

## $(q_1, q_2)$ -QUASIMETRIC SPACES. COVERING MAPPINGS AND COINCIDENCE POINTS. A REVIEW OF THE RESULTS

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**Abstract.** In their recent papers, A.V. Arutyunov and A.V. Greshnov introduced  $(q_1, q_2)$ -quasimetric spaces and studied their properties: investigated covering mappings between  $(q_1, q_2)$ -quasimetric spaces, established sufficient conditions for the existence of a coincidence point for two mappings acting between  $(q_1, q_2)$ -quasimetric spaces such that one is a covering mapping and the other is Lipschitz continuous, proved Banach's fixed point theorem, obtained generalizations for multivalued mappings. The class of  $(q_1, q_2)$ -quasimetric spaces is sufficiently wide; it includes quasimetric spaces, b-metric spaces, Carnot-Carathéodory spaces with Box-quasimetrics,  $L_p$ -spaces with  $p \in (0, 1)$ , etc. The development of the theory of coincidence points of mappings on  $(q_1, q_2)$ -quasimetric spaces initiated interest in the study of more general  $f$ -quasimetric spaces and in generalizing Banach's fixed point theorem to such spaces. The present paper is a review of these results.  
**Key Words and Phrases:**  $(q_1, q_2)$ -quasimetric space, covering mapping, coincidence points, Lipschitz mapping, contraction mapping, fixed point, multivalued mapping, Hausdorff deviation,  $f$ -quasimetric space.

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