Fixed Point Theory, 20(2019), No. 2, 683-702
DOI: 10.24193/fpt-ro.2019.2.45
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

MODIFIED INERTIAL METHODS FOR FINDING COMMON SOLUTIONS TO VARIATIONAL INEQUALITY PROBLEMS

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Abstract. It is our aim in this paper to propose modified inertial versions of both subgradient extragradient method and the extragradient method for solving common solutions to variational inequality problems involving monotone and Lipschitz continuous operators and obtain weak convergence results in real Hilbert spaces. We give several numerical illustrations of our proposed methods and give numerical comparisons of our methods with subgradient extragradient and extragradient methods.
Key Words and Phrases: Variational inequality, monotone operator, inertial terms, weak convergence, Hilbert spaces.
2010 Mathematics Subject Classification: 47H06, 47H09, 47J05, 47J25.

1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let C_k, $k = 1, 2, \dots, N$ be nonempty, closed and convex subset of H such that $\cap_{k=1}^N C_k \neq \emptyset$. Let $f_k : C_k \to H$ be a continuous mapping on C_k. Let us consider the following common solutions to variational inequality problems (CSVIP) introduced in [11]: Find $x^* \in \cap_{k=1}^N C_k$ such that

$$\langle f_k(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C_k. \tag{1.1}$$

If $N = 1$ then CSVIP (1.1) reduces to the classical variational inequality problem (for short, VI(f, C)): find $x \in C$ such that

$$\langle f(x), y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

Denote by $SOL(f_k, C_k)$ the solution set of the Variational Inequality Problem VI(f_k, C_k) corresponding to the mapping f_k and the set C_k and

$$SOL = \cap_{k=1}^N SOL(f_k, C_k),$$

the common solution set.

Variational inequality theory is an important tool in studying a wide class of obstacle, unilateral, and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework (see, for example, [3, 4, 15, 21, 22]). This dynamic field is experiencing an explosive growth in both theory and applications. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see the monographs [14, 22] and references therein. The extragradient method, introduced in 1976 by Korpelevich [23], which is given by

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \gamma f(x_n)) \\ x_{n+1} = P_C(x_n - \gamma f(y_n)), n \geq 1, \end{cases} \tag{1.3}$$

where $\gamma \in (0, \frac{1}{L})$ for a finite-dimensional space, provides an iterative process converging to a solution of $\mathrm{VI}(f, C)$ by only assuming that $C \subseteq \mathbb{R}^n$ is nonempty, closed, and convex, and $f : C \to \mathbb{R}^n$ is monotone and L-Lipschitz continuous. The extragradient method was further extended to infinite-dimensional spaces by many authors; see, for instance, [2, 7, 9, 16, 19, 20, 24, 25, 26, 27, 28]. In the setting of Hilbert spaces, this method is only known to be weakly convergent. Note that the extragradient method needs two projections onto the set C and two evaluations of f per iteration.

A crucial feature regarding the design of numerical methods related to the extragradient method is to minimize the number of evaluations of P_C per iteration. So the extragradient method needs to be improved in situations, where a projection onto C is hard to evaluate and therefore computationally expensive. An attempt in this direction was initiated by Censor et al. [10], who modified Korpelevich's method (1.3) by replacing the second projection onto the closed and convex subset C with the one onto a subgradient half-space. Their method, which therefore uses only one projection onto C, is called the subgradient extragradient method. This subgradient extragradient method is shown to be weakly convergent to a solution of the variational inequality $\mathrm{VI}(f, C)$.

Recently, Dong *et al.* [12] introduced an algorithm for solving variational inequality problem $\mathrm{VI}(f, C)$ (1.2) by incorporating inertial terms in the extragradient algorithm. They established a weak convergence result using their proposed algorithm and gave some numerical advantage of their method. Quite recently, Dong *et al.* [13] introduced a modified inertial Mann algorithm by combining the accelerated Mann algorithm with the inertial extrapolation and obtained weak convergence result. They presented some numerical experiments to show that the modified inertial Mann algorithm has some numerical advantages by speeding up the convergence of the given algorithms.

Motivated by the works of Dong *et al.* [12], Dong *et al.* [13], Censor *et al.*[10] and Censor *et al.* [11], we propose a modified inertial subgradient extragradient algorithm and modified inertial extragradient algorithm for solving CSVIP (1.1) and give weak convergence results of the sequence of iterates generated by our methods. Furthermore, we give several numerical examples to illustrate our methods and compare our proposed methods with the already obtained methods for solving variational inequality problem in the literature.

2. Preliminaries

This section contains some definitions and basic results that will be used in our subsequent analysis. The letter H always denotes a real Hilbert space.
We first state the formal definition of some classes of functions that play an essential role in our analysis.
Definition 2.1. Let $X \subseteq H$ be a nonempty subset. Then a mapping $A : X \to H$ is called

(a) *monotone* on X if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in X$;
(b) *Lipschitz continuous* on X if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\| \text{ for all } x, y \in X.$$

We next recall some properties of the projection, cf. [5] for more details. To this end, let $C \subseteq H$ be a nonempty, closed, and convex subset of H. For any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\| \forall y \in C.$$

P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \forall x, y \in H. \tag{2.1}$$

Furthermore, $P_C x$ is characterized by the properties

$$P_C x \in C \quad \text{and} \quad \langle x - P_C x, P_C x - y\rangle \geq 0 \forall y \in C. \tag{2.2}$$

This characterization implies that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \forall x \in H, \forall y \in C. \tag{2.3}$$

The following elementary lemma will be used in our convergence analysis.
Lemma 2.2. *In H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H.$$

Lemma 2.3. *The following identities hold in H:*
(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H$;
(ii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \quad \forall x, y \in H, \quad \lambda \in \mathbb{R}$.
Lemma 2.4. (see [1, Lem. 3]) *Let $\{\psi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be sequences in $[0, +\infty)$ such that $\psi_{n+1} \leq \psi_n + \alpha_n(\psi_n - \psi_{n-1}) + \delta_n$ for all $n \geq 1$, $\sum_{n=1}^{\infty} \delta < +\infty$ and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \geq 1$. Then the following hold:*
(i) $\sum_{n\geq 1} [\psi_n - \psi_{n-1}]_+ < +\infty$, *where* $[t]_+ = \max\{t, 0\}$;
(ii) there exists $\psi^* \in [0, +\infty)$ *such that* $\lim_{n\to+\infty} \psi_n = \psi^*$.
Lemma 2.5. *(see* [5, Lem. 2.39]*) Let C be a nonempty set ofH and $\{x_n\}$ be a sequence in H such that the following two conditions hold:*
(i) for any $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;
(ii) every sequential weak cluster point of $\{x_n\}$ is in C.
Then $\{x_n\}$ converges weakly to a point in C.

We finally restate a result which essentially states the equivalence between a primal and a dual variational inequality for continuous, monotone operators.

Lemma 2.6. ([6, Lem. 7.1.7]) *Let C be a nonempty, closed, and convex subset of H. Let $f : C \to H$ be a continuous, monotone mapping and $z \in C$. Then*

$$z \in \mathrm{SOL}(f, C) \Longleftrightarrow \langle f(x), x - z\rangle \geq 0 \quad \forall x \in C.$$

3. Modified inertial subgradient extragradient method

In this section, we give a precise statement of our modified inertial subgradient extragradient method and discuss some of its elementary properties. Its convergence analysis is postponed to the next section. We first state the assumptions that we will assume to hold through the rest of this paper.

Assumptions 3.1.

(a) The set $C_k, k = 1, 2, \dots, N$ is a nonempty, closed, and convex subset of the real Hilbert space H and the set $\cap_{k=1}^{N} C_k$ is assumed to be a nonempty subset of H.

(b) The operator $f_k : C_k \to H, k = 1, 2, \dots, N$ is monotone and L_k-Lipschitz continuous on H.

(c) The common solutions set SOL of CSVIP (1.1) is nonempty.

Assumption (a) implies that projections onto C_k are well-defined. Condition (b) is slightly stronger than continuity of f_k; the same (or very similar) condition is also used, e.g., in [9, 11, 17, 18]. Note that this assumption can be weakened to continuous operators in finite-dimensional Hilbert spaces $H = \mathbb{R}^n$. This Condition (b) also holds for the large class of bounded linear operators A on a general Hilbert space H.

Since our method depends on the choice of some sequences of parameters, we next summarize the conditions regarding these sequences in the assumption below.

Assumptions 3.2. Suppose the real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(a) $\{\alpha_n\} \subset [0, \alpha]$ is nondecreasing with $\alpha_1 = 0$ and $0 \leq \alpha < 1$.

(b) $\sum_{n=1}^{\infty} \beta_n < \infty$.

(c) $\delta > \dfrac{\alpha^2(1+\alpha) + \alpha\sigma}{1-\alpha^2}$ and $0 < 1 - \mu\gamma \leq 1 - \mu\gamma_n \leq \dfrac{\delta - \alpha[\alpha(1+\alpha) + \alpha\delta + \sigma]}{\delta[1 + \alpha(1+\alpha) + \alpha\delta + \sigma]}$,

where γ, σ, δ, $\mu > 0$.

These conditions are satisfied, e.g., for $\alpha_n = \alpha\dfrac{n}{n+1}$, $0 \leq \alpha < 1$, $\beta_n = \dfrac{1}{(n+1)^2}$, and $\gamma_n = \gamma - \dfrac{1}{\mu}\Big[\dfrac{n(\theta - (1 - \mu\gamma))}{n+1}\Big]$, for all $n \in \mathbb{N}$ where $\theta := \dfrac{\delta - \alpha[\alpha(1+\alpha) + \alpha\delta + \sigma]}{\delta[1 + \alpha(1+\alpha) + \alpha\delta + \sigma]}$.

Throughout this paper, $\{\delta_n^k\}_{n=0}^{\infty}$ is a sequence in (0,1] such that $\delta_n^k \geq \epsilon > 0$ and $\sum_{k=1}^{N} \delta_n^k = 1, \forall n \in \mathbb{N}$. Take, for example, $\delta_n^k := \dfrac{1}{N}$ for all k.

We next give a precise statement of our first modified inertial subgradient extragradient method for solving CSVIP (1.1).

Algorithm 3.3.

- **Initialization.** Define $L := \max_{1\le k\le N} L_k$ and choose $\{\tau_n\} \subset [c,d]$ for some $c,d \in (0, \frac{1}{L})$. Choose $\mu \in (0,1]$, $\lambda > 0$ and $x_0, x_1 \in H$ arbitrarily. For each $k = 1,2,\ldots,N$, define $u_0^k := P_{C_k}(x_0 - \tau_0 f_k(x_0))$ and set
$$T_0^k := \{w \in H : \langle x_0 - \tau_0 f_k(x_0) - u_0^k, w - u_0^k\rangle \le 0\}.$$
Compute $z_0^k := P_{T_0^k}(x_0 - \tau_0 f_k(u_0^k))$ and $z_0 = \sum_{k=1}^N \delta_0^k z_0^k$. Compute
$$d_1 = \frac{(z_0 - x_0)}{\lambda}.$$
Choose sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ such that the conditions from Assumption 3.2 hold. Set $n := 1$
- **Step 1.** Compute:
$$\begin{aligned} w_n : &= x_n + \alpha_n(x_n - x_{n-1}),\\ u_n^k : &= P_{C_k}(w_n - \tau_n f_k(w_n)), \forall k = 1,2,\ldots,N. \end{aligned}$$
- **Step 2.** Compute
$$\begin{aligned} z_n^k : &= P_{T_n^k}(w_n - \tau_n f_k(u_n^k)), \forall k = 1,2,\ldots,N,\\ z_n : &= \sum_{k=1}^N \delta_n^k z_n^k, \end{aligned}$$
where $T_n^k = \{w \in H : \langle w_n - \tau_n f_k(w_n) - u_n^k, w - u_n^k\rangle \le 0\}$.
- **Step 3.** Compute
$$\begin{aligned} d_{n+1} : &= \frac{1}{\lambda}(z_n - w_n) + \beta_n d_n,\\ y_n : &= w_n + \lambda d_{n+1},\\ x_{n+1} : &= \mu\gamma_n w_n + (1 - \mu\gamma_n) y_n. \end{aligned} \tag{3.1}$$
- **Step 4.** Set $n \leftarrow n+1$ and **goto Step 1**.

We further make the following assumptions on the the sequences $\{w_n\}$ and $\{z_n\}$ generated in Algorithm 3.3.

Assumptions 3.4. Suppose the sequences $\{w_n\}$ and $\{z_n\}$ satisfy the following conditions:

(d) $\{z_n - w_n\}$ is bounded.
(e) $\{z_n - y\}$ is bounded for any $y \in SOL$.

4. Convergence analysis

Here we show that Algorithm 3.3 generates a sequence $\{x_n\}$ which converges weakly to a solution of the underlying common solution to variational inequality under Assumption 3.1, 3.2 and 3.4.

Theorem 4.1. *Suppose that Assumption 3.1, 3.2 and 3.4 hold in H and $y \in SOL$. Let $\{x_n\}$ be generated by Algorithm 3.3. Then the following results hold:*

(i) $\{d_n\}$ is bounded;

(ii) $\sum \|x_{n+1} - x_n\|^2 < \infty$ *and*
(iii) the sequence $\{x_n\}$ *converges weakly to a point* $y \in SOL$.

Proof. Let $x^* \in SOL$. It follows from Assumption 3.2 (b) that $\beta_n \to 0, n \to \infty$ and so there exists $n_0 \in \mathbb{N}$ such that $\beta_n \leq \frac{1}{2}, \forall n \geq n_0$. Let

$$M_1 := \max\Big\{ \max_{1\leq k\leq n_0} \|d_k\|, \frac{\lambda}{2} \sup_{n\in\mathbb{N}} \|z_n - w_n\|\Big\}.$$

Then Assumption 3.4 (d) implies that $M_1 < \infty$. Suppose that $\|d_n\| \leq M_1$ for some $n \geq n_0$. Then by triangle inequality, we get

$$\|d_{n+1}\| = \|\frac{1}{\lambda}(z_n - w_n) + \beta_n d_n\| \leq \frac{1}{\lambda}\|z_n - w_n\| + \beta_n\|d_n\| \leq M_1. \tag{4.1}$$

This implies that $\|d_{n+1}\| \leq M_1$. Therefore, $\{d_n\}$ is bounded and (i) is obtained. From (3.1), we have

$$\begin{aligned} x_{n+1} &= \mu\gamma_n w_n + (1-\mu\gamma_n)y_n \\ &= \mu\gamma_n w_n + (1-\mu\gamma_n)(w_n + \lambda d_{n+1}) \\ &= \mu\gamma_n w_n + (1-\mu\gamma_n)(w_n + z_n - w_n + \lambda\beta_n d_n) \\ &= \mu\gamma_n w_n + (1-\mu\gamma_n)(z_n + \lambda\beta_n d_n) \\ &= w_n + (1-\mu\gamma_n)(z_n - w_n + \lambda\beta_n d_n). \end{aligned} \tag{4.2}$$

Let $v_n^k = w_n - \tau_n f_k(u_n^k)$. Then $z_n^k = P_{T_n^k}(v_n^k)$ and by Lemma 2.3 (i), we get

$$\begin{aligned} \|z_n^k - x^*\|^2 &= \|P_{T_n^k}(v_n^k) - x^*\|^2 \\ &= \langle P_{T_n^k}(v_n^k) - v_n^k + v_n^k - x^*, P_{T_n^k}(v_n^k) - v_n^k + v_n^k - x^*\rangle \\ &= \|v_n^k - x^*\|^2 + \|v_n^k - P_{T_n^k}(v_n)\|^2 + 2\langle P_{T_n^k}(v_n^k) - v_n^k, v_n^k - x^*\rangle. \end{aligned} \tag{4.3}$$

Since $x^* \in SOL \subseteq C_k \subseteq T_n^k$, and by the characterization of the metric projection (2.2), we derive

$$\begin{aligned} &2\|v_n^k - P_{T_n^k}(v_n^k)\|^2 + 2\langle P_{T_n^k}(v_n^k) - v_n^k, v_n^k - x^*\rangle \\ &\quad = 2\langle v_n^k - P_{T_n^k}(v_n^k), x^* - P_{T_n^k}(v_n^k)\rangle \leq 0. \end{aligned} \tag{4.4}$$

Thus,

$$\|v_n^k - P_{T_n^k}(v_n^k)\|^2 + 2\langle P_{T_n^k}(v_n^k) - v_n^k, v_n^k - x^*\rangle \leq -\|v_n^k - P_{T_n^k}(v_n^k)\|^2. \tag{4.5}$$

Using (2.3) in Algorithm 3.3 and by Lemma 2.3 (i), we obtain

$$\begin{aligned} \|z_n^k - x^*\|^2 &\leq \|v_n^k - x^*\|^2 - \|v_n^k - P_{T_n^k}(v_n^k)\|^2 \\ &= \|w_n - \tau_n f_k(u_n^k) - x^*\|^2 - \|w_n - \tau_n f_k(u_n^k) - z_n^k\|^2 \\ &= \|w_n - x^*\|^2 - \|w_n - z_n^k\|^2 + 2\tau_n\langle x^* - z_n^k, f_k(u_n^k)\rangle. \end{aligned} \tag{4.6}$$

The monotonicity of f_k and the fact that $x^* \in SOL$ imply that

$$\begin{aligned} 0 &\leq \langle f_k(u_n^k) - f_k(x^*), u_n^k - x^*\rangle = \langle f_k(u_n^k), u_n^k - x^*\rangle \\ &\quad -\langle f_k(x^*), u_n^k - x^*\rangle \\ &\leq \langle f_k(u_n^k), u_n^k - x^*\rangle = \langle f_k(u_n^k), u_n^k - z_n^k\rangle \\ &\quad +\langle f_k(u_n^k), z_n^k - x^*\rangle \end{aligned}$$

thus,

$$\langle x^* - z_n^k, f_k(u_n^k)\rangle \leq \langle f_k(u_n^k), u_n^k - z_n^k\rangle. \tag{4.7}$$

Using (4.7) in (4.6), we get

$$\begin{aligned} \|z_n^k - x^*\|^2 &\leq \|w_n - x^*\|^2 - \|w_n - z_n^k\|^2 + 2\tau_n\langle u_n^k - z_n^k, f_k(u_n^k)\rangle \\ &= \|w_n - x^*\|^2 + 2\tau_n\langle u_n^k - z_n^k, f_k(u_n^k)\rangle \\ &\quad -2\langle w_n - u_n^k, u_n^k - z_n^k\rangle - \|w_n - u_n^k\|^2 - \|u_n^k - z_n^k\|^2 \\ &= \|w_n - x^*\|^2 - \|w_n - u_n^k\|^2 - \|u_n^k - z_n^k\|^2 \\ &\quad +2\langle z_n^k - u_n^k, w_n - \tau_n f_k(u_n^k) - u_n^k\rangle. \end{aligned} \tag{4.8}$$

Observe that from the definition of T_n^k, we have

$$\begin{aligned} \langle w_n - \tau_n f_k(u_n^k) - u_n^k, z_n^k - u_n^k\rangle &= \langle w_n - \tau_n f_k(w_n) - u_n^k, z_n^k - u_n^k\rangle \\ &\quad +\tau_n\langle f_k(w_n) - f_k(u_n^k), z_n^k - u_n^k\rangle \\ &\leq \tau_n\langle f_k(w_n) - f_k(u_n^k), z_n^k - u_n^k\rangle. \end{aligned}$$

Using the last inequality in (4.8), we get

$$\begin{aligned} \|z_n^k - x^*\|^2 &\leq \|w_n - x^*\|^2 - \|w_n - u_n^k\|^2 - \|u_n^k - z_n^k\|^2 \\ &\quad + 2\tau_n\langle z_n^k - u_n^k, f_k(w_n) - f_k(u_n^k)\rangle \\ &\leq \|w_n - x^*\|^2 + 2\tau_n\|f_k(w_n) - f_k(u_n^k)\|\|z_n^k - u_n^k\| \\ &\quad - \|w_n - u_n^k\|^2 - \|u_n^k - z_n^k\|^2 \\ &\leq \|w_n - x^*\|^2 + 2\tau_n L\|w_n - u_n^k\|\|z_n^k - u_n^k\| \\ &\quad - \|w_n - u_n^k\|^2 - \|u_n^k - z_n^k\|^2 \\ &\leq \|w_n - x^*\|^2 + \tau_n L(\|w_n - u_n^k\|^2 + \|z_n^k - u_n^k\|^2) \\ &\quad - \|w_n - u_n^k\|^2 - \|u_n^k - z_n^k\|^2 \\ &\leq \|w_n - x^*\|^2 - (1 - \tau_n L)\|w_n - u_n^k\|^2 - (1 - \tau_n L)\|z_n^k - u_n^k\|^2. \end{aligned} \tag{4.9}$$

By (4.9) we get

$$\begin{aligned}
\|z_n - x^*\|^2 &= \Big\| \sum_{k=1}^N \delta_n^k z_n^k - x^* \Big\|^2 = \Big\| \sum_{k=1}^N \delta_n^k (z_n^k - x^*) \Big\|^2 \\
&\leq \sum_{k=1}^N \delta_n^k \|z_n^k - x^*\|^2 \\
&\leq \sum_{k=1}^N \delta_n^k \Big[\|w_n - x^*\|^2 - (1-\tau_n L)\|w_n - u_n^k\|^2 \\
&\quad -(1-\tau_n L)\|u_n^k - z_n^k\|^2 \Big] \\
&= \|w_n - x^*\|^2 - (1-\tau_n L) \sum_{k=1}^N \delta_n^k \|w_n - u_n^k\|^2 \\
&\quad -(1-\tau_n L) \sum_{k=1}^N \delta_n^k \|u_n^k - z_n^k\|^2. \qquad (4.10)
\end{aligned}$$

Now, we have from (3.1), Lemma 2.3 and (4.10) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \mu\gamma_n \|w_n - x^*\|^2 + (1-\mu\gamma_n)\|z_n - x^* + \lambda\beta_n d_n\|^2 \\
&\quad - \mu\gamma_n(1-\mu\gamma_n)\|z_n - w_n + \lambda\beta_n d_n\|^2 \\
&= \mu\gamma_n \|w_n - x^*\|^2 + (1-\mu\gamma_n)(\|z_n - x^*\|^2 + 2\lambda\beta_n\langle z_n - x^*, d_n\rangle \\
&\quad + \lambda^2\beta_n^2\|d_n\|^2) - \mu\gamma_n(1-\mu\gamma_n)\|z_n - w_n + \lambda\beta_n d_n\|^2 \\
&\leq \|w_n - x^*\|^2 + (1-\mu\gamma_n)\left\{2\lambda\beta_n\langle z_n - x^*, d_n\rangle + \lambda^2\beta_n^2\|d_n\|^2\right\} \\
&\quad - \mu\gamma_n(1-\mu\gamma_n)\|z_n - w_n + \lambda\beta_n d_n\|^2 \\
&= \|w_n - x^*\|^2 - \mu\gamma_n(1-\mu\gamma_n)\|z_n - w_n + \lambda\beta_n d_n\|^2 + \beta_n\varphi_n, \qquad (4.11)
\end{aligned}$$

where

$$\varphi_n = (1-\mu\gamma_n)(2\lambda\beta_n\langle z_n - x^*, d_n\rangle + \lambda^2\beta_n\|d_n\|^2). \qquad (4.12)$$

Using Assumption 3.4, it follows that $\{\varphi_n\}$ is bounded. Thus, there exists $M_2 > 0$ such that $\varphi_n \leq M_2$ for all $n \geq 1$. By Lemma 2.3 (ii), we get

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|(1+\alpha_n)(x_n - x^*) - \alpha_n(x_{n-1} - x^*)\|^2 \\
&= \|(1+\alpha_n)(x_n - x^*) - \alpha_n\|(x_{n-1} - x^*)\|^2 \\
&= (1+\alpha_n)\|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 \\
&\quad +\alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2, \qquad (4.13)
\end{aligned}$$

which by (4.11) implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 - (1+\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|x_{n-1} - x^*\|^2 \\
\leq -\mu\gamma_n(1-\mu\gamma_n)\|z_n - w_n + \lambda\beta_n d_n\|^2 \\
+\alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 + \beta_n\varphi_n. \qquad (4.14)
\end{aligned}$$

From (3.1) and (4.2), we get

$$\|z_n - w_n + \lambda\beta_n d_n\|^2 = \left\|\frac{x_{n+1} - w_n}{1-\mu\gamma_n}\right\|^2$$

$$= \left\|\frac{x_{n+1} - x_n - \alpha_n(x_n - x_{n-1})}{1-\mu\gamma_n}\right\|^2$$

$$= \frac{\|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle}{(1-\mu\gamma_n)^2}$$

$$\geq \frac{1}{(1-\mu\gamma_n)^2}\Big\{\|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2$$

$$+\alpha_n(-\rho_n\|x_{n+1} - x_n\|^2 - \frac{1}{\rho_n}\|x_n - x_{n-1}\|^2)\Big\}, \tag{4.15}$$

where $\rho_n = \dfrac{1}{(\alpha_n + \delta(1-\mu\gamma_n))}$. It follows from (4.14) and (4.15) that

$$\|x_{n+1} - x^*\|^2 - (1+\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|x_{n-1} - x^*\|^2$$
$$\leq \frac{\mu\gamma_n(\alpha_n\rho_n - 1)}{1-\mu\gamma_n}\|x_{n+1} - x_n\|^2 + \theta_n\|x_n - x_{n-1}\|^2 + \beta_n\varphi_n, \tag{4.16}$$

where

$$\theta_n = \alpha_n(1+\alpha_n) + \alpha_n\mu\gamma_n\frac{1-\rho_n\alpha_n}{\rho_n(1-\mu\gamma_n)} > 0, \text{for} n > 1. \tag{4.17}$$

Since $\rho_n\alpha_n < 1$ and $(1-\mu\gamma_n) \in (0,1)$. Now, taking into account the choice of ρ_n, we obtain $\delta = (1-\rho_n\alpha_n)/\rho_n(1-\mu\gamma_n)$ and, from (4.17), it follows

$$\theta_n = \alpha_n(1+\alpha_n) + \alpha_n\mu\gamma_n\delta \leq \alpha(1+\alpha) + \alpha\delta \tag{4.18}$$

Define the sequences $\{\phi_n\}$ and $\{\psi_n\}$ by

$$\phi_n := \|x_n - x^*\|^2, \quad \psi_n := \phi_n - \alpha_n\phi_{n-1} + \theta_n\|x_n - x_{n-1}\|^2$$

for all $n \geq 1$. Using the monotonicity of $\{\alpha_n\}$ and the fact that $\phi_n \geq 0$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\psi_{n+1} - \psi_n \quad \leq \quad & \phi_{n+1} - (1+\alpha_n)\phi_n + \alpha_n\phi_{n-1} + \theta_{n+1}\|x_{n+1} - x_n\|^2 \\ & -\theta_n\|x_n - x_{n-1}\|^2.\end{aligned} \tag{4.19}$$

By (4.16), we know

$$\begin{aligned}\psi_{n+1} - \psi_n \quad &\leq \quad \frac{\mu\gamma_n(\alpha_n\rho_n - 1)}{1-\mu\gamma_n}\|x_{n+1} - x_n\|^2 + \theta_{n+1}\|x_{n+1} - x_n\|^2 \\ &= \quad \left(\frac{\mu\gamma_n(\alpha_n\rho_n - 1)}{1-\mu\gamma_n} + \theta_{n+1}\right)\|x_{n+1} - x_n\|^2.\end{aligned} \tag{4.20}$$

Now, we claim that

$$\frac{\mu\gamma_n(\alpha_n\rho_n - 1)}{1-\mu\gamma_n} + \theta_{n+1} \leq -\sigma \tag{4.21}$$

Indeed, by the choice of ρ_n, we have

$$\begin{aligned}\frac{\mu\gamma_n(\alpha_n\rho_n-1)}{1-\mu\gamma_n}+\theta_{n+1}\le-\sigma \quad&\Longleftrightarrow\quad (1-\mu\gamma_n)(\theta_{n+1}+\sigma)+\mu\gamma_n(\alpha_n\rho_n-1)\le 0\\ &\Longleftrightarrow\quad (1-\mu\gamma_n)(\theta_{n+1}+\sigma)-\frac{\delta(1-\mu\gamma_n)\mu\gamma_n}{\alpha_n+\delta(1-\mu\gamma_n)}\le 0\\ &\Longleftrightarrow\quad (\alpha_n+\delta(1-\mu\gamma_n))(\theta_{n+1}+\sigma)+\delta(1-\mu\gamma_n)\le\delta\end{aligned}$$

Using (4.18), we have

$$\begin{aligned}(\alpha_n+\delta(1-\mu\gamma_n))(\theta_{n+1}+\delta)+\delta(1-\mu\gamma_n) \quad&\le\quad (\alpha+\delta(1-\mu\gamma_n))(\alpha(1+\alpha)\\ &\qquad +\alpha\delta+\sigma)+\delta(1-\mu\gamma_n)\\ &\le\quad \delta \end{aligned}\tag{4.22}$$

where the last inequality follows by using the upper bound for $\{1-\mu\gamma_n\}$ in Assumption 3.2 (c). Hence the claim in (4.21) is true. Thus it follows from (4.20) and (4.21) that

$$\psi_{n+1}-\psi_n\le-\sigma\|x_{n+1}-x_n\|^2, \tag{4.23}$$

which implies that

$$\psi_{n+1}\le\psi_n.$$

The boundedness of $\{\alpha_n\}$ delivers

$$-\alpha\phi_{n-1}\le\phi_n-\alpha\phi_{n-1}\le\psi_n\le\psi_1.$$

We then obtain

$$\begin{aligned}\phi_n \quad&\le\quad \alpha^n\phi_0+\psi_1\sum_{n=0}^{k-1}\alpha^n\\ &\le\quad \alpha^n\phi_0+\frac{1}{1-\alpha}\psi_1.\end{aligned}$$

Using (4.23) and the boundedness of $\{\psi_n\}$, we have

$$\begin{aligned}\sigma\sum_{k=1}^{n}\|x_{k+1}-x_k\|^2 \quad&\le\quad \psi_1-\psi_{n+1}\\ &\le\quad \psi_1+\alpha\phi_n\\ &\le\quad \psi_1+\alpha^n\phi_0+\frac{1}{1-\alpha}\psi_1,\end{aligned}$$

which implies that $\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2<\infty$. This establishes (ii).

Using (4.16), (4.18) and Lemma 2.4, we have that $\lim_{n\to\infty}\|x_n-x^*\|$ exists (since $\alpha_n\rho_n<1$ in (4.16)).

From (4.2), we have

$$\begin{aligned}\|z_n - w_n\| &= \left\|\frac{x_{n+1} - w_n}{1-\mu\gamma_n} - \lambda\beta_n d_n\right\| \\ &= \left\|\frac{x_{n+1} - x_n - \alpha_n(x_n - x_{n-1})}{1-\mu\gamma_n} - \lambda\beta_n d_n\right\| \\ &\le \frac{\|x_{n+1} - x_n\| + \alpha_n\|x_n - x_{n-1}\|}{1-\mu\gamma_n} + \lambda\beta_n\|d_n\|.\end{aligned}$$

By Assumption 3.2 (b) and the fact that $\{d_n\}$ is bounded and $\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2 < \infty$,

$$\|z_n - w_n\| \to 0, n \to \infty. \tag{4.24}$$

Observe that for some $M_4 > 0$,

$$\begin{aligned}\Big|\|w_n - x^*\|^2 - \|z_n - x^*\|^2\Big| &= \Big|\Big(\|w_n - x^*\| - \|z_n - x^*\|\Big)\Big(\|w_n - x^*\| + \|z_n - x^*\|\Big)\Big| \\ &\le \|w_n - z_n\|\Big(\|w_n - x^*\| + \|z_n - x^*\|\Big) \\ &\le \|w_n - z_n\|M_4 \to 0, n \to \infty.\end{aligned}$$

Then, from (4.10), we get

$$(1-\tau_n L)\sum_{k=1}^{N}\delta_n^k\|w_n - u_n^k\|^2 \to 0, n \to \infty.$$

Since $\tau_n < \frac{1}{L}$ and $\delta_n^k \ge \epsilon > 0, \forall k$, we get

$$\lim_{n\to\infty}\|w_n - u_n^k\| = 0, \forall k.$$

Similarly from (4.10) again, we have

$$\lim_{n\to\infty}\|u_n^k - z_n^k\| = 0, \forall k.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in H$. We show that $z \in SOL$. From that $\lim\|x_{n+1} - x_n\| = 0$ and

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$

we get $w_n - x_n \to 0, n \to \infty$. Since $x_{n_j} \rightharpoonup z$, we obtain $w_{n_j} \rightharpoonup z$. Now, $w_n - u_n^k \to 0$ implies $u_{n_j}^k \rightharpoonup z$ and since $u_n^k \in C_k$, we have that $z \in C_k$. For all $x \in C_k$ and using (2.2), we have (since f_k is monotone)

$$\begin{aligned}0 &\le \langle u_{n_j}^k - w_{n_j} + \tau_{n_j} f_k(w_{n_j}), x - u_{n_j}^k\rangle \\ &= \langle u_{n_j}^k - w_{n_j}, x - u_{n_j}^k\rangle + \tau_{n_j}\langle f_k(w_{n_j}), w_{n_j} - u_{n_j}^k\rangle \\ &\quad +\tau_{n_j}\langle f_k(w_{n_j}), x - w_{n_j}\rangle \\ &\le \langle u_{n_j}^k - w_{n_j}, x - w_{n_j}\rangle + \tau_{n_j}\langle f_k(w_{n_j}), w_{n_j} - u_{n_j}^k\rangle \\ &\quad +\tau_{n_j}\langle f_k(x), x - w_{n_j}\rangle.\end{aligned}$$

Passing to the limit, we get (recall that $0 < c \leq \tau_{n_j} \leq d < \frac{1}{L}$)

$$\langle f_k(x), x - z\rangle \geq 0, \forall x \in C_k.$$

By Lemma 2.6, we have that $z \in SOL(f_k, C_k), k = 1, 2, \ldots, N$. Hence, $z \in SOL$. Now, using Lemma 2.5, it follows that $\{x_n\}$ converges weakly to $z \in SOL$. This completes the proof. □

Remark 4.2. If the set $C_k, k = 1, 2, \ldots, N$ are simple enough, so that projections onto C_k are easily executed, then this Modified Inertial Extragradient Method below is particularly useful.

Algorithm 4.3.

- **Initialization.** Define $L := \max_{1\leq k\leq N} L_k$ and and choose $\{\tau_n\} \subset [c, d]$ for some $c, d \in (0, \frac{1}{L})$. Choose $\mu \in (0, 1]$, $\lambda > 0$ and $x_0, x_1 \in H$ arbitrarily. Define $u_0^k := P_{C_k}(x_0 - \tau_0 f_k(x_0))$. Let $z_0^k := P_{C_k}(x_0 - \tau_0 f_k(u_0^k))$ and $z_0 = \sum_{k=1}^{N} \delta_0^k z_0^k$. Compute $d_1 = \frac{(z_0 - x_0)}{\lambda}$. Choose sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ such that the conditions from Assumption 3.2 hold. Set $n := 1$
- **Step 1.** Compute

$$\begin{aligned} w_n : &= x_n + \alpha_n(x_n - x_{n-1}), \\ u_n^k : &= P_{C_k}(w_n - \tau_n f_k(w_n)), \forall k = 1, 2, \ldots, N. \end{aligned}$$

- **Step 2.** Compute

$$\begin{aligned} z_n^k : &= P_{C_k}(w_n - \tau_n f_k(u_n^k)), \forall k = 1, 2, \ldots, N, \\ z_n : &= \sum_{k=1}^{N} \delta_n^k z_n^k. \end{aligned}$$

- **Step 3.** Compute

$$\begin{aligned} d_{n+1} : &= \frac{1}{\lambda}(z_n - w_n) + \beta_n d_n, \\ y_n : &= w_n + \lambda d_{n+1}, \\ x_{n+1} &:= \mu\gamma_n w_n + (1 - \mu\gamma_n) y_n. \end{aligned} \tag{4.25}$$

- **Step 4.** Set $n \leftarrow n + 1$ and **goto Step 1**.

Following the same method of proof in Theorem 4.1, we can prove the following result using our Modified Inertial Extragradient Algorithm 4.3.

Theorem 4.4. *Suppose that Assumptions 3.1 3.2 and 3.4 hold in H and $y \in SOL$. Let $\{x_n\}$ be generated by Algorithm 4.3. Then the following results hold:*

(i) $\{d_n\}$ is bounded;

(ii) $\sum \|x_{n+1} - x_n\|^2 < \infty$ and

(iii) the sequence $\{x_n\}$ converges weakly to a point $y \in SOL$.

Next, we give another Modified Inertial Subgradient Extragradient Method for solving CSVIP (1.1). Here, the Step 3 in Algorithm 3.3 is modified and replaced. This algorithm very useful especially when N is large in Algorithm 3.3.

Algorithm 4.5.

- **Initialization.** Define $L := \max_{1\le k\le N} L_k$ and choose $\{\tau_n\} \subset [c,d]$ for some $c,d \in (0,\frac{1}{L})$. Choose $\mu \in (0,1]$, $\lambda > 0$ and $x_0, x_1 \in H$ arbitrarily. For each $k = 1,2,\dots,N$, define $u_0^k := P_{C_k}(x_0 - \tau_0 f_k(x_0))$ and set
$$T_0^k := \{w \in H : \langle x_0 - \tau_0 f_k(x_0) - u_0, w - u_0\rangle \le 0\}.$$
Compute $z_0^k := P_{T_0}(x_0 - \tau_0 f_k(u_0^k))$ and
$$k_n := \operatorname{argmax}\{\|z_0^k - w_0\| : k = 1,2,\dots,N\},\ z_0 := z_0^{k_n}.$$
Compute $d_1 = \frac{(z_0 - x_0)}{\lambda}$. Choose sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ such that the conditions from Assumption 3.2 hold. Set $n := 1$
- **Step 1.** Compute
$$\begin{aligned} w_n :&= x_n + \alpha_n(x_n - x_{n-1}),\\ u_n^k :&= P_{C_k}(w_n - \tau_n f_k(w_n)), \forall k = 1,2,\dots,N. \end{aligned}$$
- **Step 2.** Compute
$$\begin{aligned} z_n^k :&= P_{T_n^k}(w_n - \tau_n f_k(u_n^k)), \forall k = 1,2,\dots,N,\\ k_n :&= \operatorname{argmax}\{\|z_n^k - w_n\| : k = 1,2,\dots,N\}, z_n := z_n^{k_n}, \end{aligned}$$
where $T_n^k = \{w \in H : \langle w_n - \tau_n f_k(w_n) - u_n^k, w - u_n^k\rangle \le 0\}$.
- **Step 3.** Compute
$$\begin{aligned} d_{n+1} :&= \frac{1}{\lambda}(z_n - w_n) + \beta_n d_n,\\ y_n :&= w_n + \lambda d_{n+1},\\ x_{n+1} :&= \mu\gamma_n w_n + (1-\mu\gamma_n) y_n. \end{aligned} \tag{4.26}$$
- **Step 4.** Set $n \leftarrow n+1$ and **goto Step 1**.

Following the same method of proof in Theorem 4.1, we can prove the following result using our Algorithm 4.5.

Theorem 4.6. *Suppose that Assumptions 3.1 3.2 and 3.4 hold in H and $y \in SOL$. Let $\{x_n\}$ be generated by Algorithm 4.5. Then the following results hold:*
(i) $\{d_n\}$ is bounded;
(ii) $\sum \|x_{n+1} - x_n\|^2 < \infty$ and
(iii) the sequence $\{x_n\}$ converges weakly to a point $y \in SOL$.

Proof. From (4.24) and the definition of z_n, we have
$$\|z_n^k - w_n\| \to 0, n \to \infty, k = 1,2\dots,N.$$
From (4.9), we have
$$\begin{aligned} (1-\tau_n L)\|w_n - u_n^k\|^2 &\le \|w_n - x^*\|^2 - \|z_n^k - x^*\|^2\\ &\le \|w_n - z_n^k\|\Big(\|w_n - x^*\| + \|z_n^k - x^*\|\Big) \to 0,\\ &\quad n \to \infty,\ k = 1,2,\dots,N. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \|w_n - u_n^k\| = 0, \forall k.$$

Similarly, we can get from (4.9) that

$$\lim_{n\to\infty} \|u_n^k - z_n^k\| = 0, \forall k.$$

The rest of the proof follows as in the proof of Theorem 4.1. □

Remark 4.7. Suppose that common solutions set SOL of CSVIP (1.1) is bounded. Then it is possible to get a bounded, closed convex set C (e.g., C is a closed ball with a large enough radius) that contains SOL such that P_C can be easily computed. In this case, we can replace x_{n+1} in (3.1) with

$$x_{n+1} = P_C(\mu\gamma_n w_n + (1 - \mu\gamma_n)y_n).$$

Therefore, boundedness of C implies that $\{x_n\}$ is bounded and furthermore implies that $\{w_n\}$ and $\{z_n\}$ are all bounded. Hence, Assumption 3.4 is satisfied. Suppose common solutions set SOL of CSVIP (1.1) is not bounded, then we need to verify Assumption 3.4 is satisfied before applying our algorithm. Alternatively, we can set $\beta_n = 0, \forall n \geq 1$ in our Algorithm 3.3 and apply the scheme.

5. Numerical examples

In this section, we provide some concrete example including numerical results of the problem considered in Section 3 of this paper. All codes were written in Matlab 2012b and run on Hp $i-5$ Dual-Core 8.00 GB (7.78 GB usable) RAM laptop.
In all these examples, we choose $\lambda = 1$, $\beta_n = \frac{1}{n^2}$ and $\gamma_n = 0.8$.
Example 5.1. We first compare our Algorithm 3.3 for $N = 1$ with the subgradient extragradient algorithm of Censor *et al.* [10] using the following numerical example. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) := (x + y + \sin(x), -x + y + \sin(y)), x, y \in \mathbb{R}^2.$$

It was proved in Dong *et al.* [12] that f is monotone (in fact, strongly monotone) and Lipschitz with $L = \sqrt{10}$. Let $C := \{(x_1, x_2) \in \mathbb{R}^2 : -10 \leq x_1 \leq 100, 10 \leq x_2 \leq 100\}$. Take $\tau_n = \tau = \frac{1}{2L}$. Here, C is a box constraint which is obviously closed and convex since it is the Cartesian product of closed intervals, i.e.,

$$C = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] = [-10, 100] \times [10, 100].$$

Using Lemma 1.2.8 of [8], we have

$$\begin{aligned} P_C(x) = P_{[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]}(x_1, x_2) &= \Big(P_{[\alpha_1,\beta_1]}x_1, P_{[\alpha_2,\beta_2]}x_2\Big) \\ &= \Big(\max\{\min\{x_1, \beta_1\}, \alpha_1\}, \max\{\min\{x_2, \beta_2\}, \alpha_2\}\Big) \end{aligned}$$

Using randomly generated initial points $x_0, x_1 \in \mathbb{R}^2$ in MATLAB, $\mu = 0.01$ and using termination criterion $\frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|} < 10^{-4}$, we compare the proposed Algorithm in 3.3 with that of Censor *et al.* [10]. The results are reported in Table (1) and Figure (1).
Example 5.2. Furthermore, we also compare our Algorithm 4.3 for $N = 1$ with the Algorithm (7) of Dong *et al.* [12] using Example 4.2 of Dong *et al.* [12]. This

TABLE 1. Comparison between Algorithm 3.3 and Censor *et al.* [10] Algorithm

	No. of Iterations	CPU (Time)
Proposed Algorithm 3.3	18	1.5828×10^{-3}
Censor *et al.* [10] Algorithm	44	2.2433×10^{-3}

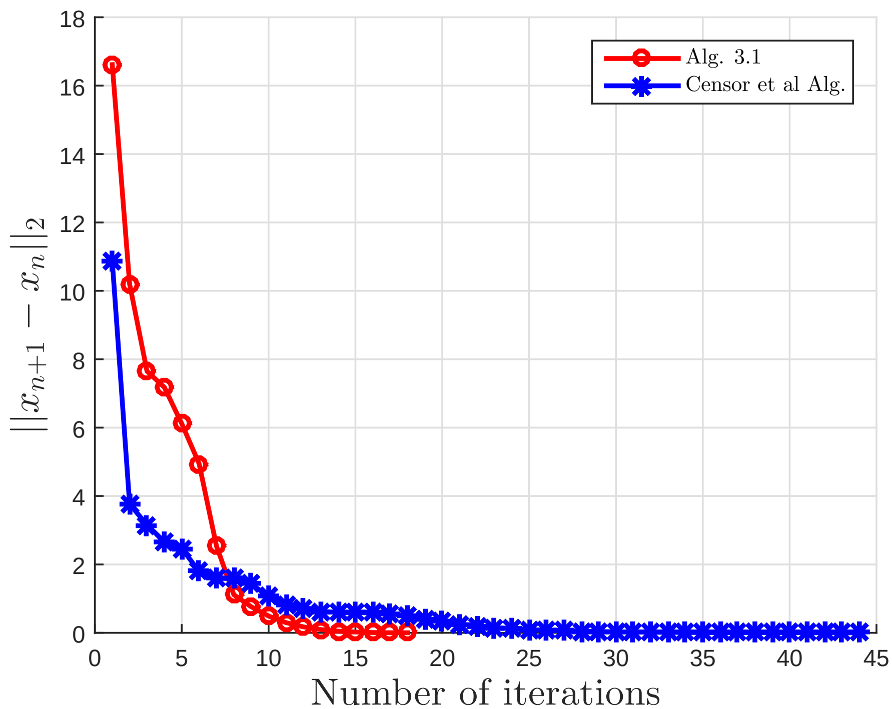

FIGURE 1. Comparison between Algorithm 3.3 and Censor *et al.* [10] Algorithm

numerical comparison makes sense since C is simple and P_C can be easily computed. Now, let $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x) = Ax + b$, where $A = Z^T Z, Z = (z_{ij})n \times n$ and $b = (b_i) \in \mathbb{R}^n$, where $z_{ij} \in [1, 100]$ and $b_i \in [-100, 0]$ are generated randomly. It is known from [29] that f is monotone and Lipschitz continuous with $L = \max(eig(A))$. Take $C := \{x \in \mathbb{R}^n : \|x - d\| \leq 2\}$ with $d = (1, -1)$. Suppose the initial point $x_0 = (c_i) \in \mathbb{R}^n$, where $c_i \in [1, 100]$ is generated randomly and $n = 100$ and $\tau_n = \tau = \frac{1}{(1.05L)}$.
Using randomly generated initial points $x_0 = (c_i) \in \mathbb{R}^{100}$, $x_1 = (u_i) \in \mathbb{R}^{100}$ in MATLAB where $c_i \in [1, 100]$ and $u_i \in [1, 5]$, $\mu = 0.01$, $\lambda_n = 0.8$ and using termination

criterion $\frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|} < 10^{-4}$, we compare the proposed Algorithm 4.3 with that of Dong *et al.* [12] Algorithm. The results are reported in Table (2) and Figure (2).

TABLE 2. Comparison between Algorithm 4.3 and Dong *et al.* [12] Algorithm

	No. of Iterations	CPU (Time)
Proposed Algorithm 4.3	11	0.057167
Dong *et al.* [12] Algorithm	97	0.52698

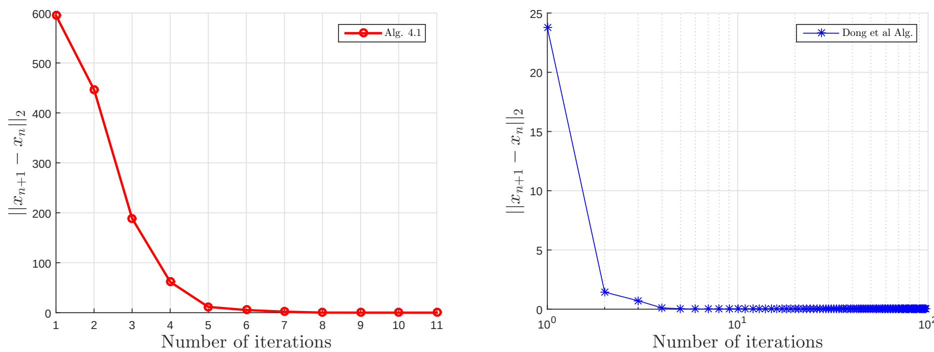

FIGURE 2. Comparison between Algorithm 4.3 and Dong *et al.* [12] Algorithm

Example 5.3. Using the same example in Censor *et al.* [11], we consider a two-disc convex feasibility problem in $\mathbb{R}^2$ and provide an explicit formulation of our Algorithm 3.3 for $N = 2$, as well as some numerical results. More explicitly, let $C_k := \{(x, y) \in \mathbb{R}^2 : (x - a_k)^2 + (y - b_k)^2 \leq r_k^2\}$ with $C_1 \cap C_2 \neq \emptyset$. Consider the problem of finding a point $(x^*, y^*) \in \mathbb{R}^2$ such that $(x^*, y^*) \in C_1 \cap C_2$. In this case $f_1 = f_2 = 0$. Let $\tau_n = \frac{1}{2}$. Now, let

$$C_1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \text{ and } C_2 := \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 \leq 1\}.$$

Using randomly generated initial points $x_0, x_1 \in \mathbb{R}^2$ in MATLAB, $\lambda_n = 0.8$, $\delta_n^k = 0.5$ for all n and $k = 1, 2$, and using termination criterion $\frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|} < 10^{-4}$, we examine our proposed Algorithm 3.3 with different values of μ: $\mu = 0.01$, $\mu = 0.5$ and $\mu = 0.9$. The results are reported in Table 3 and Figures 3-6.

TABLE 3. Algorithm 3.3 with different values of μ

	No. of Iterations	CPU (Time)
$\mu = 0.01$	13	1.1353×10^{-3}
$\mu = 0.1$	13	9.2373×10^{-4}
$\mu = 0.5$	11	9.5006×10^{-4}
$\mu = 0.9$	28	2.9693×10^{-3}

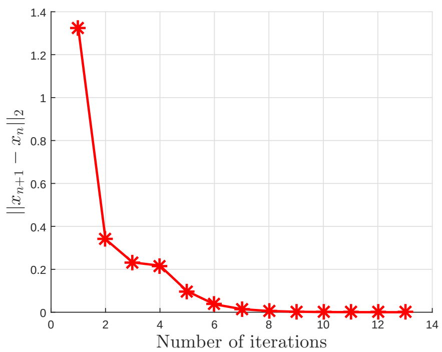

FIGURE 3. Algorithm 3.3 with $\mu = 0.01$

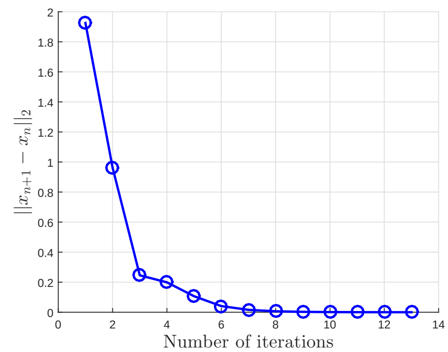

FIGURE 4. Algorithm 3.3 with $\mu = 0.1$

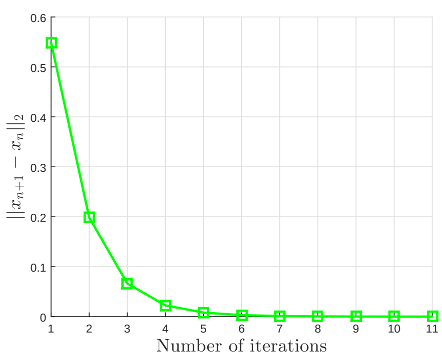

FIGURE 5. Algorithm 3.3 with $\mu = 0.5$

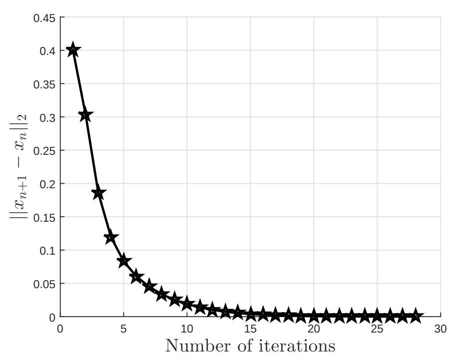

FIGURE 6. Algorithm 3.3 with $\mu = 0.9$

Remark 5.4.

(1) Over all, from all the three Examples, we could observe that our Algorithms are robust and easy to implement.
(2) From Example 5.1, our Algorithm 3.3 approaches the termination criterion faster than Censor *et al.* [10] Algorithm with less number of iterations.
(3) From Example 5.2, our Algorithm 4.3 approaches the termination criterion much faster with very small number of iterations when compared with Dong *et al.* [12] Algorithm.
(4) Furthermore in Example 5.3, we observe that, on average, for most of the choices of μ, number of iterations and CPU time taken to reach the termination criterion are the same. However, when approaching $\mu = 1$, there is a slight increase in the number of iterations and CPU time taken to reach the termination criterion.

6. Conclusion

In this paper, we establish weak convergence results of modified inertial versions of the subgradient extragradient method of Censor *et al.* [10] and the extragradient method of Korpelevich for solving Common Solutions to Variational Inequalities Problem (CSVIP) in real Hilbert spaces. We give numerical illustrations of our results and show the numerical improvements over existing results in the literature. In our next project, we will establish convergence rate of the iterative sequences generated by the modified inertial subgradient extragradient method. Also, extensions of the results in this paper will be considered in uniformly convex Banach spaces in the future.

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Received: June 7, 2017; Accepted: December 15, 2017.