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GENERALIZATIONS OF EDELSTEIN'S FIXED POINT THEOREM IN COMPACT METRIC SPACES

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Abstract. We study deeply some fixed point theorem in compact metric spaces proved very recently, generalizing this theorem. We also show this theorem is a generalization of the famous Edelstein's fixed point theorem.
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1. Introduction

In 2008, Kohsaka and Takahashi [12] introduced the concept of nonspreading mapping. Let T be a mapping on a subset C of a smooth Banach space E. T is said to be *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) \tag{1.1}$$

for any $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$. They proved the existence of fixed points of T provided C is bounded, closed and convex, and E is reflexive and strictly convex. In the case where E is Hilbertian, (1.1) is equivalent to

$$2\,\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2. \tag{1.2}$$

We note the concept of nonspreading mapping is very important because of useful applications. Inspired by this, many new nonlinear mappings have been introduced; see [1, 2, 9, 11, 14, 20] and others. Condition (CC) is one of them, which is strictly weaker than (1.2).

Theorem 1.1 (Corollary 5.3 in [10])**.** *Let T be a mapping on a weakly compact subset C of a Banach space E which satisfies Condition* (CC)*, that is, there exist a continuous, strictly increasing function η from $[0, \infty)$ into itself with $\eta(0) = 0$ and $r, s \in [0, 1)$ such that $r + 2\,s = 1$ and*

$$\eta\big(\|Tx - Ty\|\big) \leq r\,\eta\big(\|x - y\|\big) + s\,\eta\big(\|x - Ty\|\big) + s\,\eta\big(\|Tx - y\|\big)$$

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for any $x, y \in X$. Assume that C has the Opial property. Then $\{T^n x\}$ converges weakly to a fixed point of T for any $x \in C$.

Theorem 1.1 deduces the following.

Theorem 1.2 (Corollary 5.5 in [10])**.** *Let (X, d) be a compact metric space and let T be a mapping on X. Assume that there exist a continuous, strictly increasing function η from $[0, \infty)$ into itself with $\eta(0) = 0$ and $r, s \in [0, 1)$ such that $r + 2s = 1$ and*

$$\eta\big(d(Tx, Ty)\big) \le r\,\eta\big(d(x, y)\big) + s\,\eta\big(d(x, Ty)\big) + s\,\eta\big(d(Tx, y)\big)$$

for any $x, y \in X$. Then $\{T^n x\}$ converges to a fixed point of T for any $x \in X$.

In this paper, Theorem 1.2 becomes more important because Theorem 1.2 can deduce the famous Edelstein's fixed point theorem (Theorem 6.3 below); see Section 6. In other words, Theorem 1.2 is one of generalizations of Theorem 6.3; see also [13, 16, 17, 19]. Motivated by this fact, we will study Theorem 1.2 more deeply. Indeed in Theorem 1.2, $\eta \circ d$ does not satisfy the triangle inequality. So we do not have to assume that the underlying space X is a metric space. We will give an answer to the question of what works well in the proof of Theorem 1.2.

2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

The following lemma was crucial in the proofs of main theorems in [10].

Lemma 2.1 (Lemma 4.6 in [10])**.** *Put $I_0 = \big\{(m, n) : m, n \in \mathbb{N} \cup \{0\},\ m \le n\big\}$ and $I = \{(m, n) : m, n \in \mathbb{N},\ m < n\}$. Let B be a function from I_0 into $[0, \infty)$ satisfying the following*:

- *$B(0, n) \le 1$ for $n \in \mathbb{N}$.*
- *$B(n, n) = 0$ for $n \in \mathbb{N} \cup \{0\}$.*
- *There exist $r, s \in [0, 1)$ such that $r + 2s = 1$ and*

$$B(m, n) \le r\,B(m-1, n-1) + s\,B(m-1, n) + s\,B(m, n-1)$$

for $(m, n) \in I$.

Then $\lim_n B(n, n+1) = 0$ holds.

The following three lemmas are obvious.

Lemma 2.2 ([18])**.** *Let η be a continuous, strictly increasing function from $[0, \infty)$ into itself with $\eta(0) = 0$. Then the following holds*:

(H1) *For any sequence $\{a_n\}$ in $[0, \infty)$, $\lim_n \eta(a_n) = 0$ iff $\lim_n a_n = 0$.*

Lemma 2.3 ([18])**.** *Let η be a function from $[0, \infty)$ into itself satisfying* (H1)*. Then $\eta^{-1}(0) = \{0\}$ holds, that is, $\eta(\alpha) = 0 \Leftrightarrow \alpha = 0$.*

Lemma 2.4 ([18])**.** *Let η be a function from $[0, \infty)$ into itself. Then the following are equivalent*:

(i) *η satisfies* (H1)*.*

(ii) *There exist a function* φ *from* $[0, \infty)$ *into itself,* $\delta > 0$ *and a function* ψ *from* $[0, \delta]$ *into* $[0, \infty)$ *such that* φ *and* ψ *are continuous and strictly increasing,* $\varphi(0) = \psi(0) = 0$*,* $\varphi(a) < \eta(a)$ *for any* $a \in (0, \infty)$ *and* $\eta(a) < \psi(a)$ *for any* $a \in (0, \delta]$.

Assuming something additionally, we obtain the following lemma, which is simpler than Lemma 2.4.

Lemma 2.5. *Let* η *be a function from* $[0, \infty)$ *into itself. Assume* $\sup\{\eta(a) : a \in [0, \alpha]\} < \infty$ *for any* $\alpha > 0$*. Then the following are equivalent*:

(i) η *satisfies* (H1).

(ii) *There exist continuous, strictly increasing functions* φ *and* ψ *from* $[0, \infty)$ *into itself such that* $\varphi(0) = \psi(0) = 0$ *and* $\varphi(a) < \eta(a) < \psi(a)$ *for any* $a \in (0, \infty)$.

A mapping T on a metric space (X, d) is said to be *quasinonexpansive* [4] if

$$d(Tx, z) \le d(x, z)$$

for any $x, z \in C$ with $Tz = z$.

Proposition 2.6 (Proposition 4.1 in [10])**.** *Assume that a mapping* T *on a metric space* (X, d) *satisfies Condition* (CC)*. Assume also that* T *has a fixed point. Then* T *is a quasinonexpansive mapping.*

Jachymski proved the following, very interesting lemma.

Lemma 2.7 (Lemma 3 in Jachymski [6])**.** *Let* φ *be a continuous, strictly increasing function from* $[0, \infty)$ *into itself satisfying* $\varphi(a) < a$ *for any* $a \in (0, \infty)$ *and* $\lim_{a \to \infty} \varphi(a) = \infty$*. Then for any* $r \in (0, 1)$*, there exists a continuous, strictly increasing function* η *from* $[0, \infty)$ *into itself satisfying* $\eta\big(\varphi(a)\big) = r\, \eta(a)$ *for any* $a \in [0, \infty)$.

Remark 2.8. In [6], the statement is "η is nondecreasing". However, from the proof in [6], we have that η is strictly increasing.

Jachymski and Jóźwik characterized Browder contraction; see [6, 7, 15] and others.

Theorem 2.9 (Theorem 4 in Jachymski and Jóźwik [8])**.** *Let* T *be a mapping on a metric space* (X, d)*. Then the following are equivalent*:

(i) T *is a Browder contraction*; *see* [3].

(ii) *There exists a continuous, strictly increasing function* φ *from* $[0, \infty)$ *into itself satisfying* $d(Tx, Ty) \le \varphi\big(d(x, y)\big)$ *for any* $x, y \in X$.

(iii) *There exists a function* φ *from* $[0, \infty)$ *into itself satisfying* $d(Tx, Ty) \le \varphi\big(d(x, y)\big)$ *for any* $x, y \in X$ *and*

$$\limsup_{b \to a} \varphi(b) < a \quad \text{for any } a \in (0, \infty). \tag{2.1}$$

(iv) *A function* φ *from* $[0, \infty)$ *into itself defined by*

$$\varphi(a) = \max\big\{0, \sup\{d(Tx, Ty) : x, y \in X, d(x, y) = a\}\big\}$$

for $a \in [0, \infty)$ *satisfies* (2.1).

(v) *If* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *satisfying* $\lim_n d(x_n, y_n) = a$ *for some* $a \in (0, \infty)$*, then* $\limsup_n d(Tx_n, Ty_n) < a$ *holds.*

Proof. We have proved (i) ⇔ (ii) ⇔ (iii) in [8]. (iii) ⇔ (iv) ⇔ (v) is obvious. □

3. Some generalized quasimetric space

We consider some condition, which is weaker than that of quasimetric space.

Definition 3.1. Let X be a set and let p be a function from $X \times X$ into $[0, \infty)$. Then (X, p) is called a *metric space* if the following hold:

(D1) $p(x, x) = 0$
(D2) $p(x, y) = 0 \Rightarrow x = y$
(D3) $p(x, y) = p(y, x)$ (symmetry)
(D4) $p(x, z) \leq p(x, y) + p(y, z)$ (subadditivity or triangle inequality)

Remark 3.2. Let X be a set and let p be a function from $X \times X$ into $[0, \infty)$.

- If X satisfies (D1), then X is a premetric space.
- If X satisfies (D1), (D2) and (D3), then X is a semimetric space.
- If X satisfies (D1), (D2) and (D4), then X is a quasimetric space.
- If X satisfies (D1), (D3) and (D4), then X is a pseudometric space.
- If X satisfies (D2), (D3) and (D4), then X is a metametric space.

Let (X, p) satisfy (D1), (D2) and the following:

(D5) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(x, y) < \delta$ and $p(y, z) < \delta$ imply $p(x, z) < \varepsilon$.

If we name the condition on the space X, 'generalized quasimetric space' seems to be appropriate because (D5) is weaker than (D4). However this name has been already used for some other condition. So in order to avoid confusion, we do not name this condition in this paper.

We can define some concepts of 'convergence' and others naturally.

Definition 3.3. Let (X, p) satisfy (D1), (D2) and (D5).

- A sequence $\{x_n\}$ in X is said to *converge* to $x \in X$ if $\lim_n p(x_n, x) = 0$.
- X is said to be *Hausdorff* if $\lim_n p(x_n, x) = \lim_n p(x_n, y) = 0$ implies $x = y$.
- X is said to be *sequentially compact* if every sequence in X has a subsequence which converges.
- X is said to be *bounded* if $\sup\{p(x, y) : x, y \in X\} < \infty$.

Lemma 3.4. *Let* (X, p) *satisfy* (D1), (D2) *and* (D5). *Then for any* $n \in \mathbb{N}$ *with* $n \geq 2$, *the following holds:*

$(\mathrm{D5})_n$ *For* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $p(x_j, x_{j+1}) < \delta$ *for* $j \in \{1, 2, \cdots, n-1\}$ *implies* $p(x_i, x_j) < \varepsilon$ *for* $i, j \in \{1, 2, \cdots, n\}$ *with* $i < j$.

Proof. $(\mathrm{D5})_2$ is obvious and $(\mathrm{D5})_3$ becomes (D5) itself. We only show $(\mathrm{D5})_4$ because we can show similarly $(\mathrm{D5})_n$ with $n \geq 5$. Fix $\varepsilon > 0$. Then by (D5), there exists $\alpha > 0$ satisfying

- $p(x, y) < \alpha$ and $p(y, z) < \alpha$ imply $p(x, z) < \varepsilon$.

By (D5) again, there exists $\beta > 0$ satisfying

- $p(x, y) < \beta$ and $p(y, z) < \beta$ imply $p(x, z) < \min\{\alpha, \varepsilon\}$.

Put $\delta = \min\{\alpha, \beta, \varepsilon\}$ and fix $x, y, z, w \in X$ with $p(x, y) < \delta$, $p(y, z) < \delta$ and $p(z, w) < \delta$. Then since $p(x, y) < \beta$ and $p(y, z) < \beta$, we have $p(x, z) < \min\{\alpha, \varepsilon\}$. Since $p(z, w) < \delta \leq \alpha$, we have $p(x, w) < \varepsilon$. □

Lemma 3.5. *Let* (X, p) *satisfy* (D1), (D2) *and* (D5). *Let* $\{x_n^{(j)}\}_{n=1}^{\infty}$ *be a sequence in* X *for* $j \in \{1, 2, \cdots, \nu\}$. *Assume* $\lim_n p(x_n^{(j)}, x_n^{(j+1)}) = 0$ *for* $j \in \{1, 2, \cdots, \nu - 1\}$. *Then* $\lim_n p(x_n^{(i)}, x_n^{(j)}) = 0$ *holds for* $i, j \in \{1, 2, \cdots, \nu\}$ *with* $i < j$.

Proof. The conclusion follows from Lemma 3.4. □

4. Fixed point theorem

In this section, we generalize Theorem 1.2.

Theorem 4.1. *Let* (X, p) *satisfy* (D1), (D2) *and* (D5). *Assume that* X *is Hausdorff, bounded and sequentially compact. Let* T *be a mapping on* X *satisfying that there exist* $r, s \in [0, 1)$ *such that* $r + 2\,s = 1$ *and*

$$p(Tx, Ty) \leq r\,p(x, y) + s\,p(x, Ty) + s\,p(Tx, y)$$

for any $x, y \in X$. *Then* $\{T^n x\}$ *converges to a fixed point of* T *for any* $x \in X$.

Remark 4.2. The author thinks that Theorem 4.1 is better than Theorem 1.2 when we would like to know the mathematical structure on them.

Proof. Put

$$M := \sup\{p(x, y) : x, y \in X\} + 1 < \infty.$$

Fix $x \in X$ and define a sequence $\{x_n\}$ in X by $x_n = T^n x$. Define a function B by

$$B(m, n) = p(x_m, x_n)/M$$

for $(m, n) \in I_0$, where I_0 is as in Lemma 2.1. Then by Lemma 2.1, we have $\lim_n B(n, n+1) = 0$ and hence $\lim_n p(x_n, x_{n+1}) = 0$. Similarly, putting $B(n, m) = p(x_m, x_n)/M$ for $(m, n) \in I_0$, we obtain $\lim_n p(x_{n+1}, x_n) = 0$. So by Lemma 3.5, we have

$$\lim_{n\to\infty} p(x_{n+j}, x_{n+k}) = 0 \quad \text{for any } j, k \in \mathbb{N}. \tag{4.1}$$

Since X is sequentially compact, there exists a subsequence $\{f(n)\}$ of the sequence $\{n\}$ in $\mathbb{N}$ such that $\{x_{f(n)}\}$ converges to some $z \in X$. By Lemma 3.5 again, we have

$$\lim_{n\to\infty} p(x_{f(n)+k}, z) = 0 \quad \text{for any } k \in \mathbb{N}. \tag{4.2}$$

From the assumption, we have

$$p(x_{f(n)+1}, Tz) \leq r\,p(x_{f(n)}, z) + s\,p(x_{f(n)+1}, z) + s\,p(x_{f(n)}, Tz)$$

and

$$\begin{aligned} p(x_{f(n)+2}, Tz) \leq{}& r\,s\,p(x_{f(n)}, z) + (r + s^2)\,p(x_{f(n)+1}, z) \\ &+ s\,p(x_{f(n)+2}, z) + s^2\,p(x_{f(n)}, Tz). \end{aligned}$$

We will show by induction

$$\begin{aligned} & p(x_{f(n)+k}, Tz) \\ & \leq r\, s^{k-1}\, p(x_{f(n)}, z) + \sum_{j=1}^{k-1} (r+s^2)\, s^{k-1-j}\, p(x_{f(n)+j}, z) \\ & \qquad + s\, p(x_{f(n)+k}, z) + s^k\, p(x_{f(n)}, Tz) \end{aligned} \tag{4.3}$$

for any $k \in \mathbb{N}$. Indeed (4.3) holds when $k = 1, 2$. We assume that (4.3) holds for some $k \in \mathbb{N}$ with $k \geq 2$. Then we have

$$\begin{aligned} & p(x_{f(n)+k+1}, Tz) \\ & \leq r\, p(x_{f(n)+k}, z) + s\, p(x_{f(n)+k+1}, z) + s\, p(x_{f(n)+k}, Tz) \\ & \leq r\, p(x_{f(n)+k}, z) + s\, p(x_{f(n)+k+1}, z) \\ & \qquad + s \left(r\, s^{k-1}\, p(x_{f(n)}, z) + \sum_{j=1}^{k-1} (r+s^2)\, s^{k-1-j}\, p(x_{f(n)+j}, z) \right. \\ & \qquad \left. + s\, p(x_{f(n)+k}, z) + s^k\, p(x_{f(n)}, Tz) \right) \\ & = r\, s^k\, p(x_{f(n)}, z) + \sum_{j=1}^{k} (r+s^2)\, s^{k-j}\, p(x_{f(n)+j}, z) \\ & \qquad + s\, p(x_{f(n)+k+1}, z) + s^{k+1}\, p(x_{f(n)}, Tz). \end{aligned}$$

Therefore by induction we have shown (4.3) for any $k \in \mathbb{N}$. Fix $\varepsilon > 0$ and choose $\delta > 0$ appearing in (D5). We also choose $k \in \mathbb{N}$ satisfying $s^k M < \delta$. By (4.2) and (4.3), we have

$$p(x_{f(n)+k}, Tz) < \delta \quad \text{for sufficiently large } n \in \mathbb{N}. \tag{4.4}$$

We also have by (4.1)

$$p(x_{f(n)}, x_{f(n)+k}) < \delta \quad \text{for sufficiently large } n \in \mathbb{N}. \tag{4.5}$$

By (4.4) and (4.5), we obtain $p(x_{f(n)}, Tz) < \varepsilon$ for sufficiently large $n \in \mathbb{N}$. Therefore we have shown that $\{x_{f(n)}\}$ also converges to Tz. Since X is Hausdorff, we obtain $Tz = z$. Using this, we have

$$p(x_{n+1}, z) \leq r\, p(x_n, z) + s\, p(x_{n+1}, z) + s\, p(x_n, z)$$

and hence $p(x_{n+1}, z) \leq p(x_n, z)$ for any $n \in \mathbb{N}$. Since

$$\liminf_{n \to \infty} p(x_n, z) \leq \lim_{n \to \infty} p(x_{f(n)}, z) = 0,$$

we obtain $\lim_n p(x_n, z) = 0$. □

Remark 4.3. The identity mapping on X satisfies the assumption of Theorem 4.1. So there can be plural fixed points.

Remark 4.4. From the proof of Theorem 4.1, we know that (D5) plays a very important role.

5. Metric space case

Theorem 4.1 deduces the following fixed point theorem in compact metric spaces, which is a generalization of Theorem 1.2.

Theorem 5.1. *Let (X, d) be a compact metric space and let T be a mapping on X. Assume that there exist a bounded function η from $[0, \infty)$ into itself with* (H1) *and $r, s \in [0, 1)$ such that $r + 2\,s = 1$ and*

$$\eta\big(d(Tx, Ty)\big) \le r\,\eta\big(d(x, y)\big) + s\,\eta\big(d(x, Ty)\big) + s\,\eta\big(d(Tx, y)\big)$$

for any $x, y \in X$. Then $\{T^n x\}$ converges to a fixed point of T for any $x \in X$.

Remark 5.2. Since X is compact, X is bounded. In this theorem, the boundedness of η is essentially equivalent to the following:

- $\sup\{\eta(a) : a \in [0, \alpha]\} < \infty$ for any $\alpha > 0$.

So every continuous function η is considered to be bounded in this context.

Proof. We choose functions φ and ψ appearing in Lemma 2.5. Since both functions are strictly increasing, we note that φ^{-1} and ψ^{-1} exist. Also, for any $a \in (0, \infty)$, since $\varphi(a) < \eta(a)$, we note $a < \varphi^{-1}\big(\eta(a)\big)$ provided $\eta(a)$ belongs to the domain of φ^{-1}. Define p by $p = \eta \circ d$. Then since $\eta^{-1}(0) = \{0\}$ by Lemma 2.3, (X, p) satisfies (D1) and (D2). In order to show (D5), we fix $\varepsilon > 0$. We consider the following two cases:

- $p(x, y) < \varepsilon$ for any $x, y \in X$
- $\varepsilon \le p(x, y)$ for some $x, y \in X$

In the first case, we can choose any $\delta > 0$. In the second case, we note that ε belongs to the domain of ψ^{-1}. Because $\varepsilon \le \eta\big(d(x, y)\big) < \psi\big(d(x, y)\big)$. We put δ by

$$\delta = \varphi\big(\psi^{-1}(\varepsilon)/2\big).$$

Let $x, y, z \in X$ satisfy $p(x, y) < \delta$ and $p(y, z) < \delta$. Then we have

$$\eta\big(d(x, y)\big) < \varphi\big(\psi^{-1}(\varepsilon)/2\big)$$

and hence

$$d(x, y) < \varphi^{-1}\Big(\eta\big(d(x, y)\big)\Big) < \psi^{-1}(\varepsilon)/2.$$

Similarly we can show $d(y, z) < \psi^{-1}(\varepsilon)/2$. Thus, we have $d(x, z) < \psi^{-1}(\varepsilon)$ and hence

$$p(x, z) = \eta\big(d(x, z)\big) < \psi\big(d(x, z)\big) < \varepsilon.$$

Therefore we have shown that (X, p) satisfies (D5). We note that (X, d) is sequentially compact. From the definition of (H1), $\lim_n d(x_n, x) = 0$ is equivalent to $\lim_n p(x_n, x) = 0$. This implies that (X, p) is Hausdorff and that (X, p) is sequentially compact. Since η is bounded, (X, p) is bounded. We have shown the assumption of Theorem 4.1. So by Theorem 4.1, we obtain the desired result. □

Problem 5.3. We do not know whether Theorem 5.1 is still valid without assuming of the boundedness of η.

In order to show that Theorem 5.1 is a real generalization of Theorem 1.2, we give an example.

Example 5.4. Let f be a injective function from $\mathbb{N}$ into $\mathbb{N}$, which is not monotone. Define a compact subset X of the Euclidean space $(\mathbb{R}^1, d)$ by

$$X = \{0\} \cup \{10^{-f(n)} : n \in \mathbb{N}\}.$$

Define a mapping T on X by

$$T(0) = 0 \quad \text{and} \quad T(10^{-f(n)}) = 10^{-f(n+1)}$$

for $n \in \mathbb{N}$. Then the assumption of Theorem 5.1 holds. However, the assumption of Theorem 1.2 does not hold.

Proof. Put $x_n = 10^{-f(n)}$ for $n \in \mathbb{N}$. It is obvious that 0 is a fixed point of T and $Tx_n = x_{n+1}$ holds for $n \in \mathbb{N}$. We note

$$|10^{-j} - 10^{-k}| = 0.\underbrace{000\cdots0}_{j}\underbrace{999\cdots9}_{k-j}$$

and

$$|10^{-j} - 0| = 0.\underbrace{000\cdots0}_{j-1}1$$

for any $j, k \in \mathbb{N}$ with $j < k$. So the above values are all different. Thus, the values of $d(x, y)$ $(x, y \in X,\ x \neq y)$ are all different. We next note $\lim_n f(n) = \infty$ because f is injective. So

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0$$

holds. Therefore there exists a continuous function η from $[0, \infty)$ into itself satisfying (H1) and

$$\eta\big(d(x_n, x_m)\big) = |r^n - r^m| \quad \text{and} \quad \eta\big(d(x_n, 0)\big) = |r^n|$$

for any $m, n \in \mathbb{N}$, where $r \in (0, 1)$ is arbitrary. We have

$$\begin{aligned}\eta\big(d(Tx, Ty)\big) &= r\,\eta\big(d(x, y)\big) \\ &\leq r\,\eta\big(d(x, y)\big) + s\,\eta\big(d(x, Ty)\big) + s\,\eta\big(d(Tx, y)\big)\end{aligned}$$

for any $x, y \in X$, where $s = (1-r)/2$. Hence the assumption of Theorem 5.1 holds. On the other hand, since f is not monotone, there exists $k \in \mathbb{N}$ such that $f(k) > f(k+1)$. Then we have

$$d(Tx_k, 0) = 10^{-f(k+1)} > 10^{-f(k)} = d(x_k, 0),$$

which implies that T is not quasinonexpansive. By Proposition 2.6, the assumption of Theorem 1.2 does not hold. □

6. Edelstein's fixed point theorem

In this section, we give a proof of the famous Edelstein's fixed point theorem by using Theorem 1.2.

Proposition 6.1. *Let (X, d) be a compact metric space and let T be a mapping on X. Then the following are equivalent*:

(i) *T is an Edelstein contraction, that is, for any $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) < d(x, y) \tag{6.1}$$

holds.

(ii) *For any $r \in (0, 1)$, there exists a continuous, strictly increasing function η from $[0, \infty)$ into itself satisfying $\eta(0) = 0$ and*

$$\eta\big(d(Tx, Ty)\big) \leq r\, \eta\big(d(x, y)\big) \tag{6.2}$$

for any $x, y \in X$.

(iii) *There exist $r \in (0, 1)$ and a continuous, strictly increasing function η from $[0, \infty)$ into itself satisfying $\eta(0) = 0$ and (6.2) for any $x, y \in X$.*

Remark 6.2. See Theorem 2 in Jachymski [6]. In a compact metric space, T is an Edelstein contraction iff T is a Browder contraction.

Proof. Both (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are obvious. Let us prove (i) $\Rightarrow$ (ii). We assume (i). Arguing by contradiction, we assume that T is not a Browder contraction. Then by Theorem 2.9 (v), there exist $a \in (0, \infty)$ and sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} d(x_n, y_n) = a \quad \text{and} \quad \limsup_{n\to\infty} d(Tx_n, Ty_n) \geq a.$$

Since X is compact, taking subsequences, without loss of generality, we may assume that $\{x_n\}$ and $\{y_n\}$ converge to x and y, respectively. Then we have

$$d(Tx, Ty) \geq d(x, y) = a > 0,$$

which implies a contradiction. By Theorem 2.9 (ii), there exists a continuous, strictly increasing function ψ from $[0, \infty)$ into itself satisfying $\psi(a) < a$ for any $a \in (0, \infty)$ and $d(Tx, Ty) \leq \psi\big(d(x, y)\big)$ for any $x, y \in X$. Define a function φ from $[0, \infty)$ into itself by $\varphi(a) = \big(\psi(a) + a\big)/2$. Then φ is continuous, strictly increasing and $\lim_{a\to\infty} \varphi(a) = \infty$. We can choose η satisfying the conclusion of Lemma 2.7. Then we have

$$\eta\big(d(Tx, Ty)\big) \leq \eta\Big(\psi\big(d(x, y)\big)\Big) \leq \eta\Big(\varphi\big(d(x, y)\big)\Big) = r\, \eta\big(d(x, y)\big)$$

for any $x, y \in X$. Therefore we obtain (ii). □

Theorem 6.3 (Edelstein [5])**.** *Let (X, d) be a compact metric space and let T be a mapping on X satisfying* (i) *of Proposition 6.1. Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for any $x \in X$.*

Proof. Let $r \in (0, 1)$ be arbitrary and put $s = (1 - r)/2$. By Proposition 6.1, there exists a function η satisfying (6.2). Then we have

$$\begin{aligned} \eta\big(d(Tx, Ty)\big) &\leq r\, \eta\big(d(x, y)\big) \\ &\leq r\, \eta\big(d(x, y)\big) + s\, \eta\big(d(x, Ty)\big) + s\, \eta\big(d(Tx, y)\big) \end{aligned}$$

for any $x, y \in X$. So by Theorem 1.2, $\{T^n x\}$ converges to a fixed point for any $x \in X$. The uniqueness of fixed point follows from (6.1). □

From the proofs of Theorems 5.1 and 6.3, the following theorem is a generalization of Theorem 6.3.

Theorem 6.4. *Let* (X, p) *satisfy* (D1)*,* (D2) *and* (D5)*. Assume that* X *is Hausdorff, bounded and sequentially compact. Let* T *be a mapping on* X*. Assume that there exists* $r \in [0, 1)$ *such that*

$$p(Tx, Ty) \leq r\, p(x, y)$$

for any $x, y \in X$*. Then* $\{T^n x\}$ *converges to a fixed point of* T *for any* $x \in X$*.*

Proof. The conclusion follows from Theorem 4.1. □

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