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BOUNDARY VALUE PROBLEMS FOR SEQUENTIAL HILFER FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH INTEGRO-MULTISTRIP-MULTIPOINT BOUNDARY CONDITIONS

BASHIR AHMAD*, SOTIRIS K. NTOUYAS**,* AND FAWZIAH M. ALOTAIBI*

*Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail: bashirahmad_qau@yahoo.com, foozalotaibi_22@hotmail.com

**Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece
E-mail: sntouyas@uoi.gr

Abstract. We study a novel fractional model of boundary value problems in the setting of Hilfer fractional derivative operators. Precisely, sequential Hilfer fractional differential equations and inclusions with integro-multistrip-multi-point boundary conditions are considered. Existence and uniqueness results are established for the proposed problems by using the techniques of fixed point theory. In the single-valued case, the classical theorems due to Banach and Krasnosel'skiĭ are used, while the multi-valued case is investigated with the aid of Leray-Schauder nonlinear alternative for multi-valued maps, and Covitz and Nadler's fixed point theorem for multi-valued contractions. The obtained results are well-illustrated by numerical examples.

Key Words and Phrases: Fractional differential equation and inclusion, boundary value problem, fractional derivative, fractional integral, fixed point theorem, multi-valued map.
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1. Introduction

Fractional calculus and fractional differential equations have received considerable attention of many scientists in view of their extensive applications in the mathematical modelling of real world phenomena in a variety of fields such as physics, applied mathematics, control theory, etc. For a detailed account of the subject, we refer the reader to the books [2], [21], [26], [30] and the references cited therein. For application of fractional calculus to the other fields of science, for instance, see [11], [17], [22], [25], [28].

Differential equations and inclusions equipped with various types of boundary conditions have been widely investigated by many researchers, for instance, see the papers [1], [3], [4], [6], [7], [13], [27].

Keeping in mind the utility of fractional order operators, different forms of such operators were proposed, for example, see Kilbas et al. in [21]. Hilfer in [18] introduced a new fractional derivative operator as

$$^{H}D^{\alpha,\beta}u(t) = I^{\beta(n-\alpha)}D^{n}I^{(1-\beta)(n-\alpha)}u(t), n-1<\alpha<n, 0\leq\beta\leq 1, t>a, D=\frac{d}{dt},$$

where I^{ξ} denotes the Riemann-Liouville fractional integral of order ξ defined by

$$I^{\xi}u(t) = \frac{1}{\Gamma(\xi)}\int_a^t (t-s)^{\xi-1}u(s)ds.$$

Here ξ stands for $\beta(n-\alpha)$ or $(1-\beta)(n-\alpha))$. In passing, we remark that the Hilfer fractional derivative corresponds to the one due to Riemann-Liouville and Caputo for $\beta=0$ and $\beta=1$ respectively.

Many authors studied initial value problems involving Hilfer fractional derivatives, for example, see [12, 16, 30] and references therein. Some recent works on the nonlocal boundary value problems involving Hilfer fractional derivatives can be found in the articles [5, 31, 32].

In the present paper, our aim is to enrich the existing literature on nonlocal boundary value problems involving Hilfer fractional derivatives, by studying the nonlinear sequential fractional differential equation of the form:

$$\left(^{H}D^{\alpha,\beta} + k\ ^{H}D^{\alpha-1,\beta}\right)x(t) = f(t,x(t)), \quad t\in J:=[a,b], \quad a\geq 0, \tag{1.1}$$

supplemented with integro-multistrip-multipoint boundary conditions:

$$\begin{aligned} &x^{(i)}(a)=0, i=0,1,2,\ldots,n-2, \\ &\int_a^b x(s)ds = \sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} x(s)ds + \sum_{j=1}^{q}\mu_j\ x(\rho_j), \end{aligned} \tag{1.2}$$

where $^{H}D^{\alpha,\beta}$ denotes the fractional derivative operator of Hilfer type of order α, $n-1<\alpha\leq n$ with $n\geq 3$, and type β, $0\leq\beta\leq 1$, $f: J\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $a<\eta_1<\eta_2<\ldots<\eta_p<\rho_1<\rho_2<\ldots<\rho_q<b$, and $k,\lambda_i,\mu_j>0$, $i=2,3,\ldots,p$, $j=1,2,\ldots,n$ with $p,q\in\mathbb{N}$.

The multi-valued version of the problem (1.1)-(1.2) is also studied by considering the following inclusion problem:

$$\begin{cases} \left(^{H}D^{\alpha,\beta} + k\ ^{H}D^{\alpha-1,\beta}\right)x(t) \in F(t,x(t)), \quad t\in J:=[a,b], \\ x^{(i)}(a)=0, \ i=0,1,2,\ldots,n-2, \\ \displaystyle\int_a^b x(s)ds = \sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} x(s)ds + \sum_{j=1}^{q}\mu_j\ x(\rho_j), \end{cases} \tag{1.3}$$

where $F: J\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multi-valued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$).

We prove the existence results for the inclusion boundary value problem (1.3) by using Leray-Schauder nonlinear alternative for multi-valued maps and Covitz-Nadler fixed point theorem for multi-valued contractions.

The rest of the paper is composed as follows. Section 2 contains some preliminary material related to our study. In Section 3, we apply Krasnosel'skiĭ and Banach fixed point theorems to prove the existence and uniqueness of solutions for the boundary value problem (1.1)-(1.2). Section 4 is devoted to the study of the inclusion boundary value problem (1.3) by means of the standard fixed point theorems for multi-valued maps. The application of the obtained results is demonstrated by numerical examples to indicate their efficacy.

2. Preliminaries

Let $C(J,\mathbb{R})$ represent the Banach space of all continuous functions from J into $\mathbb{R}$ endowed with the norm $\|f\| = \sup\{|f(t)| : t \in J\}$, while $L^1(J,\mathbb{R})$ denotes the Banach space of functions $y : J \longrightarrow \mathbb{R}$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} = \int_0^1 |y(t)|dt.$$

In the sequel, we use the notation:

$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$,

$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and

$\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

Here $(X, \|\cdot\|)$ is a Banach space. For further details on multi-valued maps, see the books [10], [13] and [20].

The composition of Riemann-Liouville fractional integral operator with the Hilfer fractional derivative operator is presented in the following lemma.

Lemma 2.1. ([19]) *Let $f \in L(a,b)$, $n-1 < \alpha \le n, n \in \mathbb{N}$, $0 \le \beta \le 1$, $I^{(n-\alpha)(1-\beta)} f \in AC^k[a,b]$. Then*

$$\left(I^{\alpha}\ {}^{H}D^{\alpha,\beta} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)} \lim_{t\to a^+} \left(I^{(1-\beta)(n-\alpha)} f\right)(t).$$

3. Existence and uniqueness results for the problem (1.1)-(1.2)

We begin this section with an auxiliary lemma that plays a key role to transform the problem (1.1)-(1.2) into a fixed point problem.

Lemma 3.1. *Let $h \in C(J,\mathbb{R})$ and*

$$\Lambda := \frac{b-a}{\Gamma(\gamma+1)} - \sum_{i=2}^{p} \lambda_{i-1} \frac{[(\eta_i - a)^{\gamma} - (\eta_{i-1} - a)^{\gamma}]}{\Gamma(\gamma+1)} - \sum_{j=1}^{q} \mu_j \frac{(\rho_j - a)^{\gamma-1}}{\Gamma(\gamma)} \neq 0. \tag{3.1}$$

Then $x \in C(J,\mathbb{R})$ is a solution of the linear boundary value problem

$$\begin{cases} \left({}^{H}D^{\alpha,\beta} + k\ {}^{H}D^{\alpha-1,\beta}\right)x(t) = h(t), \quad t \in J := [a,b], \\ x^{(i)}(a) = 0,\ i = 0,1,2,\ldots,n-2, \\ \displaystyle\int_a^b x(s)ds = \sum_{i=2}^{p} \lambda_{i-1} \int_{\eta_{i-1}}^{\eta_i} x(s)ds + \sum_{j=1}^{q} \mu_j\ x(\rho_j), \end{cases} \tag{3.2}$$

if and only if

$$\begin{aligned} x(t) &= I^{\alpha}h(t) - k\int_a^t x(s)ds \\ &\quad + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ -\int_a^b I^{\alpha}h(s)ds + \sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^{\alpha}h(s)ds + \sum_{j=1}^{q}\mu_j I^{\alpha}h(\rho_j) \\ &\quad - k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds - k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds \\ &\quad + k\int_a^b\int_a^s x(u)duds\Bigg\}. \end{aligned} \tag{3.3}$$

Proof. Operating fractional integral I^{α} on both sides of equation (3.2) and using Lemma 2.1, we obtain

$$\begin{aligned} x(t) &= c_0\frac{(t-a)^{-(n-\alpha)(1-\beta)}}{\Gamma(1-(n-\alpha)(1-\beta))} + c_1\frac{(t-a)^{1-(n-\alpha)(1-\beta)}}{\Gamma(2-(n-\alpha)(1-\beta))} + I^{\alpha}h(t) \\ &\quad + c_3\frac{(t-a)^{2-(n-\alpha)(1-\beta)}}{\Gamma(3-(n-\alpha)(1-\beta))} + \ldots + c_{n-1}\frac{(t-a)^{n-1-(n-\alpha)(1-\beta)}}{\Gamma(n-(n-\alpha)(1-\beta))} + I^{\alpha}h(t) \\ &= c_0\frac{(t-a)^{\gamma-n}}{\Gamma(\gamma-(n-1))} + c_1\frac{(t-a)^{\gamma-(n-1)}}{\Gamma(\gamma-(n-2))} \\ &\quad + c_3\frac{(t-a)^{\gamma-(n-2)}}{\Gamma(\gamma-(n-3))} + \ldots + c_{n-1}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} - kI^1x(t) + I^{\alpha}h(t), \end{aligned} \tag{3.4}$$

where $\gamma = n-(n-\alpha)(1-\beta)$ and $c_i \in \mathbb{R}, i = 0, 1, \ldots, n-1$.
Using the boundary conditions $x^{(i)}(a) = 0, i = 0, 1, 2, \ldots, n-2$ in (3.4), we obtain $c_0 = c_1 = \ldots = c_{n-2} = 0$. Thus (3.4) reduces to

$$x(t) = c_{n-1}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} - kI^1x(t) + I^{\alpha}h(t), \tag{3.5}$$

which, together with the last boundary condition, yields

$$\begin{aligned} c_{n-1} &= \frac{1}{\Lambda}\Bigg\{ -\int_a^b I^{\alpha}h(s)ds + \sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^{\alpha}h(s)ds + \sum_{j=1}^{q}\mu_j I^{\alpha}h(\rho_j) \\ &\quad - k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds - k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds \\ &\quad + k\int_a^b\int_a^s x(u)duds\Bigg\}. \end{aligned}$$

Inserting the value of c_{n-1} in (3.5), we obtain the solution (3.3). We can obtain the converse of this lemma by direct computation. The proof is finished. □

Having in mind Lemma 3.1, we introduce an operator $\mathcal{G} : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ as follows

$$\begin{aligned}
&(\mathcal{G}x)(t) \\
= \;& I^{\alpha} f(t, x(t)) - k\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ -\int_a^b I^{\alpha} f(s, x(s))ds \\
&+\sum_{i=2}^{p} \lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^{\alpha} f(s, x(s))ds + \sum_{j=1}^{q} \mu_j I^{\alpha} f(\rho_j, x(\rho_j)) \qquad (3.6)\\
&-k\sum_{i=2}^{p} \lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds - k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds + k\int_a^b\int_a^s x(u)duds\Bigg\}.
\end{aligned}$$

Obviously the problem (1.1)-(1.2) is equivalent to the fixed point problem: $x = \mathcal{G}x$. Let us now introduce the following notations for the sake of computational convenience:

$$\begin{aligned}
Q \;=\;& \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{i=2}^{p}|\lambda_{i-1}|\frac{(\eta_i-a)^{\alpha+1}-(\eta_{i-1}-a)^{\alpha+1}}{\Gamma(\alpha+2)} \\
&+\sum_{j=1}^{q}|\mu_j|\frac{(\rho_j-a)^{\alpha}}{\Gamma(\alpha+1)}\Bigg], \qquad (3.7)
\end{aligned}$$

and

$$\begin{aligned}
Q_1 \;=\;& |k|(b-a) + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[|k|\sum_{j=1}^{q}|\mu_j|(\rho_j-a) \\
&+|k|\sum_{i=1}^{p}|\lambda_{i-1}|\frac{(\eta_i-a)^2-(\eta_{i-1}-a)^2}{2} + |k|\frac{(b-a)^2}{2}\Bigg]. \qquad (3.8)
\end{aligned}$$

3.1. **Existence result via Krasnosel'skiĭ fixed point theorem.** In the following result, we prove the existence of solutions for the problem (1.1)-(1.2) by means of Krasnosel'skiĭ fixed point theorem for a sum of two operators [23].

Theorem 3.2. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the conditions:*

(A_1) $|f(t, x) - f(t, y)| \le L|x - y|$, *for all* $t \in J$, $L > 0$, $x, y \in \mathbb{R}$;
(A_2) $|f(t, u)| \le \mu(t)$ *for all* $(t, u) \in J \times \mathbb{R}$, $\mu \in C(J, \mathbb{R}^+)$.

Then the sequential Hilfer fractional boundary value problem (1.1)-(1.2) has at least one solution on J provided that $LQ_1 < 1$, where Q_1 is given by (3.8).

Proof. By the assumption (A_2), (3.7) and (3.8), we fix $\overline{r} \ge (Q\|\mu\|)/(1 - Q_1)$ and consider a closed ball $B_{\overline{r}} = \{x \in C(J, \mathbb{R}) : \|x\| \le \overline{r}\}$. Next we define operators $\mathcal{G}_1$ and

$\mathcal{G}_2$ on $B_{\overline{r}}$ as follows

$$\begin{aligned}(\mathcal{G}_1x)(t) &= I^\alpha f(t,x(t)) + \frac{(t-a)^{\gamma-1}}{\Lambda}\bigg\{-\int_a^b I^\alpha f(s,x(s))ds \\ &\quad + \sum_{i=2}^p \lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^\alpha f(s,x(s))ds + \sum_{j=1}^q \mu_j I^\alpha f(\rho_j, x(\rho_j))\bigg\},\end{aligned}$$

$$\begin{aligned}(\mathcal{G}_2x)(t) &= -k\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\bigg\{-k\sum_{i=2}^p \lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds \\ &\quad -k\sum_{j=1}^q \mu_j\int_a^{\rho_j} x(s)ds + k\int_a^b\int_a^s x(u)duds\bigg\}.\end{aligned}$$

Notice that $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$. For $x, y \in B_{\overline{r}}$, we find that

$$\begin{aligned}&\|\mathcal{G}_1x + \mathcal{G}_2y\| = \sup_{t\in J}|\mathcal{G}_1x + \mathcal{G}_2y| \\ \leq\ & I^\alpha|f(s,x(s))| + \frac{(t-a)^{\gamma-1}}{|\Lambda|}\bigg[\int_a^b I^\alpha|f(s,x(s))|ds \\ & + \sum_{i=2}^p|\lambda_{i-1}|\int_{\eta_{i-1}}^{\eta_i} I^\alpha|f(s,x(s))|ds + \sum_{j=1}^q|\mu_j|I^\alpha|f(\rho_j,x(\rho_j))|\bigg] \\ & + |k|\int_a^t|y(s)|ds + \frac{(t-a)^{\gamma-1}}{|\Lambda|}\bigg\{|k|\sum_{j=1}^q\mu_j\int_a^{\rho_j}|y(s)|ds \\ & + |k|\sum_{i=2}^p|\lambda_{i-1}|\int_{\eta_{i-1}}^{\eta_i}\int_a^s|y(u)|duds + |k|\int_a^b\int_a^s|y(u)|duds\bigg\} \\ \leq\ & \|\mu\|\bigg\{\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\bigg[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} \\ & + \sum_{i=2}^p|\lambda_{i-1}|\frac{(\eta_i-a)^{\alpha+1}-(\eta_{i-1}-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{j=1}^q|\mu_j|\frac{(\rho_j-a)^\alpha}{\Gamma(\alpha+1)}\bigg]\bigg\} \\ & + \|x\|\bigg\{|k|(b-a) + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\bigg[|k|\sum_{j=1}^q|\mu_j|(\rho_j-a) \\ & + |k|\sum_{i=1}^p|\lambda_{i-1}|\frac{(\eta_i-a)^2-(\eta_{i-1}^2-a)}{2} + |k|\frac{(b-a)^2}{2}\bigg]\bigg\} \\ =\ & \|\mu\|Q + \overline{r}Q_1 \leq \overline{r}.\end{aligned}$$

This shows that $\mathcal{G}_1x + \mathcal{G}_2y \in B_{\overline{r}}$.
By (A_2) and the condition $LQ_1 < 1$, it can easily be shown that $\mathcal{G}_2$ is a contraction mapping.

The operator $\mathcal{G}_1$ is continuous, since f is continuous, and uniformly bounded on $B_{\overline{r}}$ as

$$\begin{aligned}\|\mathcal{G}_1 x\| \quad &\leq \quad \|\mu\|\Bigg\{\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\\ &\quad +\sum_{i=2}^{p}|\lambda_{i-1}|\frac{(\eta_i-a)^{\alpha+1}-(\eta_{i-1}-a)^{\alpha+1}}{\Gamma(\alpha+2)}+\sum_{j=1}^{q}|\mu_j|\frac{(\rho_j-a)^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\Bigg\}.\end{aligned}$$

The compactness of the operator $\mathcal{G}_1$ will be proved next. For $a \leq t_1 < t_2 \leq b$, we have

$$\begin{aligned}&|\mathcal{G}_1 x(t_2)-\mathcal{G}_1 x(t_1)|\\ &\leq \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]f(s,x(s))ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s))ds\right|\\ &+\frac{|(t_2-a)^{\gamma-1}-(t_1-a)^{\gamma-1}|}{|\Lambda|}\Bigg\{\int_a^b I^{\alpha}|f(s,x(s))|ds\\ &+\sum_{i=2}^{p}|\lambda_{i-1}|\int_{\eta_{i-1}}^{\eta_i} I^{\alpha}|f(s,x(s))|ds+\sum_{j=1}^{q}|\mu_j|I^{\alpha}|f(\rho_j,x(\rho_j))|\Bigg\}\\ &\leq \frac{\|\mu\|}{\Gamma(\alpha+1)}[|(t_2-a)^{\alpha}-(t_1-a)^{\alpha}|+2(t_2-t_1)^{\alpha}]\\ &+\frac{|(t_2-a)^{\gamma-1}-(t_1-a)^{\gamma-1}|}{|\Lambda|}\Bigg[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\\ &+\sum_{i=2}^{p}|\lambda_{i-1}|\frac{(\eta_i-a)^{\alpha+1}-(\eta_{i-1}-a)^{\alpha+1}}{\Gamma(\alpha+2)}+\sum_{j=1}^{q}|\mu_j|\frac{(\rho_j-a)^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\|\mu\|\to 0,\end{aligned}$$

as $t_1-t_2\to 0$, independent of x. Hence, by the Arzelá-Ascoli theorem, $\mathcal{G}_1$ is compact on $B_{\overline{r}}$. Thus all the assumptions of Krasnosel'skiĭ's fixed point theorem hold true and hence its conclusion implies that the problem (1.1)-(1.2) has at least one solution on J, which ends the proof. □

3.2. **Uniqueness result via Banach's fixed point theorem.**

Theorem 3.3. *Let $f: J\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the assumption (A_1). Then the sequential Hilfer fractional boundary value problem (1.1)-(1.2) has a unique solution on J if $LQ+Q_1<1$, where Q and Q_1 are given by (3.7) and (3.8) respectively.*

Proof. Let us first show that $\mathcal{G}B_r\subset B_r$, where $\mathcal{G}$ is the operator defined by (3.6) and $r\geq MQ/(1-LQ-Q_1)$ with $M=\sup_{t\in J}|f(t,0)|$ and $B_r=\{x\in C(J,\mathbb{R}):\|x\|\leq r\}$. By the assumption (A_1), we have that $|f(t,x)|\leq Lr+M$. Hence, for any $x\in B_r$, we find that

$$
\begin{aligned}
&\|\mathcal{G}x\| = \sup_{t\in J}|\mathcal{G}x(t)| \\
\leq\ & (Lr+M)\Bigg\{\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} \\
& +\sum_{i=2}^{p}|\lambda_{i-1}|\frac{(\eta_i-a)^{\alpha+1}-(\eta_{i-1}-a)^{\alpha+1}}{\Gamma(\alpha+2)}+\sum_{j=1}^{q}|\mu_j|\frac{(\rho_j-a)^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\Bigg\} \\
& +r\Bigg\{|k|(b-a)\frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[|k|\sum_{j=1}^{q}|\mu_j|(\rho_j-a) \\
& +|k|\sum_{i=1}^{p}|\lambda_{i-1}|\frac{(\eta_i-a)^2-(\eta_{i-1}-a)^2}{2}+|k|\frac{(b-a)^2}{2}\Bigg]\Bigg\} \\
=\ & (Lr+M)Q+rQ_1\leq r,
\end{aligned}
$$

which implies that $\mathcal{G}B_r\subset B_r$. Next, we will prove that $\mathcal{G}$ is a contraction. For $x,y\in C(J,\mathbb{R})$ and $t\in J$, we get

$$
\begin{aligned}
&|\mathcal{G}x(t)-\mathcal{G}y(t)| \\
&\leq \sup_{t\in J}\Bigg\{I^{\alpha}|f(t,x(t))-f(t,y(t))|+\Bigg[\frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{\int_a^b I^{\alpha}|f(s,x(s))-f(s,y(s))|ds \\
&\quad +\sum_{i=2}^{p}|\lambda_{i-1}|\int_{\eta_{i-1}}^{\eta_i} I^{\alpha}|f(s,x(s))-f(s,y(s))|\ ds \\
&\quad +\sum_{j=1}^{q}\mu_j I^{\alpha}|f(\rho_j,x(\rho_j))-f(\rho_j,y(\rho_j))|\Bigg\}\Bigg] \\
&\quad +|k|\int_a^t|x(s)-y(s)|ds+\frac{(t-a)^{\gamma-1}}{|\Lambda|}\Bigg\{|k|\sum_{j=1}^{q}\mu_j\int_a^{\rho_j}|x(s)-y(s)|ds \\
&\quad +|k|\sum_{i=2}^{p}|\lambda_{i-1}|\int_{\eta_{i-1}}^{\eta_i}\int_a^s|x(u)-y(u)|duds+|k|\int_a^b\int_a^s|x(u)-y(u)|duds\Bigg\} \\
&\leq L\Bigg\{\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} \\
&\quad +\sum_{i=2}^{p}|\lambda_{i-1}|\frac{(\eta_i-a)^{\alpha+1}-(\eta_{i-1}-a)^{\alpha+1}}{\Gamma(\alpha+2)}+\sum_{j=1}^{q}|\mu_j|\frac{(\rho_j-a)^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\Bigg\}\|x-y\| \\
&\quad +\Bigg\{|k|(b-a)\frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[|k|\sum_{j=1}^{q}|\mu_j|(\rho_j-a) \\
&\quad +|k|\sum_{i=1}^{p}|\lambda_{i-1}|\frac{(\eta_i-a)^2-(\eta_{i-1}-a)^2}{2}+|k|\frac{(b-a)^2}{2}\Bigg]\Bigg\}\|x-y\|=(LQ+Q_1)\|x-y\|.
\end{aligned}
$$

Therefore $\|\mathcal{G}x - \mathcal{G}y\| \leq (LQ + Q_1)\|x - y\|$. Since $LQ + Q_1 < 1$, the operator $\mathcal{G}$ is a contraction. Hence the operator $\mathcal{G}$ has a unique fixed point by contraction mapping principle. Consequently the sequential Hilfer fractional boundary value problem (1.1)-(1.2) has a unique solution on J. This completes the proof. □

Example 3.4. Consider the nonlinear sequential Hilfer fractional differential equation:

$$\left({}^H D^{7/2,1/2} + 1/2\ {}^H D^{5/2,1/2}\right)x(t) = \frac{1}{18} tan^{-1}x + e^t,\ t \in J := [0,1], \tag{3.9}$$

subject to integro-multistrip-multipoint boundary conditions:

$$\begin{aligned} & x(0) = x'(0) = x''(0) = 0, n = 4 \\ & \int_0^1 x(s)ds = \sum_{i=2}^{4} \lambda_{i-1} \int_{\eta_{i-1}}^{\eta_i} x(s)ds + \sum_{j=1}^{2} \mu_j\ x(\rho_j), \end{aligned} \tag{3.10}$$

where $\eta_1 = 1/12, \eta_2 = 1/4, \eta_3 = 1/3, \eta_4 = 5/12, \rho_1 = 7/12, \rho_2 = 2/3, \gamma = 15/4$. It is clear that $L = 1/18$ and (A_1) is satisfied. Let $\lambda_1 = \lambda_2 = \lambda_3 = \mu_1 = \mu_2 = 1$. Then we have $\Lambda = 12.374118, Q = 0.09427, Q_1 = 0.583335$, and $LQ+Q_1 = 0.588572 < 1$. Thus all the assumptions of Theorem 3.3 are satisfied and hence its conclusion implies that the problem (3.9)-(3.10) has a unique solution on $[0, 1]$.

Remark 3.5. Notice that the above example illustrates Theorem 3.2 as

$$|f(t,x)| \leq \mu(t) = e^t + \pi/36 \text{ and } LQ_1 = 0.583335/18 < 1.$$

4. Existence results for the problem (1.3)

Before stating and proving our main existence results for problem (1.3), we will give the definition of its solution.

Definition 4.1. A function $x \in AC^{(n-1)}(J, \mathbb{R})$ is said to be a solution of the problem (1.3) if there exists a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, x)$ a.e. on J such that

$$\begin{aligned} x(t) \quad = \quad & I^\alpha v(t) - k\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ - \int_a^b I^\alpha v(s)ds \\ & + \sum_{i=2}^{p} \lambda_{i-1} \int_{\eta_{i-1}}^{\eta_i} I^\alpha v(s)ds + \sum_{j=1}^{q} \mu_j I^\alpha v(\rho_j) - k\sum_{i=2}^{p} \lambda_{i-1} \int_{\eta_{i-1}}^{\eta_i} \int_a^s x(u)duds \\ & - k\sum_{j=1}^{q} \mu_j \int_a^{\rho_j} x(s)ds + k\int_a^b \int_a^s x(u)duds \Bigg\}. \end{aligned}$$

4.1. **The upper semicontinuous case.** Assuming that F has convex values and is L^1-Carathéodory, we prove an existence result for the inclusion problem (1.3) by applying Leray-Schauder nonlinear alternative for multi-valued maps [15].

The following lemma is used in forthcoming result.

Lemma 4.2. ([24]) *Let $F : [a,b] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([a,b],\mathbb{R})$ to $C([a,b],\mathbb{R})$. Then the operator*

$$\Theta \circ S_F : C([a,b],\mathbb{R}) \to \mathcal{P}_{cp,c}(C([a,b],\mathbb{R})),\ x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([a,b],\mathbb{R}) \times C([a,b],\mathbb{R})$.

Theorem 4.3. *Assume that $Q_1 < 1$ and the following conditions hold:*

- (H_1) *$F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;*
- (H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C(J,\mathbb{R}^+)$ such that*
  $\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|)$ for each $(t,x) \in J \times \mathbb{R}$;
- (H_3) *there exists a positive constant M satisfying*

$$\frac{(1-Q_1)M}{\psi(M)\|p\|Q} > 1,$$

where Q and Q_1 are given by (3.7) and (3.8) respectively.

Then the sequential Hilfer inclusion fractional boundary value problem (1.3) has at least one solution on J.

Proof. Introduce a multi-valued map: $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ as

$$N(x) = \left\{ \begin{array}{l} h \in C(J,\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} I^{\alpha}v(t) - k\displaystyle\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ -\displaystyle\int_a^b I^{\alpha}v(s)ds \\ +\displaystyle\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^{\alpha}v(s)ds + \sum_{j=1}^{q}\mu_j I^{\alpha}v(\rho_j) \\ -k\displaystyle\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds \\ -k\displaystyle\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds + k\int_a^b\int_a^s x(u)duds\Bigg\},\ v \in S_{F,x}, \end{array} \right. \end{array} \right\}$$

where $S_{F,x} = \{f \in L^1([a,b],\mathbb{R}) : f(t) \in F(t,x)\ for\ a.e.\ t \in [a,b]\}$. Notice that the fixed points of N are solutions of the problem (1.3). It will be shown in several steps that the map N satisfies the hypothesis of the Leray-Schauder nonlinear alternative for multi-valued maps [15].

Step 1. *For each $x \in C(J,\mathbb{R})$, $N(x)$ is convex.*

Let $h_1, h_2 \in N(x)$. Then there exist $v_1, v_2 \in S_{F,x}$ such that, for each $t \in J$, we have

$$\begin{aligned} h_i(t) = I^{\alpha}v_i(t) - k\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ -\int_a^b I^{\alpha}v_i(s)ds \\ + \sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^{\alpha}v_i(s)ds + \sum_{j=1}^{q}\mu_j I^{\alpha}v_i(\rho_j) - k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds \\ - k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds + k\int_a^b\int_a^s x(u)duds\Bigg\},\ i = 1, 2. \end{aligned}$$

Let $0 \le \theta \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} & [\theta h_1 + (1-\theta)h_2](t) \\ = \ & I^{\alpha}[\theta v_1(t) + (1-\theta)v_2(t)] - k\int_a^t x(s)ds \\ & + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ -\int_a^b I^{\alpha}[\theta v_1(s) + (1-\theta)v_2(s)]ds \\ & + \sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^{\alpha}[\theta v_1(s) + (1-\theta)v_2(s)]ds + \sum_{j=1}^{q}\mu_j I^{\alpha}[\theta v_1(\rho_j) + (1-\theta)v_2(\rho_j)] \\ & - k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds - k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds + k\int_a^b\int_a^s x(u)duds\Bigg\}. \end{aligned}$$

Since F has convex values, that is, $S_{F,x}$ is convex, we have $\theta h_1 + (1-\theta)h_2 \in N(x)$.

Step 2. *Bounded sets are mapped by N into bounded sets in $C(J, \mathbb{R})$.*

For a fixed $r > 0$, let $B_r = \{x \in C(J, \mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $C(J, \mathbb{R})$. Then, for each $h \in N(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) \ = \ & I^{\alpha}v(t) - k\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ -\int_a^b I^{\alpha}v(s)ds \\ & + \sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^{\alpha}v(s)ds + \sum_{j=1}^{q}\mu_j I^{\alpha}v(\rho_j) - k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds \\ & - k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds + k\int_a^b\int_a^s x(u)duds\Bigg\}. \end{aligned}$$

For $t \in J$, we obtain

$$\begin{aligned}
& |h(t)| \\
\leq\ & I^{\alpha}|v(t)| + \frac{(t-a)^{\gamma-1}}{|\Lambda|}\Bigg[\int_a^b I^{\alpha}|v(s)|ds + \sum_{i=2}^{p}|\lambda_{i-1}|\int_{\eta_{i-1}}^{\eta_i} I^{\alpha}|v(s)|ds \\
& + \sum_{j=1}^{q}|\mu_j|I^{\alpha}|v(\rho_j)|\Bigg] + |k|\int_a^t |x(s)|ds + \frac{(t-a)^{\gamma-1}}{|\Lambda|}\Bigg\{|k|\sum_{j=1}^{q}\mu_j\int_a^{\rho_j}|x(s)|ds \\
& +|k|\sum_{i=2}^{p}|\lambda_{i-1}|\int_{\eta_{i-1}}^{\eta_i}\int_a^s |x(u)|duds + |k|\int_a^b\int_a^s |x(u)|duds\Bigg\} \\
\leq\ & \|p\|\psi(\|x\|)\Bigg\{\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} \\
& + \sum_{i=2}^{p}|\lambda_{i-1}|\frac{(\eta_i-a)^{\alpha+1}-(\eta_{i-1}-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{j=1}^{q}|\mu_j|\frac{(\rho_j-a)^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\Bigg\} \\
& +\|x\|\Bigg\{|k|(b-a) + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[|k|\sum_{j=1}^{q}|\mu_j|(\rho_j-a) \\
& +|k|\sum_{i=1}^{p}|\lambda_{i-1}|\frac{(\eta_i-a)^2-(\eta_{i-1}-a)^2}{2} + |k|\frac{(b-a)^2}{2}\Bigg]\Bigg\} \\
\leq\ & \|p\|\psi(\|x\|)Q + \|x\|Q_1,
\end{aligned}$$

and consequently

$$\|h\| \leq \|p\|\psi(r)Q + rQ_1.$$

Step 3. *Bounded sets are mapped by N into equicontinuous sets of $C(J,\mathbb{R})$.*

Let $x \in B_r$ and $h \in N(x)$. Then there exists $v \in S_{F,x}$ such that, for each $t \in J$, we have

$$\begin{aligned}
h(t) &= I^{\alpha}v(t) - k\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{-\int_a^b I^{\alpha}v(s)ds \\
&\quad + \sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^{\alpha}v(s)ds + \sum_{j=1}^{q}\mu_j I^{\alpha}v(\rho_j) - k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds \\
&\quad -k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds + k\int_a^b\int_a^s x(u)duds\Bigg\}.
\end{aligned}$$

Let $t_1,\ t_2 \in J,\ t_1 < t_2$. Then

$$\begin{aligned}
& |h(t_2) - h(t_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left| \int_a^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]v(s)ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} v(s)ds \right| \\
& + |k|(t_2 - t_1) + \frac{|(t_2 - a)^{\gamma-1} - (t_1 - a)^{\gamma-1}|}{|\Lambda|}\Bigg\{ \int_a^b I^\alpha |v(s)|ds \\
& + \sum_{i=2}^{p} |\lambda_{i-1}| \int_{\eta_{i-1}}^{\eta_i} I^\alpha |v(s)|ds + \sum_{j=1}^{q} |\mu_j| I^\alpha |v(\rho_j)| \\
& + |k| \sum_{i=2}^{p} \lambda_{i-1} \int_{\eta_{i-1}}^{\eta_i} \int_a^s |x(u)|duds + |k| \sum_{j=1}^{q} \mu_j \int_a^{\rho_j} |x(s)|ds \\
& + |k| \int_a^b \int_a^s x(u)duds \Bigg\} \\
\leq\ & \frac{\|p\|\psi(r)}{\Gamma(\alpha)}\left| \int_a^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \right| + |k|r(t_2 - t_1) \\
& + \frac{|(t_2 - a)^{\gamma-1} - (t_1 - a)^{\gamma-1}|}{|\Lambda|}\Bigg[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} \\
& + \sum_{i=2}^{p} |\lambda_{i-1}| \frac{(\eta_i - a)^{\alpha+1} - (\eta_{i-1} - a)^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{j=1}^{q} |\mu_j| \frac{(\rho_j - a)^\alpha}{\Gamma(\alpha+1)} \\
& + |k| r \sum_{j=1}^{q} |\mu_j|(\rho_j - a) + |k| r \sum_{i=2}^{p} |\lambda_{i-1}| \frac{(\eta_i - a)^2 - (\eta_{i-1} - a)^2}{2} \\
& + |k| r \frac{(b-a)^2}{2} \Bigg] \to 0,
\end{aligned}$$

as $t_1 \to t_2$ independently of $x \in B_r$. It follows by Arzelá-Ascoli theorem that $N : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ is completely continuous.

Next, using the fact that a completely continuous operator is upper semicontinuous if it has a closed graph [10, Proposition 1.2], it will be shown that the operator N is upper semicontinuous. This will be established in the following step.

Step 4. *N has a closed graph.*

Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$. Then we show that $h_* \in N(x_*)$. Now $h_n \in N(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that, for each $t \in J$, we have

$$h_n(t) = I^\alpha v(t) - k \int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda} \Bigg\{ - \int_a^b I^\alpha v_n(s)ds$$

$$+\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^\alpha v_n(s)ds+\sum_{j=1}^{q}\mu_j I^\alpha v_n(\rho_j)-k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds$$
$$-k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds+k\int_a^b\int_a^s x(u)duds\Bigg\}.$$

For each $t\in J$, we must have $v_*\in S_{F,x_*}$ such that

$$\begin{aligned}h_*(t) &= I^\alpha v_*(t)-k\int_a^t x(s)ds+\frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{-\int_a^b I^\alpha v_*(s)ds\\ &+\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^\alpha v_*(s)ds+\sum_{j=1}^{q}\mu_j I^\alpha v_*(\rho_j)-k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds\\ &-k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds+k\int_a^b\int_a^s x(u)duds\Bigg\}.\end{aligned}$$

Consider the continuous linear operator $\Theta : L^1(J,\mathbb{R})\to C(J)$ by

$$\begin{aligned} & v\to\Theta(v)(t)\\ = & I^\alpha v(t)-k\int_a^t x(s)ds+\frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{-\int_a^b I^\alpha v(s)ds+\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^\alpha v(s)ds\\ & +\sum_{j=1}^{q}\mu_j I^\alpha v(\rho_j)-k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds\\ & -k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds+k\int_a^b\int_a^s x(u)duds\Bigg\}.\end{aligned}$$

Observe that $\|h_n-h_*\|\to 0$ as $n\to\infty$, and thus, it follows from Lemma 4.2, that $\Theta\circ S_{F,x}$ is a closed graph operator. Moreover, we have

$$h_n\in\Theta(S_{F,x_n}).$$

Since $x_n\to x_*$, Lemma 4.2 implies that

$$\begin{aligned}h_*(t) &= I^\alpha v_*(t)-k\int_a^t x(s)ds+\frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{-\int_a^b I^\alpha v_*(s)ds\\ &+\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^\alpha v_*(s)ds+\sum_{j=1}^{q}\mu_j I^\alpha v_*(\rho_j)-k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds\\ &-k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds+k\int_a^b\int_a^s x(u)duds\Bigg\},\end{aligned}$$

for some $v_*\in S_{F,x_*}$.

Step 5. *We show that there exists an open set $U\subseteq C(J,\mathbb{R})$ with $x\notin\theta N(x)$ for any $\theta\in(0,1)$ and all $x\in\partial U$.*

Let $x \in \theta N(x)$ for some $\theta \in (0,1)$. Then there exists $v \in L^1(J,\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in J$, we have

$$\begin{aligned} x(t) &= \theta I^\alpha v(t) - \theta k\int_a^t x(s)ds + \theta\frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ - \int_a^b I^\alpha v(s)ds \\ &\quad + \sum_{i=2}^p \lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^\alpha v(s)ds + \sum_{j=1}^q \mu_j I^\alpha v(\rho_j) - k\sum_{i=2}^p \lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds \\ &\quad - k\sum_{j=1}^q \mu_j\int_a^{\rho_j} x(s)ds + k\int_a^b\int_a^s x(u)duds\Bigg\}. \end{aligned}$$

Following the computation as in Step 2, for each $t \in J$, we have

$$\begin{aligned} |x(t)| &\le \|p\|\psi(\|x\|)\Bigg\{\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} \\ &\quad + \sum_{i=2}^p |\lambda_{i-1}|\frac{(\eta_i-a)^{\alpha+1} - (\eta_{i-1}-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{j=1}^q |\mu_j|\frac{(\rho_j-a)^\alpha}{\Gamma(\alpha+1)}\Bigg]\Bigg\} \\ &\quad + \|x\|\Bigg\{|k|(b-a)\frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[|k|\sum_{j=1}^q |\mu_j|(\rho_j-a) \\ &\quad + |k|\sum_{i=1}^p |\lambda_{i-1}|\frac{(\eta_i-a)^2 - (\eta_{i-1}-a)^2}{2} + |k|\frac{(b-a)^2}{2}\Bigg]\Bigg\} \\ &= \|p\|\psi(\|x\|)Q + \|x\|Q_1. \end{aligned}$$

In consequence, we get

$$\frac{(1-Q_1)\|x\|}{\psi(\|x\|)\|p\|Q} \le 1.$$

By the assumption (H_3), we can find M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C(J,\mathbb{R}) : \|x\| < M\}.$$

Note that $N : \overline{U} \to \mathcal{P}(C([a,b],\mathbb{R}))$ is a compact, upper semicontinuous multi-valued map with convex closed values, and there is no $x \in \partial U$ such that $x \in \theta N(x)$ for some $\theta \in (0,1)$, from the choice of U. Hence, by the Leray-Schauder nonlinear alternative for multi-valued maps [15], we deduce that N has a fixed point $x \in \overline{U}$ which is a solution of the sequential Hilfer inclusion fractional boundary value problem (1.3). This completes the proof. □

Example 4.4. Consider the sequential Hilfer fractional differential inclusion:

$$\left({}^HD^{7/2,1/2} + 1/2\ {}^HD^{5/2,1/2}\right)x(t) \in F(t,x),\ t \in J := [0,1], \tag{4.1}$$

subject to integro-multistrip-multipoint boundary conditions (3.10), where

$$F(t,x) = \Big[\frac{1}{t+5}\Big(\frac{x^2}{10(1+x^2)}+x+\frac{1}{15}e^{-x^2}+\frac{1}{10}\Big), \\ (1+2e^{-t})\Big(\sin x+\frac{1}{2\pi}arc\tan(x)+\frac{1}{2}\Big)\Big]. \tag{4.2}$$

It is easy to find that $p(t)=1+2e^{-t}, \|p\|=3, \psi(\|x\|)=\|x\|+3/4$. Using the values $Q=0.09427, Q_1=0.583335$ obtained in Example 3.4, we find from the condition (H_2) that $M>M_1\approx 1.584606$. Clearly all the assumptions of Theorem 4.3 are satisfied and hence the sequential Hilfer fractional differential inclusion (4.1) supplemented with the boundary conditions (3.10) has a solution on $[0,1]$.

4.2. **The Lipschitz case.** Here we prove an existence result for the problem (1.3) with a non-convex valued multi-valued map via a fixed point theorem for multivalued maps due to Covitz and Nadler [9].

Theorem 4.5. *Assume that the following conditions hold:*

- (H_4) $F: J\times\mathbb{R}\to\mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot,x): J\to\mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x\in\mathbb{R}$*;*
- (H_5) $H_d(F(t,x),F(t,\bar{x}))\le m(t)|x-\bar{x}|$ *for almost all* $t\in J$ *and* $x,\bar{x}\in\mathbb{R}$ *with* $m\in C(J,\mathbb{R}^+)$ *and* $d(0,F(t,0))\le m(t)$ *for almost all* $t\in J$.

Then the sequential Hilfer inclusion fractional boundary value problem (1.3) has at least one solution on J *if*

$$Q\|m\|+Q_1<1,$$

where Q *and* Q_1 *are given by (3.7) and (3.8) respectively.*

Proof. We verify that the operator $N: C(J,\mathbb{R})\to\mathcal{P}(C(J,\mathbb{R}))$, defined at the beginning of the proof of Theorem 4.3, satisfies the hypothesis of Covitz and Nadler fixed point theorem [9].

Step I. N is nonempty and closed for every $v\in S_{F,x}$.

It follows by the measurable selection theorem (e.g., [8, Theorem III.6]) that the set-valued map $F(\cdot,x(\cdot))$ is measurable and hence it admits a measurable selection $v: J\to\mathbb{R}$. In view of the assumption (H_5), we get $|v(t)|\le m(t)+m(t)|x(t)|$, that is, $v\in L^1(J,\mathbb{R})$ and hence F is integrably bounded. In consequence, we deduce that $S_{F,x}\neq\emptyset$.

Now we show that $N(x)\in\mathcal{P}_{cl}(C(J,\ \mathbb{R}))$ for each $x\in C(J,\mathbb{R})$. For that, let $\{u_n\}_{n\ge 0}\in N(x)$ be such that $u_n\to u$ $(n\to\infty)$ in $C(J,\mathbb{R})$. Then $u\in C(J,\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in J$,

$$u_n(t) = I^\alpha v_n(t) - k\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ -\int_a^b I^\alpha v_n(s)ds + \sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^\alpha v_n(s)ds + \sum_{j=1}^{q}\mu_j I^\alpha v_n(\rho_j) - k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds - k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds + k\int_a^b\int_a^s x(u)duds\Bigg\}.$$

Since F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus $v \in S_{F,x}$ and for each $t \in J$, we have

$$\begin{aligned} & u_n(t) \to v(t) \\ = \ & I^\alpha v(t) - k\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ -\int_a^b I^\alpha v(s)ds + \sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^\alpha v(s)ds \\ & + \sum_{j=1}^{q}\mu_j I^\alpha v(\rho_j) - k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds \\ & -k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds + k\int_a^b\int_a^s x(u)duds\Bigg\}. \end{aligned}$$

Thus $u \in N(x)$.

Step II. We show that there exists $0 < \widehat{\theta} < 1$ ($\widehat{\theta} = Q\|m\| + Q_1$) such that

$$H_d(N(x), N(\bar{x})) \le \theta\|x - \bar{x}\| \text{ for each } x, \bar{x} \in AC^{(n-1)}(J,\mathbb{R}).$$

Let $x, \bar{x} \in AC^{(n-1)}(J,\mathbb{R})$ and $h_1 \in N(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in J$,

$$h_1(t) = I^\alpha v_1(t) - k\int_a^t x(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ -\int_a^b I^\alpha v_1(s)ds + \sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^\alpha v_1(s)ds + \sum_{j=1}^{q}\mu_j I^\alpha v_1(\rho_j) - k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s x(u)duds - k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j} x(s)ds + k\int_a^b\int_a^s x(u)duds\Bigg\}.$$

By (H_5), we have

$$H_d(F(t,x), F(t,\bar{x})) \le m(t)|x(t) - \bar{x}(t)|.$$

So there exists $w(t) \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \le m(t)|x(t) - \bar{x}(t)|, \ t \in J.$$

Let $V : J \to \mathcal{P}(\mathbb{R})$ by

$$V(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|\}.$$

There exists a function $v_2(t)$ which is a measurable selection for V, since the multivalued operator $V(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [8]). Hence $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in J$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.
For each $t \in J$, let us define

$$\begin{aligned} h_2(t) = I^\alpha v_2(t) - k\int_a^t \bar{x}(s)ds + \frac{(t-a)^{\gamma-1}}{\Lambda}\Bigg\{ -\int_a^b I^\alpha v_2(s)ds \\ + \sum_{i=2}^{p} \lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i} I^\alpha v_2(s)ds + \sum_{j=1}^{q}\mu_j I^\alpha v_2(\rho_j) - k\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s \bar{x}(u)duds \\ - k\sum_{j=1}^{q}\mu_j\int_a^{\rho_j}\bar{x}(s)ds + k\int_a^b\int_a^s \bar{x}(u)duds\Bigg\}. \end{aligned}$$

Thus

$$\begin{aligned} & |h_1(t) - h_2(t)| \\ \leq\ & I^\alpha|v_2(s) - v_1(s)|ds + |k|(b-a)|x(s) - \bar{x}(s)| \\ & + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg\{\int_a^b I^\alpha|v_2(s) - v_1(s)|ds \\ & + \sum_{i=2}^{p}|\lambda_{i-1}|\int_{\eta_{i-1}}^{\eta_i} I^\alpha|v_2(s) - v_1(s)|ds + \sum_{j=1}^{q}|\mu_j| I^\alpha|v_2(\rho_j) - v_1(\rho_j)| \\ & + |k|\sum_{i=2}^{p}\lambda_{i-1}\int_{\eta_{i-1}}^{\eta_i}\int_a^s |x(s) - \bar{x}(s)|duds + |k|\sum_{j=1}^{q}\mu_j\int_a^{\rho_j}|x(s) - \bar{x}(s)|ds \\ & + |k|\int_a^b\int_a^s |x(s) - \bar{x}(s)|duds\Bigg\} \\ \leq\ & \Bigg\{\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{i=2}^{p}|\lambda_{i-1}|\frac{(\eta_i - a)^{\alpha+1} - (\eta_{i-1} - a)^{\alpha+1}}{\Gamma(\alpha+2)} \\ & + \sum_{j=1}^{q}|\mu_j|\frac{(\rho_j - a)^\alpha}{\Gamma(\alpha+1)}\Bigg]\Bigg\}\|m\|\|x - \bar{x}\| + \Bigg\{|k|(b-a) \\ & + \frac{(b-a)^{\gamma-1}}{|\Lambda|}\Bigg[|k|\sum_{j=1}^{q}|\mu_j|(\rho_j - a) + |k|\sum_{i=1}^{p}|\lambda_{i-1}|\frac{(\eta_i - a)^2 - (\eta_{i-1} - a)^2}{2} \\ & + |k|\frac{(b-a)^2}{2}\Bigg]\Bigg\}\|x - \bar{x}\|. \end{aligned}$$

Hence

$$\|h_1 - h_2\| \leq (Q\|m\| + Q_1)\|x - \bar{x}\|.$$

By interchanging the roles of x and $\bar{x}$, we obtain

$$H_d(N(x), N(\bar{x})) \leq (Q\|m\| + Q_1)\|x - \bar{x}\|.$$

Hence N is a contraction. Consequently, by Covitz and Nadler fixed point theorem [9], the operator N has a fixed point x which is a solution of the sequential Hilfer inclusion fractional boundary value problem (1.3). This completes the proof. □

Example 4.6. Consider the sequential Hilfer fractional differential inclusion:

$$\left({}^H D^{7/2,1/2} + 1/2 \; {}^H D^{5/2,1/2}\right) x(t) \in F(t, x), \;\; t \in J := [0,1], \tag{4.3}$$

subject to integro-multistrip-multipoint boundary conditions (3.10), where

$$F(t, x(t)) = \left[\frac{1 + \tan^{-1} x}{81 + t^2}, \left(\frac{\sqrt{t+3}}{30}\right)\left(\frac{1 + 6|x|}{1 + 5|x|}\right)\right]. \tag{4.4}$$

Clearly F is measurable for all $x \in \mathbb{R}$ and that

$$H_d(F(t, x), F(t, \bar{x})) \leq \left(\frac{\sqrt{t+3}}{30}\right)|x - \bar{x}|, \; x, \bar{x} \in \mathbb{R}, \; t \in [0,1].$$

Here $m(t) = \sqrt{t+3}/30$ with $\|m\| = 1/15$ and $d(0, F(t, 0)) \leq m(t)$, $t \in [0,1]$. With the given data, it is found that $Q\|m\| + Q_1 \approx 0.589620 < 1$. Thus all the conditions of Theorem 4.5 are satisfied and consequently there exists a solution for the inclusion (4.3) complemented with the boundary conditions (3.10) on $[0,1]$.

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