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ALTERNATIVE CHARACTERIZATIONS OF AGIFSs HAVING ATTRACTOR

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Abstract. In this paper we study affine generalized iterated function systems (for short AGIFSs) which are particular cases of the concept of generalized iterated function system introduced by R. Miculescu and A. Mihail. Using a technique introduced by F. Strobin and J. Swaczyna, we associate to each $n \in \mathbb{N}^*$ and each AGIFS $\mathcal{F}$ a new AGIFS $\mathcal{F}_n$. Our main result states that the following statements are equivalent: a) $\mathcal{F}$ has attractor. b) There exists $n \in \mathbb{N}^*$ such that $\mathcal{F}_n$ has attractor. c) There exists $n \in \mathbb{N}^*$ such that $\mathcal{F}_n$ is hyperbolic. d) There exists $n \in \mathbb{N}^*$ such that $\mathcal{F}_n$ is topologically contractive.
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1. Introduction

The concept of *generalized iterated function system* (abbreviated GIFS) was introduced by R. Miculescu and A. Mihail in [8] and [9] as part of the effort to extend Hutchinson's classical theory of iterated function systems (IFS). More precisely, a GIFS of order m consists on a finite family of functions $f_1, ..., f_n : X^m \to X$, where (X, d) is a metric space. Under certain conditions, Miculescu and Mihail proved the existence and uniqueness of the attractor of a GIFS and studied its properties (an upper bound for the Hausdorff-Pompeiu distance between the attractors of two such GIFSs, an upper bound for the Hausdorff-Pompeiu distance between the attractor of such a GIFS and an arbitrary compact set of X and the continuous dependence of the attractor in the f_i). The concept of GIFS is a effective generalization of the one of IFS since there exists a set which is the attractor of a GIFS but there exists no IFS having it as attractor (see [8]). Moreover, in [12], F. Strobin proved that for any $m \geq 2$, there exists a Cantor subset of the plane which is an attractor of some GIFS of order m, but is not an attractor of a GIFS of order $m-1$. In [2], algorithms allowing to generate images of attractors of GIFSs are presented. As the main ingredients used in [8] and [9] are particular cases of the fixed point for φ-contractions, F. Strobin and J. Swaczyna (see [13]) extended these results for the more general setting of generalized φ-contractions. N. Secelean (see [10]) studied countable iterated function systems consisting of generalized contraction mappings on the product space X^I into X, where $I \subseteq \mathbb{N}$. A concept of code space for GIFSs was introduced by Strobin and Swaczyna in [14]. It was used to treat the problem of connectedness of the attractor of a GIFS. In [7], R. Miculescu and A. Mihail proved the existence of an analogue of Hutchinson measure associated with a GIFS with probabilities and presented some of its properties. A similar study can be found in [4] for generalized iterated function systems with place dependent probabilities and in [11] for countable iterated function systems with probabilities.

A. Kameyama (see [3]) introduced the concept of self-similar topological system and raised the following question: given a topological self-similar system $(K,(f_i)_{i\in\{1,2,...,N\}})$, does there exist a metric on K comparable to the topology such that all the functions f_i are contractions? R. Atkins, M. Barnsley, A. Vince and D. Wilson (see [1]) provided an affirmative answer to Kameyama's question for *self-similar sets derived from affine transformations* on $\mathbb{R}^m$. R. Miculescu and A. Mihail (see [5]) extended this result by replacing $\mathbb{R}^m$ with an arbitrary Banach space $(X, \|.\|)$ and the set $\{1, 2, ..., N\}$ with an arbitrary set I. See also [6].

In this paper we mix the above two emphasized notions by introducing the notion of affine generalized iterated function system (for short AGIFS). Our main concern is to provide alternative characterizations for such a system having attractor. With this purpose in view, using the techniques used in [14], we associate to each $n \in \mathbb{N}^*$ and each AGIFS $\mathcal{F}$ a new AGIFS $\mathcal{F}_n$ and prove that $\mathcal{F}$ has attractor if and only if there exists $n \in \mathbb{N}^*$ such that $\mathcal{F}_n$ has attractor if and only if there exists $n \in \mathbb{N}^*$ such that $\mathcal{F}_n$ is hyperbolic if and only if there exists $n \in \mathbb{N}^*$ such that $\mathcal{F}_n$ is topologically contractive.

2. Preliminaries

The Hausdorff-Pompeiu metric

Let us consider a metric space (X, d).

By $\mathbb{K}(X)$ we mean the metric space of nonempty compact subsets of X endowed with the Hausdorff-Pompeiu metric described by

$$h(A,B) = \max\{d(A,B), d(B,A)\},$$

for all $A, B \in \mathbb{K}(X)$, where $d(A,B) = \sup\limits_{x\in A}(\inf\limits_{y\in B} d(x,y))$.

We recall that $(\mathbb{K}(X), h)$ is complete provided that (X, d) is complete.

Proposition 2.1. *If $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are families of elements of $\mathbb{K}(X)$ such that $\underset{i\in I}{\cup} A_i, \underset{i\in I}{\cup} B_i \in \mathbb{K}(X)$, then*

$$h\left(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\right) \le \sup_{i\in I} h(A_i, B_i).$$

Proposition 2.2. *(see Lemma* 2.8 *from* [5]*). If the norms* $\|.\|$ *and* $|||.|||$ *on* $\mathbb{R}^m$ *are equivalent (i.e. there exist* $\alpha, \beta > 0$ *such that* $\alpha \|.\| \leq |||.||| \leq \beta \|.\|$*), then* $h_{\|.\|}$ *is equivalent with* $h_{|||.|||}$ *(i.e.* $\alpha h_{\|.\|} \leq h_{|||.|||} \leq \beta h_{\|.\|}$*), where by* $h_{\|.\|}$ *we mean the Hausdorff-Pompeiu metric associated to the metric induced by* $\|.\|$.

Generalized iterated function systems

Let us consider a metric space (X, d) and $p \in \mathbb{N}^*$.

By X^p we denote the Cartesian product of X by itself p times endowed with the metric $d_{\max}$ described by

$$d_{\max}((x_1, ..., x_p), (y_1, ..., y_p)) = \max\{d(x_1, y_1), ..., d(x_p, y_p)\},$$

for all $(x_1, ..., x_p), (y_1, ..., y_p) \in X^p$.

Definition 2.3. A generalized iterated function system (of order p) is a pair

$$\mathcal{F} = ((X, d), (f_i)_{i \in \{1,2,...,N\}}),$$

where $p, N \in \mathbb{N}^*$ and $f_i : X^p \to X$ is continuous for each $i \in \{1, 2, ..., N\}$.

The function $\mathcal{F}_{\mathcal{F}} : (\mathbb{K}(X))^p \to \mathbb{K}(X)$ described by

$$\mathcal{F}_{\mathcal{F}}(B_1, ..., B_p) = \bigcup_{i \in \{1,2,...,N\}} f_i(B_1, ..., B_p),$$

for all $(B_1, ..., B_p) \in (\mathbb{K}(X))^p$ is called the fractal operator associated to $\mathcal{F}$.

We shall use the abbreviation GIFS for a generalized iterated function system.

Remark 2.4. For $p = 1$ we get the concept of iterated function system.

The Strobin-Swaczyna generalized code space

For $p, N \in \mathbb{N}^*$ we define inductively the sets Ω_1, Ω_2, ..., Ω_k, ... in the following way:

$$\Omega_1 = \{1, 2, ..., N\} \text{ and } \Omega_{k+1} = \underbrace{\Omega_k \times \Omega_k \times ... \times \Omega_k}_{p \text{ times}}$$

for every $k \in \mathbb{N}^*$.

We also consider the sets

$$\Omega = \Omega_1 \times \Omega_2 \times ... \times \Omega_k \times ... \text{ and } {}_k\Omega = \Omega_1 \times \Omega_2 \times ... \times \Omega_k,$$

where $k \in \mathbb{N}^*$.

For $i \in \{1, 2, ..., p\}$, $k \in \mathbb{N}$, $k \geq 2$ and $\alpha = \alpha^1 \alpha^2 ... \alpha^k \in {}_k\Omega$, where $\alpha^2 = \alpha_1^2 \alpha_2^2 ... \alpha_p^2 \in \Omega_2$,..., $\alpha^k = \alpha_1^k \alpha_2^k ... \alpha_p^k \in \Omega_k$, we consider

$$\alpha(i) = \alpha_i^2 \alpha_i^3 ... \alpha_i^k \in {}_{k-1}\Omega.$$

The GIFS $\mathcal{F}_n$ associated to a GIFS $\mathcal{F}$ and to $n \in \mathbb{N}^*$

Given a metric space (X,d) and $p \in \mathbb{N}^*$, we define inductively the spaces X_1, X_2, ..., X_k, ... in the following way:

$$X_1 = \underbrace{X \times X \times ... \times X}_{p \text{ times}} \text{ and } X_{k+1} = \underbrace{X_k \times X_k \times ... \times X_k}_{p \text{ times}}$$

for every $k \in \mathbb{N}^*$. We endow X_k with the maximum metric for every $k \in \mathbb{N}^*$. Note that X_k is isometric to X^{p^k} with the maximum metric for every $k \in \mathbb{N}^*$.

In case that $(X,d) = (\mathbb{K}(X), h)$, we denote X_k by $\mathbb{X}_k$.

For a generalized iterated function system (of order p) $\mathcal{F} = ((X,d),(f_i)_{i\in\{1,2,...,N\}})$ we define inductively a family of functions $\{f_\alpha : X_k \to X \mid \alpha \in {}_k\Omega\}$ for every $k \in \mathbb{N}^*$ in the following way:

For $k = 1$, the family is $\{f_1, f_2, ..., f_N\}$.

If the functions f_α, where $\alpha \in_k \Omega$, have been defined, then, for

$$\alpha = \alpha^1\alpha^2...\alpha^k\alpha^{k+1} \in_{k+1} \Omega,$$

where $\alpha^1 \in \Omega_1$, $\alpha^2 \in \Omega_2$, ...$\alpha^k \in \Omega_k$, $\alpha^{k+1} \in \Omega_{k+1}$, we define

$$f_\alpha(x_1, x_2, ..., x_p) = f_{\alpha^1}(f_{\alpha(1)}(x_1), ..., f_{\alpha(p)}(x_p)),$$

for every $(x_1, x_2, ..., x_p) \in X_{k+1} = \underbrace{X_k \times X_k \times ... \times X_k}_{p \text{ times}}$.

Note that if $p = 1$, then ${}_k\Omega = \{1, 2, ..., N\}^k$ and if $\alpha = \alpha^1\alpha^2...\alpha^k \in_k \Omega$, then $f_\alpha = f_{\alpha^1} \circ ... \circ f_{\alpha^k}$, so the introduced families of functions are natural generalizations of compositions of functions.

For a given $n \in \mathbb{N}^*$, we introduce a new GIFS (of order p^n) given by

$$\mathcal{F}_n := ((X,d),(f_\alpha)_{\alpha\in_n\Omega}).$$

Note that $\mathcal{F}_{\mathcal{F}_n} : (\mathbb{K}(X))^{p^n} \to \mathbb{K}(X)$ is given

$$\mathcal{F}_{\mathcal{F}_n}(B_1, ..., B_p) = \bigcup_{\alpha\in_n\Omega} f_\alpha(B_1, ..., B_p),$$

for all $(B_1, ..., B_p) \in \mathbb{X}_n$.

Fixed points for functions $f : X^p \to X$

Let us consider a metric space (X,d), $p \in \mathbb{N}^*$ and $f : X^p \to X$.

For $k \in \mathbb{N}^*$ we define inductively a family of functions $f^{[k]} : X^{p^k} \to X$ in the following way: $f^{[1]} = f$; $f^{[2]}(x_1, ..., x_p) = f(f(x_1), ..., f(x_p))$ for every $(x_1, ..., x_p) \in \underbrace{X^p \times ... \times X^p}_{p \text{ times}} = X^{p^2}$; assuming that we have defined $f^{[k]}$, then

$$f^{[k+1]}(x_1, ..., x_p) = f(f^{[k]}(x_1), ..., f^{[k]}(x_p)),$$

for every $(x_1, ..., x_p) \in \underbrace{X^{p^k} \times ... \times X^{p^k}}_{p \text{ times}} = X^{p^{k+1}}$. Note that for $p = 1$, we have

$$f^{[k]} = \underbrace{f \circ ... \circ f}_{k \text{ times}}.$$

One can easily check the following:

Lemma 2.5. *In the above framework, we have:*
a) $f^{[u]}(f^{[v]}(x_1, ..., x_{p^v}), ..., f^{[v]}(x_{p^{u+v}-p^v+1}, ..., x_{p^{u+v}}))$

$$= f^{[v]}(f^{[u]}(x_1, ..., x_{p^u}), ..., f^{[u]}(x_{p^{u+v}-p^u+1}, ..., x_{p^{u+v}})),$$

for all $u, v \in \mathbb{N}^*$, $(x_1, ..., x_{p^v}) \in X^{p^v}$, ..., $(x_{p^{u+v}-p^v+1}, ..., x_{p^{u+v}}) \in X^{p^v}$, $(x_1, ..., x_{p^u})$, ..., $(x_{p^{u+v}-p^u+1}, ..., x_{p^{u+v}}) \in X^{p^u}$.

b) $f^{[u+v]}(x_1, ..., x_{p^{u+v}}) = f^{[u]}(f^{[v]}(x_1, ..., x_{p^v}), ..., f^{[v]}(x_{p^{u+v}-p^v+1}, ..., x_{p^{u+v}}))$, *for all* $u, v \in \mathbb{N}^*$, $(x_1, ..., x_{p^{u+v}}) \in X^{p^{u+v}}$, $(x_1, ..., x_{p^v})$, ..., $(x_{p^{u+v}-p^v+1}, ..., x_{p^{u+v}}) \in X^{p^v}$.

Proof. a) We are going to use the mathematical induction method to prove the above lemma.

First we treat the case $u = 1$.

We shall prove, using the same mathematical induction method, that

$$f(f^{[v]}(x_1, ..., x_{p^v}), ..., f^{[v]}(x_{p^{1+v}-p^v+1}, ..., x_{p^{1+v}}))$$
$$= f^{[v]}(f(x_1, ..., x_p), ..., f(x_{p^{1+v}-p+1}, ..., x_{p^{1+v}})), \tag{2.1}$$

for all $v \in \mathbb{N}^*$, $(x_1, ..., x_{p^v})$, ..., $(x_{p^{1+v}-p^v+1}, ..., x_{p^{1+v}}) \in X^{p^v}$, $(x_1, ..., x_p)$, ..., $(x_{p^{1+v}-p+1}, ..., x_{p^{1+v}}) \in X^p$.

The above equality is clear for $v = 1$. The inductive step is justified by the following sequence of equalities:

$$f(f^{[v+1]}(x_1, ..., x_{p^{v+1}}), ..., f^{[v+1]}(x_{p^{v+2}-p^{v+1}+1}, ..., x_{p^{v+2}}))$$
$$\overset{\text{definition of } f^{[v+1]}}{=} f(f(f^{[v]}(f(x_1, ..., x_{p^v}), ..., f^{[v]}(x_{p^{v+1}-p^v+1}, ..., x_{p^{v+1}})), ...,$$
$$f(f^{[v]}(x_{p^{v+2}-p^{v+1}+1}, ..., x_{p^{v+2}-p^{v+1}+p^v}), ..., f^{[v]}(x_{p^{v+2}-p^v+1}, ..., x_{p^{v+2}})))$$
$$\overset{(2.1)}{=} f(f^{[v]}(f(x_1, ..., x_p), ..., f(x_{p^{v+1}-p+1}, ..., x_{p^{v+1}})), ...,$$
$$f^{[v]}(f(x_{p^{v+2}-p^{v+1}+1}, ..., x_{p^{v+2}-p^{v+1}+p}), ..., f(x_{p^{v+2}-p+1}, ..., x_{p^{v+2}})))$$
$$\overset{\text{definition of } f^{[v+1]}}{=} f^{[v+1]}(f(x_1, ..., x_p), ..., f(x_{p^{v+2}-p+1}, ..., x_{p^{v+2}})).$$

Now the inductive step (over u) is justified by the following sequence of equalities:

$$f^{[u+1]}(f^{[v]}(x_1, ..., x_{p^v}), ..., f^{[v]}(x_{p^{u+v+1}-p^v+1}, ..., x_{p^{u+v+1}}))$$
$$\overset{\text{definition of } f^{[u+1]}}{=} f(f^{[u]}(f^{[v]}(x_1, ..., x_{p^v}), ..., f^{[v]}(x_{p^{u+v}-p^v+1}, ..., x_{p^{u+v}})), ...,$$
$$f^{[u]}(f^{[v]}(x_{p^{u+v+1}-p^{u+v}+1}, ..., x_{p^{u+v+1}-p^{u+v}+p^v}), ..., f^{[v]}(x_{p^{u+v+1}-p^v+1}, ..., x_{p^{u+v+1}})))$$
$$\overset{\text{inductive hypothesis}}{=} f(f^{[v]}(f^{[u]}(x_1, ..., x_{p^u}), ..., f^{[u]}(x_{p^{u+v}-p^u+1}, ..., x_{p^{u+v}})), ...,$$
$$f^{[v]}(f^{[u]}(x_{p^{u+v+1}-p^{u+v}+1}, ..., x_{p^{u+v+1}-p^{u+v}+p^u}), ..., f^{[u]}(x_{p^{u+v+1}-p^u+1}, ..., x_{p^{u+v+1}})))$$

$$\overset{\text{definition of } f^{[v+1]}}{=} f^{[v+1]}(f^{[u]}(x_1,...,x_{p^u}),...,f^{[u]}(x_{p^{u+v+1}-p^u+1},...,x_{p^{u+v+1}}))$$
$$= f^{[v]}(f(f^{[u]}(x_1,...,x_{p^u}),...,f^{[u]}(x_{p^{u+1}-p^u+1},...,x_{p^{u+1}})),...,$$
$$f(f^{[u]}(x_{p^{u+v+1}-p^{u+1}+1},...,x_{p^{u+v+1}-p^{u+1}+p^u}),...,f^{[u]}(x_{p^{u+v+1}-p^u+1},...,x_{p^{u+v+1}})))$$
$$= f^{[v]}(f^{[u+1]}(x_1,...,x_{p^{u+1}}),...,f^{[u+1]}(x_{p^{u+v+1}-p^{u+1}+1},...,x_{p^{u+v+1}})).$$

b) The proof is similar to the one of a).

Definition 2.6. In the above framework, $x \in X$ is called a fixed point of f provided that $f(x,...,x) = x$.

Proposition 2.7. *In the above framework, if $f^{[n]}$ has a unique fixed point, then f has a unique fixed point. Moreover, $f^{[n]}$ and $f^{[k]}$ have the same unique fixed point for every $k \in \mathbb{N}^*$.*

Proof. If $x \in X$ is the fixed point of $f^{[n]}$, i.e. $f^{[n]}(x,...,x) = x$, then

$$f(f^{[n]}(x,...,x),...,f^{[n]}(x,...,x)) = f(x,...,x),$$

so, using Lemma 2.5, we have $f^{[n]}(f(x,...,x),...,f(x,...,x)) = f(x,...,x)$. Consequently $f(x,...,x)$ is the fixed point of $f^{[n]}$ and, based on the uniqueness of it, we infer that $f(x,...,x) = x$, i.e. x is a fixed point of f.

In addition, $f(f(x,...,x),...,f(x,...,x)) = f(x,...,x) = x$, i.e. $f^{[2]}(x,...,x) = x$ and, using the method of mathematical induction, we conclude that $f^{[k]}(x,...,x) = x$ for every $k \in \mathbb{N}^*$.

Moreover, if $y \in X$ is a fixed point of f, then $f^{[n]}(y,...,y) = y$, i.e. y is a fixed point of $f^{[n]}$. Hence, as $f^{[n]}$ has a unique fixed point, we infer that $y = x$. Therefore f has a unique fixed point. □

Proposition 2.8. *For a GIFS $\mathcal{F}$, we have $\mathcal{F}_{\mathcal{F}}^{[n]} = \mathcal{F}_{\mathcal{F}_n}$ for every $n \in \mathbb{N}^*$.*

Proof. Note that $\mathcal{F}_{\mathcal{F}}^{[n]}, \mathcal{F}_{\mathcal{F}_n} : (\mathbb{K}(X))^{p^n} \to \mathbb{K}(X)$.

We will use the method of mathematical induction to prove that

$$\mathcal{F}_{\mathcal{F}}^{[n]}(A_1,...,A_p) = \mathcal{F}_{\mathcal{F}_n}(A_1,...,A_p),$$

for every $n \in \mathbb{N}^*$ and every $(A_1,...,A_p) \in \mathbb{X}_n$.

The statement is obvious for $n = 1$.

The statement is true for $n = 2$ since

$$\mathcal{F}_{\mathcal{F}_2}(A_1,...,A_p) = \bigcup_{\alpha \in {}_2\Omega} f_\alpha(A_1,...,A_p)$$
$$= \bigcup_{\alpha^1=1}^{N} \bigcup_{\alpha(1)\in {}_1\Omega,...,\alpha(p)\in {}_1\Omega} f_{\alpha^1}(f_{\alpha(1)}(A_1),...,f_{\alpha(p)}(A_p))$$
$$= \bigcup_{\alpha^1=1}^{N} f_{\alpha^1}\left(\bigcup_{\alpha(1)\in {}_1\Omega} f_{\alpha(1)}(A_1),..., \bigcup_{\alpha(p)\in {}_1\Omega} f_{\alpha(p)}(A_p)\right)$$
$$= \mathcal{F}_{\mathcal{F}}(\mathcal{F}_{\mathcal{F}}(A_1),...,\mathcal{F}_{\mathcal{F}}(A_p)) = \mathcal{F}_{\mathcal{F}}^{[2]}(A_1,...,A_p),$$

for all $(A_1,...,A_p) \in \mathbb{X}_2$.

Suppose now that the statement is true for every $k \in \{1,2,...,n\}$. Then it is also true for $k = n+1$ since

$$\mathcal{F}_{\mathcal{F}_{n+1}}(A_1,...,A_p) = \bigcup_{\alpha \in\ _{n+1}\Omega} f_\alpha(A_1,...,A_p)$$

$$= \bigcup_{\alpha^1=1}^{N} \bigcup_{\alpha(1)\in_n\Omega,...,\alpha(p)\in\ _n\Omega} f_{\alpha^1}(f_{\alpha(1)}(A_1),...,f_{\alpha(p)}(A_p))$$

$$= \bigcup_{\alpha^1=1}^{N} f_{\alpha^1}\left(\bigcup_{\alpha(1)\in_n\Omega} f_{\alpha(1)}(A_1),..., \bigcup_{\alpha(p)\in_n\Omega} f_{\alpha(p)}(A_p)\right)$$

$$= \mathcal{F}_{\mathcal{F}}(\mathcal{F}_{\mathcal{F}}^{[n]}(A_1),...,\mathcal{F}_{\mathcal{F}}^{[n]}(A_p)) = \mathcal{F}_{\mathcal{F}}^{[n+1]}(A_1,...,A_p),$$

for all $(A_1,...,A_p) \in \mathbb{X}_{n+1}$. □

Corollary 2.9. *For a generalized iterated function system $\mathcal{F}$, we have $\mathcal{F}_{\mathcal{F}}^{[mn]} = \mathcal{F}_{\mathcal{F}_n}^{[m]}$ for every $m,n \in \mathbb{N}^*$.*

Proof. We shall use the mathematical induction method in order to prove the above corollary.

Let us note by $P(m)$ the proposition: $\mathcal{F}_{\mathcal{F}}^{[mn]}(A_1,...,A_{p^{mn}}) = \mathcal{F}_{\mathcal{F}_n}^{[m]}(A_1,...,A_{p^{mn}})$ for all $n \in \mathbb{N}^*$ and all $A_1,...,A_{p^{mn}} \in \mathbb{K}(X)$.

In view of Proposition 2.8, $P(1)$ is true.

Now we suppose that $P(m)$ is true and prove that $P(m+1)$ is true.

Indeed, we have

$$\mathcal{F}_{\mathcal{F}}^{[(m+1)n]}(A_1,...,A_{p^{(m+1)n}}) = \mathcal{F}_{\mathcal{F}}^{[mn+n]}(A_1,...,A_{p^{(m+1)n}})$$

$$\overset{\text{Lemma 2.5, b)}}{=} \mathcal{F}_{\mathcal{F}}^{[mn]}(\mathcal{F}_{\mathcal{F}}^{[n]}(A_1,...,A_{p^n}),...,\mathcal{F}_{\mathcal{F}}^{[n]}(A_{p^{(m+1)n}-p^n+1},...,A_{p^{(m+1)n}}))$$

$$= \mathcal{F}_{\mathcal{F}_n}^{[m]}(\mathcal{F}_{\mathcal{F}_n}(A_1,...,A_{p^n}),...,\mathcal{F}_{\mathcal{F}_n}(A_{p^{(m+1)n}-p^n+1},...,A_{p^{(m+1)n}}))$$

$$= \mathcal{F}_{\mathcal{F}_n}^{[m+1]}(A_1,...,A_{p^{(m+1)n}}),$$

for all $A_1,...,A_{p^{(m+1)n}} \in \mathbb{K}(X)$. □

Affine generalized iterated function systems

Definition 2.10. An affine generalized iterated function system (of order p) is a pair $\mathcal{F} := ((\mathbb{R}^m, \|.\|), (f_i)_{i\in\{1,2,...,N\}})$, where $p,m,N \in \mathbb{N}^*$ and, for each $i \in \{1,2,...,N\}$, there exist $b_i \in \mathbb{R}^m$ and a linear function $A_i : (\mathbb{R}^m)^p \to \mathbb{R}^m$ such that $f_i = A_i + b_i$.

We shall use the abbreviation AGIFS for an affine generalized iterated function system.

Remark 2.11. Note that if $\mathcal{F}$ is a AGIFS of order p, then $\mathcal{F}_n$ is a AGIFS of order p^n, for every $n \in \mathbb{N}^*$ (see [2]).

Definition 2.12. An AGIFS $\mathcal{F} := ((\mathbb{R}^m, \|.\|), (f_i)_{i\in\{1,2,...,N\}})$ is called contractive if there exists $C \in [0,1)$ such that $\|A_i\| \le C$ for every $i \in \{1,2,...,N\}$.

Definition 2.13. An AGIFS $\mathcal{F} := ((\mathbb{R}^m, \|.\|), (f_i)_{i\in\{1,2,...,N\}})$ is called hyperbolic if there exists a norm $|||.|||$ on $\mathbb{R}^m$ such that the AGIFS $((\mathbb{R}^m, |||.|||), (f_i)_{i\in\{1,2,...,N\}})$ is contractive.

Definition 2.14. A convex body is a compact convex subset of $(\mathbb{R}^m, \|.\|)$ with non-empty interior.

Definition 2.15. We say that an AGIFS $\mathcal{F}$ has attractor if there exists (a unique) $A \in \mathbb{K}(\mathbb{R}^m)$ such that:
i) $\mathcal{F}_{\mathcal{F}}(A, ..., A) = A$;
ii) $\lim_{k\to\infty} \mathcal{F}_{\mathcal{F}}^{[k]}(B, ..., B) = A$ for every $B \in \mathbb{K}(\mathbb{R}^m)$.

Definition 2.16. An AGIFS $\mathcal{F}$ is called topologically contractive if there exists a convex body K such that $\mathcal{F}_{\mathcal{F}}(K, ..., K) \subseteq \overset{\circ}{K}$.

Definition 2.17. For A and B subsets of $(\mathbb{R}^m, \|.\|)$, we define

$$\delta_{\|.\|}(A, B) = \inf_{a\in A, b\in B} \|a - b\|.$$

Remark 2.18. If K is a convex body, then $K - K$ is a bounded, balanced, convex neighborhood of 0.

Definition 2.19. If K is a bounded, balanced, convex neighborhood of 0 from $(\mathbb{R}^m, \|.\|)$, then the Minkowski norm associated with K is described by

$$\|x\|_K = \inf\{\lambda > 0 \mid x \in \lambda K \text{ for every } x \in \mathbb{R}^m\}.$$

Remark 2.20. The norms $\|.\|$ and $\|.\|_K$ are equivalent.

Lemma 2.21. *(see Lemma 2.4 from [5]). If $\delta_{\|.\|}(A, \mathbb{R}^m \setminus B) \geq \alpha$, where $\alpha > 0$, A and B are subsets of $(\mathbb{R}^m, \|.\|)$ such that A is a bounded, balanced, convex neighborhood of 0, then $A \subseteq (1 - \theta)B$, where $\theta = \frac{\alpha}{2(\alpha+1)}$.*

Lemma 2.22. *(see Lemma 2.6 from [5]). Let A, B, A_1 and B_1 be subsets of $(\mathbb{R}^m, \|.\|)$ such that $\delta_{\|.\|}(A_1, \mathbb{R}^m \setminus A) > 0$ and $\delta_{\|.\|}(B_1, \mathbb{R}^m \setminus B) > 0$. Then*

$$\delta_{\|.\|}(A_1 - B_1, \mathbb{R}^m \setminus (A - B)) > 0.$$

Lemma 2.23. *(see Lemma 2.7 from [5]). Let A be a bounded, balanced, convex neighborhood of 0 from $(\mathbb{R}^m, \|.\|)$ and $f : \mathbb{R}^m \to \mathbb{R}^m$ a bounded linear operator such that $f(A) \subseteq \mu A$, where $\mu > 0$. Then $\|f\|_A \leq \mu$.*

3. The main result

Theorem 3.1. *Given an AGIFS $\mathcal{F}$, the following statements are equivalent:*
1. *There exists $n \in \mathbb{N}^*$ such that $\mathcal{F}_n$ is hyperbolic.*
2. *There exists $n \in \mathbb{N}^*$ such that $\mathcal{F}_n$ has attractor.*
3. *$\mathcal{F}$ has attractor.*
4. *There exists $n \in \mathbb{N}^*$ such that $\mathcal{F}_n$ is topologically contractive.*

Proof. Let us consider the AGIFS $\mathcal{F} = ((\mathbb{R}^m, \|.\|), (f_i)_{i\in\{1,2,...,N\}})$ of order p.

"1 $\Rightarrow$ 2". Since $\mathcal{F}_n$ is hyperbolic, there exists a norm $|||.|||$ on $\mathbb{R}^m$, equivalent with $\|.\|$, such that the AGIFS $((\mathbb{R}^m, |||.|||), (f_\alpha)_{\alpha\in_n\Omega})$ is contractive, i.e. there exists $C \in [0, 1)$ such that $|||A_\alpha||| \leq C$ for every $\alpha \in {}_n\Omega$. In the sequel we denote by ρ the metric on X_n, where X is $(\mathbb{R}^m, |||.|||)$.

Claim. *$\mathcal{F}_{\mathcal{F}_n}$ is a contraction.*
Justification of the claim. We have

$$h_{|||\cdot|||}(\mathcal{F}_{\mathcal{F}_n}(A_1, ..., A_p), \mathcal{F}_{\mathcal{F}_n}(B_1, ..., B_p))$$

$$= h_{|||\cdot|||}\left(\bigcup_{\alpha\in_n\Omega} f_\alpha(A_1, ..., A_p), \bigcup_{\alpha\in_n\Omega} f_\alpha(B_1, ..., B_p)\right) \overset{\text{Proposition 2.1}}{\leq}$$

$$\leq \sup_{\alpha\in_n\Omega} \max\{d_{|||.|||}(f_\alpha(A_1, ..., A_p), f_\alpha(B_1, ..., B_p)), d_{|||.|||}(f_\alpha(B_1, ..., B_p), f_\alpha(A_1, ..., A_p))\}$$

$$\leq Ch_\rho((A_1, ..., A_p), (B_1, ..., B_p)),$$

for all $(A_1, ..., A_p), (B_1, ..., B_p) \in \mathbb{X}_n$ and the justification is done.

As $(\mathbb{K}(\mathbb{R}^m), h_{|||.|||})$ is complete, based on the contraction principle, we conclude that there exists a unique $A \in \mathbb{K}(\mathbb{R}^m)$ such that $\mathcal{F}_{\mathcal{F}_n}(A, ..., A) = A$ and $\lim\limits_{k\to\infty} \mathcal{F}^{[k]}_{\mathcal{F}_n}(B, ..., B) = A$ for every $B \in \mathbb{K}(\mathbb{R}^m)$. Hence $\mathcal{F}_n$ has attractor.

"2 $\Rightarrow$ 3". By hypothesis, there exists a unique $A \in \mathbb{K}(\mathbb{R}^m)$ such that $\mathcal{F}_{\mathcal{F}_n}(A, ..., A) = A$. Therefore, according to Proposition 2.8, we get $\mathcal{F}^{[n]}_{\mathcal{F}}(A, ..., A) = A$ and, using Proposition 2.7, we conclude that $\mathcal{F}_{\mathcal{F}}(A, ..., A) = A$.

The proof of this implication is done if we prove the following implication:

$$\lim_{k\to\infty} \mathcal{F}^{[k]}_{\mathcal{F}_n}(B, ..., B) = A \Rightarrow \lim_{k\to\infty} \mathcal{F}^{[k]}_{\mathcal{F}}(B, ..., B) = A,$$

for every $B \in \mathbb{K}(\mathbb{R}^m)$.

Here is the justification of the above implication: For $r_0 \in \{1, 2, ..., n-1\}$ fixed, with the notations $q_k = nk + r_0$ and $\mathcal{F}^{[r_0]}_{\mathcal{F}}(B) = C$, we have

$$\lim_{k\to\infty} \mathcal{F}^{[q_k]}_{\mathcal{F}}(B, ..., B) = \lim_{k\to\infty} \mathcal{F}^{[nk]}_{\mathcal{F}}(\mathcal{F}^{[r_0]}_{\mathcal{F}}(B), ..., \mathcal{F}^{[r_0]}_{\mathcal{F}}(B))$$

$$= \lim_{k\to\infty} \mathcal{F}^{[nk]}_{\mathcal{F}}(C, ..., C) \overset{\text{Corollary 2.9}}{=} \lim_{k\to\infty} \mathcal{F}^{[k]}_{\mathcal{F}_n}(C, ..., C) = A,$$

for every $B \in \mathbb{K}(\mathbb{R}^m)$. Consequently the implication is true.

"3 $\Rightarrow$ 4". By hypothesis, there exists $A \in \mathbb{K}(\mathbb{R}^m)$ such that $\mathcal{F}_{\mathcal{F}}(A, ..., A) = A$ and $\lim\limits_{n\to\infty} \mathcal{F}^{[n]}_{\mathcal{F}}(K, ..., K) = A$, where $K \overset{def}{=} \{\alpha_1 x_1 + ... + \alpha_n x_n \mid n \in \mathbb{N}^*,\ \alpha_1, ..., \alpha_n \in [0, 1],$ $\alpha_1 + ... + \alpha_n = 1$ and $x_1, ..., x_n \in B[A, 1] \overset{def}{=} \{x \in \mathbb{R}^m \mid \inf\limits_{a\in A} \|x - a\| \leq 1\} \in \mathbb{K}(\mathbb{R}^m)\}$ is a convex body. Therefore there exists $n_0 \in \mathbb{N}^*$ such that $\mathcal{F}^{[n]}_{\mathcal{F}}(K, ..., K) \subseteq B(A, \frac{1}{2}) \overset{def}{=}$ $\{x \in \mathbb{R}^m \mid \inf\limits_{a\in A} \|x - a\| < \frac{1}{2}\} \subseteq \overset{\circ}{K}$ for every $n \in \mathbb{N}^*$, $n \geq n_0$. Using Proposition 2.8, we get that $\mathcal{F}_{\mathcal{F}_n}(K, ..., K) \subseteq \overset{\circ}{K}$, so $\mathcal{F}_n$ is topologically contractive, for every $n \in \mathbb{N}^*$, $n \geq n_0$.

"$4 \Rightarrow 1$". By hypothesis, there exists a convex body $K \in \mathbb{K}(\mathbb{R}^m)$ such that $\mathcal{F}_{\mathcal{F}_n}(K, ..., K) \subseteq \overset{\circ}{K}$.

Claim. $\delta_{\|.\|}(\mathcal{F}_{\mathcal{F}_n}(K, ..., K), \mathbb{R}^m \setminus K) > 0$.

Justification of the claim. Since $\mathcal{F}_{\mathcal{F}_n}(K, ..., K) \subseteq \overset{\circ}{K}$, we infer that

$$\mathcal{F}_{\mathcal{F}_n}(K, ..., K) \cap (\mathbb{R}^m \setminus K) = \emptyset.$$

With the notation $M \overset{not}{=} \mathcal{F}_{\mathcal{F}_n}(K, ..., K) \in \mathbb{K}(\mathbb{R}^m)$, we consider the continuous function $g : M \to \mathbb{R}$ given by $g(x) = d(x, \mathbb{R}^m \setminus K)$ for every $x \in M$. There exists $u \in M$ such that $g(u) = \inf_{x \in M} g(x) \overset{not}{=} \varepsilon > 0$. For every $x \in M$ we have $d(x, \mathbb{R}^m \setminus K) \geq \varepsilon$, so $d(x, y) \geq \varepsilon$ for every $y \in \mathbb{R}^m \setminus K$. Consequently $\delta_{\|.\|}(\mathcal{M}, \mathbb{R}^m \setminus K) > 0$ and the justification is done.

Note that $\|.\|_C$ is equivalent to $\|.\|$, where $C \overset{def}{=} K - K$.

Taking into account Lemma 2.22 for $A_1 = B_1 = \mathcal{F}_{\mathcal{F}_n}(K, ..., K)$ and $A = B = K$, we deduce that $\delta_{\|.\|}(\mathcal{F}_{\mathcal{F}_n}(K, ..., K) - \mathcal{F}_{\mathcal{F}_n}(K, ..., K), \mathbb{R}^m \setminus C) \overset{not}{=} \lambda > 0$.

We have

$$A_\alpha(C, ..., C) = A_\alpha(K, ..., K) - A_\alpha(K, ..., K)$$
$$= f_\alpha(K, ..., K) - f_\alpha(K, ..., K) \subseteq \mathcal{F}_{\mathcal{F}_n}(K, ..., K) - \mathcal{F}_{\mathcal{F}_n}(K, ..., K),$$

so $\delta_{\|.\|}(A_\alpha(C, ..., C), \mathbb{R}^m \setminus C) \geq \lambda$ for every $\alpha \in {}_n\Omega$.

Using Lemma 2.21 for $A = A_\alpha(C, ..., C)$ and $B = C$, we get that $A_\alpha(C, ..., C) \subseteq (1 - \theta)C$ for every $\alpha \in_n \Omega$, where $\theta = \frac{\lambda}{2(\lambda+1)}$.

Finally we take into account Lemma 2.23 for $f = A_\alpha$ and $\mu = 1 - \theta$ to obtain that $\|A_\alpha\|_C \leq 1 - \theta < 1$ for every $\alpha \in_n \Omega$.

Therefore $\mathcal{F}_n$ is hyperbolic. □

Remark 3.2. The collocation "There exists $n \in \mathbb{N}^*$" from the statement of Theorem 3.1 could be replaced by the following one: "There exists $n_0 \in \mathbb{N}^*$ such that for every $n \in \mathbb{N}^*$, $n \geq n_0$".

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