

A NOVEL HYBRID METHOD FOR EQUILIBRIUM PROBLEM AND A COUNTABLE FAMILY OF GENERALIZED NONEXPANSIVE-TYPE MAPS WITH APPLICATIONS

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Abstract. Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space E with dual space E^* . A novel hybrid method for finding a solution of an equilibrium problem and a common element of fixed points for a family of a general class of nonlinear nonexpansive maps is constructed. The sequence of the method is proved to converge strongly to a common element of the family and a solution of the equilibrium problem. Finally, an application of our theorem complements, generalizes and extends some recent important results (*Takahashi et al.*, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.*, 341 (2008), 276-286., *Nakajo and Takahashi*, Strong convergence theorems for nonexpansive mappings and nonexpansive semi-groups, *J. Math. Anal. Appl.* vol. 279 (2003), 372-379., *Qin and Su*, Strong convergence of monotone hybrid method for fixed point iteration process, *J. Syst. Sci. and Complexity* **21** (2008) 474-482., *Klin-eam et al.*, Hybrid method for the equilibrium problem and a family of generalized nonexpansive mappings in Banach spaces, *J. Nonlinear Sci. Appl.* 9 (2016), 4963-4975).

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