Fixed Point Theory, 23(2022), No. 2, 689-706
DOI: 10.24193/fpt-ro.2022.2.16
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SETS WITH STRUCTURE, MAPPINGS AND FIXED POINT PROPERTY: FIXED POINT STRUCTURES

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Abstract. In the book, *Fixed Point Structure Theory* (I.A. Rus, *Fixed Point Structure Theory*, Cluj Univ. Press, Cluj-Napoca, 2006) there are studied fixed point structures on a set with structure. In this paper we introduce the notion of the set-mapping pair $(\mathcal{U}, M)$ (i.e., $\mathcal{U}$:= a class of sets with the same type structure and for $X, Y \in \mathcal{U}$ a set of mappings, $M(X, Y)$, from X to Y is given) and the notion of fixed point structure (f.p.s.) on a such pair. After some examples of f.p.s. we study the preserving of the fixed point property by, $(\mathcal{U}, M)$-bijections, retractions, cartesian product and exponential. We give some fixed point results in terms of a f.p.s. and we consider some special f.p.s.: f.p.s. with common fixed point property, with coincidence point property and with coincidence producing mappings. Some open problems are also formulated.

Key Words and Phrases: Set with structure, set-mapping pair, category of sets with structure, fixed point structure, retraction, coretraction, coincidence point property, common fixed point property, coincidence producing mapping, cartesian product, exponential.

2020 Mathematics Subject Classification: 47H10, 54H25, 06xx, 18Axx, 54Cxx, 55M20, 58J20, 03xx, 68xx.

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Received: June 10, 2021; Accepted: January 31, 2022.