum_{i=1}^{\frac{k}{2}}(\alpha_{2i-1}-\beta_{2i})$ and Algorithm 4.12 coincides with Algorithm 3.4. Picking $\theta = \sum_{i=1}^{\frac{k}{2}}(\alpha_{2i-1}-\beta_{2i})$, (4.5) reduces to (3.17) in Corollary 3.10. Now, we observe that all the conditions of Corollary 3.10 hold, and hence, the iterative sequences $\{x_n\}$ and $\{t_i^n\}$ $(i = 1, 2, \ldots, k)$ generated by Algorithm 4.12 converge strongly to x and t_i $(i = 1, 2, \ldots, k)$, respectively, and $(x, t_1, t_2, \ldots, t_k)$ is a solution of the problem (4.4). $\square$

Remark 4.15. (a) Taking into account that M is a C_k-monotone mapping, by virtue of Definition 4.1, the constants α_i $(i = 1, 2, \ldots, k-1)$ and β_i $(i = 2, 4, \ldots, k)$ satisfy the condition $\sum_{i=1}^{\frac{k}{2}}(\alpha_{2i-1}-\beta_{2i}) > 0$, that is,

$$\sum_{i=1}^{\frac{k}{2}}\alpha_{2i-1} > \sum_{i=1}^{\frac{k}{2}}\beta_{2i}.$$

(b) By a careful reading Theorem 4.2 in [19], we found that there are two small mistakes in its context. Firstly, relying on the fact that the constants $\lambda, \delta, \tau, \gamma, \lambda_p, \mu, C, \lambda_{F_i}$ and λ_{T_i} $(i = 1, 2, \ldots, k)$ are all positive and

$$\sum_{i=1}^{\frac{k}{2}}\alpha_{2i-1} > \sum_{i=1}^{\frac{k}{2}}\beta_{2i},$$

it follows that

$$\frac{1}{\lambda\sum_{i=1}^{\frac{k}{2}}(\alpha_{2i-1}-\beta_{2i})}\big(\delta+\lambda\tau\sqrt{1+2\gamma\lambda_p^2-2\mu+64C\lambda_p^2}+\lambda\sum_{i=1}^{k}\lambda_{F_i}\lambda_{T_i}\big) > 0.$$

Hence, condition (3) in [19], that is, the condition

$$\begin{aligned}0 < {} & \frac{1}{\lambda(\alpha_1+\alpha_3+\cdots+\alpha_{k-1}-(\beta_2+\beta_4+\cdots+\beta_k))} \\ & \times \big(\delta+\lambda\tau\sqrt{1+2\gamma\lambda_p^2-2\mu+64C\lambda_p^2}+\lambda\sum_{i=1}^{k}\lambda_{F_i}\lambda_{T_i}\big) < 1\end{aligned}$$

must be replaced by condition (4.5) in Theorem 4.11. Secondly, the constants γ, λ_p, μ and C, in addition to condition (4.5), must be also satisfied the condition $2\mu < 1 + 2\gamma\lambda_p^2 + 64C\lambda_p^2$, as we have added the mentioned condition to the assumptions of Theorem 4.11.

(c) All the results in [19] have been derived based on the strict monotonicity assumption of the mapping $C_n : E \to E^*$, whereas, the aforesaid condition imposed on C_n can be replaced by a more mild condition. In fact, all definitions and results of Sections 3 and 4 of [19] have been rewritten in this section, by replacing the strict monotonicity assumption imposed on the mapping C_n in [19] by the monotonicity condition, which is more mild of the strict monotonicity condition, and then the corresponding results have been deduced by using the results given in Section 3.

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Received: December 8, 2021; Accepted: February 26, 2022.