Fixed Point Theory, 20(2019), No. 2, 663-682
DOI: 10.24193/fpt-ro.2019.2.44
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

NONLINEAR ITERATION METHOD FOR MONOTONE VARIATIONAL INEQUALITY AND FIXED POINT PROBLEM

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Abstract. Using the subgradient extragradient and Halpern methods, we prove two strong convergence results for finding a solution of a variational inequality problem involving Lipschitz continuous monotone with the Lipschitz constant unknown and the solution is also a fixed point a quasi-nonexpansive mapping in real Hilbert space.
Key Words and Phrases: Monotone mappings, subgradient extragradient method, Halpern method, quasi-nonexpansive mapping, strong convergence, fixed point, Hilbert spaces.
2010 Mathematics Subject Classification: 49J53, 65K10, 49M37, 90C25, 47H10.

1. Introduction

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$. Let C be a nonempty closed convex subset of H and A be a mapping of C into H. Then A is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \forall x, y \in C. \tag{1.1}$$

We say that A is L-Lipschitz continuous if there exists a positive constant L such that

$$\|Ax - Ay\| \leq L\|x - y\|, \ \forall x, y \in C.$$

In this paper, we consider the following variational inequality (for short, VI(A,C)): find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \forall y \in C. \tag{1.2}$$

Let Γ be the set of solutions of VI(A,C) (1.2). It is well known that x solves the VI(A,C) (1.2) if and only if x solves the fixed point equation (see [12] for the details)

$$x = P_C(x - \gamma Ax), \gamma > 0 \text{ and } r_\gamma(x) := x - P_C(x - \gamma Ax) = 0.$$

Therefore, the knowledge of fixed-point algorithms (see [10, 29]) can be used to solve VI(A,C) (1.2).
Variational inequality theory is an important tool in studying a wide class of obstacle, unilateral, and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework [3, 4, 12, 18, 19, 29]. This field is dynamic

and is experiencing an explosive growth in both theory and applications. Several numerical methods have been developed for solving variational inequality and related optimization problems, see books [5, 10, 19] and the references therein.
The extragradient method, introduced in 1976 by Korpelevich [20] and Antipin [1] for a finite-dimensional space, provides an iterative process converging to a solution of VI(A,C) by only assuming that $C \subset \mathbb{R}^n$ is nonempty, closed and convex and $A : C \to \mathbb{R}^n$ is monotone and L-Lipschitz continuous. Some methods have been introduced in the literature for finding a solution to VI(A,C) (1.1) when the monotone operator A is continuous in $\mathbb{R}^n$ (see, for example, [13, 31]). Quite recently, Mainge [24] introduced the following projected reflected gradient-type method in $\mathbb{R}^n$ for VI(A,C) (1.2) by incorporating a linesearch procedure that does not require any additional evaluation of P_C when A is *monotone and continuous mapping* in $\mathbb{R}^n$. The extragradient method was further extended to infinite dimensional spaces by many authors; see for instance, [2, 7, 8, 9, 14, 15, 16, 22, 17, 25, 23, 28, 26, 31, 33].
Before proceeding, we recall the following definitions.
A mapping $S : C \to C$ is called
• *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C;$$

and
• *quasi-nonexpansive* if

$$\|Sx - p\| \leq \|x - p\|, \ \forall x \in C, p \in F(S),$$

where $F(S)$ denotes its fixed point set, i.e.,

$$F(S) := \{x \in C : Sx = x\}.$$

In [27], Nadezhkina and Takahashi introduced an iterative process for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping and in [28], they introduced an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping using the two well-known methods of hybrid and extragradient and obtained a strong convergence theorem for the sequences generated by this process. Similarly, weak and strong convergence results have been obtained for finding a common element of the set of fixed points of a nonexpansive mapping (quasi-nonexpansive) and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping using the subgradient extragradient method in [8, 9, 21].
Inspired by the subgradient extragradient method studied by Censor *et al.* in [8, 9], Kraikaew and Saejung [21] proved the strong convergence of the iterative sequence generated by a combination of subgradient extragradient method and Halpern method for the problem of finding a common element of the solution set of a variational inequality and the fixed-point set of a quasi-nonexpansive mapping in real Hilbert spaces. In particular, they proved the following theorem.
Theorem 1.1. *Let $S : H \to H$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $A : H \to H$ a monotone and L-Lipschitz mapping on C.*

Let λ be a positive real number such that $\lambda L < 1$. Suppose that $F(S) \cap \Gamma \neq \emptyset$. Let $\{x_n\} \subset H$ be a sequence generated by $x_1 \in H$,

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n := \{w \in H : \langle x_n - \lambda A x_n - y_n, w - y_n \rangle \leq 0\}, \\ z_n = \alpha_n x_1 + (1-\alpha_n) P_{T_n}(x_n - \lambda A y_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S z_n, \end{cases}$$

where $\{\beta_n\} \subset [a,b] \subset]0,1[$ for some $a, b \in]0,1[$ and $\{\alpha_n\}$ is a sequence in $]0,1[$ satisfying $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{F(S)\cap\Gamma} x_1$.

We remark here that the framework presented by Kraikaew and Saejung [21] requires the Lipschitz constant of A as an input parameter. Thus, the result cannot be applied to the case when A is L-Lipschitz continuous but the Lipschitz constant L is unknown. It is our aim in this paper to establish strong convergence results for approximating a solution of VI(A,C) (1.2) when A is a Lipschitz continuous monotone operator but the Lipschitz constant is unknown and the solution is also a fixed point of a quasi-nonexpansive mapping in real Hilbert spaces. We propose two convergence methods and prove strong convergence of the sequences generated by our proposed methods. Our proposed algorithms are based on known processes of subgradient extragradient and Halpern methods and our results complement most of the existing known results on this subject, including [8, 9, 21, 27, 28]. Finally, we give some applications of our results.

The paper is therefore organized as follows: We first recall some basic results which will be used in the sequel in Section 2 and the main contribution of the paper is given in Section 3. In Section 4, we give some applications of our result and finally in Section 5, we conclude with some final remarks on our next focus on monotone variational inequalities.

2. Preliminaries

We state the following well-known lemmas which will be used in the sequel.

Lemma 2.1. *Let H be a real Hilbert space. The following well-known results hold:*

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H$;

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \forall x, y \in H$;

(iii) $\|tx+sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - st\|x-y\|^2, \forall x, y \in H, \forall t, s \in \mathbb{R}$.

Lemma 2.2. (Xu, [34]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \ n \geq 0,$$

where

(i) $\{a_n\} \subset [0,1], \ \sum \alpha_n = \infty$;

(ii) $\limsup \sigma_n \leq 0$;

(iii) $\gamma_n \geq 0; \ (n \geq 1), \ \sum \gamma_n < \infty$.

Then, $a_n \to 0$ as $n \to \infty$.

Let H be a real Hilbert space and C a nonempty, closed and convex subset of H. For any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\|, \ \forall y \in C.$$

P_C is called the *metric projection* of H onto K. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{2.1}$$

for all $x, y \in H$. Furthermore, $P_C x$ is characterized by the properties $P_C x \in C$,

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \tag{2.2}$$

for all $y \in C$ and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.3}$$

for all $x \in H$ and $y \in C$.

Lemma 2.3. (Lemma 7.1.7 of [32]) *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $A : C \to H$ be a monotone and hemicontinuous mapping and $z \in C$. Then*

$$z \in VI(C, A) \Leftrightarrow \langle Ax, x - z\rangle \geq 0 \text{ for all } x \in C.$$

The following lemma was proved in $\mathbb{R}^n$ by Fang and Chen [11] and it can be easily extended to infinite dimensional real Hilbert space H. The proof is similar to the proof of Lemma 3.1 in [6].

Lemma 2.4. (Lemma 6.3 of [11]) *For any $x \in H$ and $\beta > 0$,*

$$\min\{1, \beta\}\|r_1(x)\| \leq \|r_\beta(x)\| \leq \max\{1, \beta\}\|r_1(x)\|,$$

where $r_\beta(x) := x - P_C(x - \beta Ax)$ is the residual.

3. Main results

3.1. **The first Halpern type extragradient method.** In this subsection, we propose our first Halpern type extragradient-like method and prove that the sequences generated by the proposed method converge strongly to an element of Γ which is also a fixed point of a quasi-nonexpansive mapping. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $S : H \to H$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at the origin (i.e., if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$ and $Sx_n - x_n \to 0$, as $n \to 0$, then $x = Sx$). Let $A : H \to H$ be a Lipschitz continuous monotone mapping but the Lipschitz constant is unknown and $F(S) \cap \Gamma \neq \emptyset$. Suppose $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are sequences generated by the following manner:

Algorithm 3.1. Given $\rho \in (0, 1)$, $\mu \in (0, 1)$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be real sequences in (0,1). Let $x_1 \in H$ be arbitrary and given a fixed $u \in H$.

Step 1. Compute

$$y_n = P_C(x_n - \lambda_n Ax_n), \ \forall n \geq 1,$$

where $\lambda_n = \rho^{l_n}$ and l_n is the smallest non-negative integer l such that

$$\lambda_n\|Ax_n - Ay_n\| \leq \mu\|r_{\rho^{l_n}}(x_n)\| = \mu\|x_n - y_n\| \tag{3.1}$$

Step 2. Compute

$$\begin{cases} z_n = \alpha_n u + (1-\alpha_n)P_{T_n}(x_n - \lambda_n A y_n) \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S z_n,\ n \geq 1, \end{cases} \tag{3.2}$$

where $T_n := \{z \in H : \langle x_n - \lambda_n A x_n - y_n, z - y_n\rangle \leq 0\}$.
Set $n \leftarrow n+1$ and go to Step 1.
We first show that Algorithm 3.1 is well defined and implementable in this lemma.
Lemma 3.1. *There exists a non-negative integer l_n satisfying* (3.1).
Proof. Suppose $r_{\rho^{n_0}}(x_n) = 0$ for some $n_0 \geq 1$. Take $l_n = n_0$, which satisfies (3.1). Suppose that $r_{\rho^{n_1}}(x_n) \neq 0$ for some $n_1 \geq 1$ and assume the contrary that for all l, $y_l = P_C(x_n - \rho^l A x_n)$, $\rho^l \|Ax_n - Ay_l\| > \mu \|r_{\rho^l}(x_n)\|$. Then, by Lemma 2.4 and the fact that $\rho \in (0,1)$, we obtain

$$\begin{aligned} \|Ax_n - Ay_l\| &> \frac{\mu}{\rho^l}\|r_{\rho^l}(x_n)\| \\ &\geq \frac{\mu}{\rho^l}\min\{1,\rho^l\}\|r_1(x_n)\| \\ &= \mu\|r_1(x_n)\|. \end{aligned} \tag{3.3}$$

Using the fact that P_C is continuous, we have that

$$y_l = P_C(x_n - \rho^l A x_n) \to P_C(x_n),\ l \to \infty.$$

We consider two case: $x_n \in C$ and $x_n \notin C$.

(i) If $x_n \in C$, then $x_n = P_C(x_n)$. Now, since $r_{\rho^{n_1}}(x_n) \neq 0$ and $\rho^{n_1} \leq 1$, it follows from Lemma 2.4 that

$$\begin{aligned} 0 &< \|r_{\rho^{n_1}}(x_n)\| \leq \max\{1,\rho^{n_1}\}\|r_1(x_n)\| \\ &= \|r_1(x_n)\|. \end{aligned}$$

Letting $l \to \infty$ in (3.3), we have that

$$0 = \|Ax_n - Ax_n\| \geq \mu\|r_1(x_n)\| > 0.$$

This is a contradiction and hence (3.1) is valid.

(ii) If $x_n \notin C$, then

$$\rho^l\|Ax_n - Ay_n\| \to 0,\ l \to \infty$$

while

$$\lim_{l\to\infty}\mu\|r_{\rho^l}(x_n)\| = \mu\lim_{l\to\infty}\|x_n - P_C(x_n - \rho^l A x_n)\| = \mu\|x_n - P_c(x_n)\| > 0.$$

This is a contradiction. Therefore, Algorithm 3.1 is well defined and implementable.
Remark 3.2. We observe that since A is L-Lipschitz continuous on H in Lemma 3.1, then $\sup_{n\geq 1} l_n < \infty$. Indeed, $\forall x, y$, we have that $\rho^l\|Ax - Ay\| \leq \rho^l L\|x - y\|$ and it suffices to take l such that $\rho^l \leq \frac{\mu}{L}$. This does not depend on x and y. Also, note that $\sup_{n\geq 1} l_n < \infty$ implies that $\inf_{n\geq 1}\lambda_n > 0$. This is important for our convergence analysis.
We now prove the following theorem.
Theorem 3.3. *Assume that*

(*a*) $\lim_{n\to\infty}\alpha_n = 0$;

(*b*) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(*c*) $0 < a \leq \beta_n \leq b < 1$.

Then the sequences $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *generated by Algorithm 3.1 strongly converge to* $z \in \Gamma$, *where* $z = P_{F(S)\cap\Gamma}u$.

Proof. Let $z = P_{F(S)\cap\Gamma}u$ and $t_n = P_{T_n}(u_n)$ with $u_n = (x_n - \lambda_n Ay_n),\ \forall n \geq 1$. Then, by Lemma 2.1 (i), we have

$$\begin{aligned} \|t_n - z\|^2 &= \|P_{T_n}(u_n) - z\|^2 \\ &= \langle P_{T_n}(u_n) - u_n + u_n - z, P_{T_n}(u_n) - u_n + u_n - z\rangle \\ &= \|u_n - z\|^2 + \|u_n - P_{T_n}(u_n)\|^2 + 2\langle P_{T_n}(u_n) - u_n, u_n - z\rangle. \end{aligned} \tag{3.4}$$

since $z \in \Gamma \subseteq C \subseteq T_n$ and, by the characterization of the metric projection, we derive

$$\begin{aligned} 2\|u_n - P_{T_n}(u_n)\|^2 &+ 2\langle P_{T_n}(u_n) - u_n, u_n - z\rangle \\ &= 2\langle u_n - P_{T_n}(u_n), z - P_{T_n}(u_n)\rangle \leq 0 \end{aligned} \tag{3.5}$$

that

$$\|u_n - P_{T_n}(u_n)\|^2 + 2\langle P_{T_n}(u_n) - u_n, u_n - z\rangle \leq -\|u_n - P_{T_n}(u_n)\|^2. \tag{3.6}$$

We then obtain from Algorithm 3.1 and (2.3) that

$$\begin{aligned} \|t_n - z\|^2 &\leq \|u_n - z\|^2 - \|u_n - P_{T_n}(u_n)\|^2 \\ &= \|(x_n - \lambda_n Ay_n) - z\|^2 - \|(x_n - \lambda_n Ay_n) - t_n\|^2 \\ &= \|x_n - z\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle z - t_n, Ay_n\rangle. \end{aligned} \tag{3.7}$$

The monotonicity of the operator A implies that

$$\begin{aligned} 0 &\leq \langle Ay_n - Az, y_n - z\rangle = \langle Ay_n, y_n - z\rangle - \langle Az, y_n - z\rangle \\ &\leq \langle Ay_n, y_n - z\rangle = \langle Ay_n, y_n - t_n\rangle + \langle Ay_n, t_n - z\rangle. \end{aligned}$$

Thus,

$$\langle z - t_n, Ay_n\rangle \leq \langle Ay_n, y_n - t_n\rangle. \tag{3.8}$$

Using (3.8) in (3.7), we obtain

$$\begin{aligned} \|t_n - z\|^2 &\leq \|x_n - z\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &= \|x_n - z\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &- 2\langle x_n - y_n, y_n - t_n\rangle - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &= \|x_n - z\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle \\ &- \|x_n - y_n\|^2 - \|y_n - t_n\|^2. \end{aligned} \tag{3.9}$$

Observe that

$$\begin{aligned} \langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle &= \langle x_n - \lambda_n Ax_n - y_n, t_n - y_n\rangle + \langle \lambda_n Ax_n - \lambda_n Ay_n, t_n - y_n\rangle \\ &\leq \langle \lambda_n Ax_n - \lambda_n Ay_n, t_n - y_n\rangle. \end{aligned}$$

Using the last inequality in (3.9), we have that

$$\begin{aligned}
\|t_n - z\|^2 &\leq \|x_n - z\|^2 + 2\langle \lambda_n Ax_n - \lambda_n Ay_n, t_n - y_n\rangle - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\leq \|x_n - z\|^2 + 2\lambda_n \|Ax_n - Ay_n\| \|t_n - y_n\| - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\leq \|x_n - z\|^2 + 2\mu \|x_n - y_n\| \|t_n - y_n\| - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\leq \|x_n - z\|^2 + \mu(\|x_n - y_n\|^2 + \|t_n - y_n\|^2) - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&= \|x_n - z\|^2 - (1-\mu)\|x_n - y_n\|^2 - (1-\mu)\|y_n - t_n\|^2. \qquad (3.10)
\end{aligned}$$

We then obtain from (3.2) and (3.10) that

$$\begin{aligned}
\|x_{n+1} - z\| &\leq \beta_n \|x_n - z\| + (1-\beta_n)\|Sz_n - z\| \\
&\leq \beta_n \|x_n - z\| + (1-\beta_n)\|z_n - z\| \\
&= \beta_n \|x_n - z\| + (1-\beta_n)\|\alpha_n(u - z) + (1-\alpha_n)(t_n - z)\| \\
&\leq \beta_n \|x_n - z\| + (1-\beta_n)(\alpha_n \|u - z\| + (1-\alpha_n)\|t_n - z\|) \\
&\leq \beta_n \|x_n - z\| + (1-\beta_n)(\alpha_n \|u - z\| + (1-\alpha_n)\|x_n - z\|) \\
&\leq \max\Big\{\|x_n - z\|, \|u - z\|\Big\} \\
&\ \ \vdots \\
&\leq \max\Big\{\|x_1 - z\|, \|u - z\|\Big\}.
\end{aligned}$$

This implies that $\{x_n\}$ is bounded. Consequently, $\{t_n\}, \{y_n\}$ and $\{z_n\}$ are also bounded.
Then using Lemma 2.1 (ii), (iii) and (3.10), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\beta_n(x_n - z) + (1-\beta_n)(Sz_n - z)\|^2 \\
&= \beta_n \|x_n - z\|^2 + (1-\beta_n)\|Sz_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - Sz_n\|^2 \\
&\leq \beta_n \|x_n - z\|^2 + (1-\beta_n)\|z_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - Sz_n\|^2 \\
&= \beta_n \|x_n - z\|^2 + (1-\beta_n)\|\alpha_n(u - z) + (1-\alpha_n)(t_n - z)\|^2 \\
&\quad -\beta_n(1-\beta_n)\|x_n - Sz_n\|^2 \\
&\leq \beta_n \|x_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - Sz_n\|^2 \\
&\quad +(1-\beta_n)((1-\alpha_n)^2\|t_n - z\|^2 + 2\alpha_n\langle u - z, z_n - z\rangle) \\
&\leq \beta_n \|x_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - Sz_n\|^2 \\
&\quad +(1-\beta_n)((1-\alpha_n)\|t_n - z\|^2 + 2\alpha_n\langle u - z, z_n - z\rangle) \\
&\leq (1 - \alpha_n(1-\beta_n))\|x_n - z\|^2 + 2\alpha_n(1-\beta_n)\langle u - z, z_n - z\rangle \\
&\quad -\beta_n(1-\beta_n)\|x_n - Sz_n\|^2. \qquad (3.11)
\end{aligned}$$

Furthermore, we obtain

$$\|x_{n+1} - z\|^2 \leq (1 - \alpha_n(1-\beta_n))\|x_n - z\|^2 + 2\alpha_n(1-\beta_n)\langle u - z, z_n - z\rangle. \qquad (3.12)$$

The rest of the proof will be divided into two parts.
Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - z\|\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{\|x_n - z\|\}_{n=1}^{\infty}$ converges and $\|x_n - z\|^2 - \|x_{n+1} - z\|^2 \to 0,\ n \to \infty$. From

(3.11), we have that

$$\beta_n(1-\beta_n)\|x_n - Sz_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M_1, \tag{3.13}$$

for some $M_1 > 0$. Thus,

$$\|x_n - Sz_n\| \to 0,\ n \to \infty.$$

Furthermore, we have from (3.2) and (3.10) that

$$\begin{aligned} 0 &= \liminf_{n\to\infty}(\|x_{n+1} - z\| - \|x_n - z\|) \\ &\le \liminf_{n\to\infty}(\beta_n\|x_n - z\| + (1-\beta_n)\|Sz_n - z\| - \|x_n - z\|) \\ &\le \liminf_{n\to\infty}(1-\beta_n)(\alpha_n\|u - z\| + (1-\alpha_n)\|t_n - z\| - \|x_n - z\|) \\ &= \liminf_{n\to\infty}(1-\beta_n)(\|t_n - z\| - \|x_n - z\|) \\ &\le (1-a)\liminf_{n\to\infty}(\|t_n - z\| - \|x_n - z\|) \\ &\le (1-a)\limsup_{n\to\infty}(\|t_n - z\| - \|x_n - z\|) \\ &\le 0. \end{aligned}$$

So,

$$\limsup_{n\to\infty}(\|t_n - z\| - \|x_n - z\|) = 0.$$

We obtain from (3.10) that

$$\begin{aligned} (1-\mu)\|x_n - y_n\|^2 &\le \|x_n - z\|^2 - \|t_n - z\|^2 \\ &= (\|x_n - z\| - \|t_n - z\|)(\|x_n - z\| + \|t_n - z\|) \\ &\le (\|x_n - z\| - \|t_n - z\|)M_2, \end{aligned}$$

for some $M_2 > 0$. Thus

$$\limsup_{n\to\infty}\|x_n - y_n\| = 0$$

and this implies that

$$\|x_n - y_n\| \to 0,\ n \to \infty.$$

From (3.10) again, we have

$$\begin{aligned} (1-\mu)\|y_n - t_n\|^2 &\le \|x_n - z\|^2 - \|t_n - z\|^2 \\ &= (\|x_n - z\| - \|t_n - z\|)(\|x_n - z\| + \|t_n - z\|) \\ &\le (\|x_n - z\| - \|t_n - z\|)M_2, \end{aligned} \tag{3.14}$$

from which we have

$$\|y_n - t_n\| \to 0,\ n \to \infty.$$

Furthermore,

$$\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\| \to 0,\ n \to \infty$$

and from (3.2), we get

$$\|z_n - t_n\| = \alpha_n\|u - t_n\| \to 0,\ n \to \infty$$

and

$$\|x_n - z_n\| \le \|x_n - t_n\| + \|z_n - t_n\| \to 0,\ n \to \infty.$$

Also

$$\|z_n - Sz_n\| \leq \|z_n - t_n\| + \|x_n - Sz_n\| + \|x_n - t_n\| \to 0,\ n \to \infty$$

and

$$\|x_{n+1} - x_n\| = (1 - \beta_n)\|x_n - Sz_n\| \to 0,\ n \to \infty.$$

Since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_j}\}$ such that $\{x_{n_j}\}$ converges weakly to some $w \in H$ and $\limsup_{n\to\infty}\langle u-z, z-x_n\rangle = \lim_{j\to\infty}\langle u-z, z-x_{n_j}\rangle$. We show that $w \in \Gamma$. Now, $x_n - y_n \to 0$ implies that $y_{n_j} \rightharpoonup w$ and since $y_n \in C$, we then have that $w \in C$. For all $x \in C$ and using (2.2), we have that (since A is monotone)

$$\begin{aligned} 0 &\leq \langle y_{n_j} - x_{n_j} + \lambda_{n_j} A x_{n_j}, x - y_{n_j}\rangle \\ &= \langle y_{n_j} - x_{n_j}, x - y_{n_j}\rangle + \lambda_{n_j}\langle A x_{n_j}, x_{n_j} - y_{n_j}\rangle \\ &\quad + \lambda_{n_j}\langle A x_{n_j}, x - x_{n_j}\rangle \\ &\leq \langle y_{n_j} - x_{n_j}, x - y_{n_j}\rangle + \lambda_{n_j}\langle A x_{n_j}, x_{n_j} - y_{n_j}\rangle \\ &\quad + \lambda_{n_j}\langle Ax, x - x_{n_j}\rangle. \end{aligned}$$

Passing to the limit, we get

$$\langle Ax, x - w\rangle \geq 0,\ \forall x \in C.$$

By Lemma 2.3, we have that $w \in \Gamma$.

Since $\{x_{n_j}\}$ converges weakly to some $w \in H$ and $x_n - z_n \to 0,\ n \to \infty$, we have that $\{z_{n_j}\}$ converges weakly to some $w \in H$. By demiclosedness of $I - S$ at origin and the fact that $\|z_n - Sz_n\| \to 0,\ n \to \infty$, we have that $w \in F(S)$. Hence, $w \in F(S) \cap \Gamma$. Since $z = P_{F(S)\cap\Gamma}u$, we have that

$$\begin{aligned} \limsup_{n\to\infty}\langle u - z, z - z_n\rangle &= \lim_{j\to\infty}\langle u - z, z - z_{n_j}\rangle \\ &= \langle u - z, z - w\rangle \\ &\geq 0. \end{aligned}$$

Since $\|x_{n+1} - x_n\| \to 0,\ n \to \infty$, we have that

$$\limsup_{n\to\infty}\langle u - z, x_{n+1} - z\rangle \leq 0.$$

Using Lemma 2.2 in (3.12), we obtain $\lim_{n\to\infty}\|x_n - z\| = 0$. Thus, $x_n \to z, n \to \infty$.

Case 2. Assume that $\{\|x_n - z\|\}$ is not monotonically decreasing sequence. Set $\Gamma_n = \|x_n - z\|^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough)by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0.$$

This implies that $\|x_{\tau(n)} - z\| \leq \|x_{\tau(n)+1} - z\|, \forall n \geq n_0$. Thus $\lim_{n\to\infty}\|x_{\tau(n)} - z\|$ exists. Following the arguments in Case 1, we can show that

$$\|x_{\tau(n)} - Sz_{\tau(n)}\| \to 0,\ n \to \infty,$$

$$\limsup_{n\to\infty}(\|t_{\tau(n)} - z\| - \|x_n - \tau(n)\|) = 0,$$

$$\|x_{\tau(n)} - y_{\tau(n)}\| \to 0,\ n \to \infty,$$
$$\|y_{\tau(n)} - t_{\tau(n)}\| \to 0,\ n \to \infty,$$
$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0,\ n \to \infty$$

and

$$\|z_{\tau(n)} - Sz_{\tau(n)}\| \to 0,\ n \to \infty.$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$ which converges weakly to w. Observe that since $\lim_{n\to\infty}\|x_{\tau(n)} - y_{\tau(n)}\| = 0$, we also have $y_{\tau(n)} \rightharpoonup w$. By similar argument in Case 1, we can show that $w \in F(S) \cap \Gamma$ and

$$\limsup_{n\to\infty}\langle u - z, z - z_{\tau(n)}\rangle \geq 0.$$

By (3.12), we obtain that

$$\|x_{\tau(n)+1} - z\|^2 \leq (1 - \alpha_{\tau(n)}(1 - \beta_{\tau(n)}))\|x_{\tau(n)} - z\|^2 + 2\alpha_{\tau(n)}(1 - \beta_{\tau(n)})\langle u - z, z_{\tau(n)} - z\rangle.$$

which implies that (noting that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)}(1 - \beta_{\tau(n)}) > 0$)

$$\|x_{\tau(n)} - z\|^2 \leq 2\langle u - z, z_{\tau(n)} - z\rangle.$$

This implies that

$$\limsup_{n\to\infty}\|x_{\tau(n)} - z\| \leq 0.$$

Thus,

$$\lim_{n\to\infty}\|x_{\tau(n)} - z\| = 0.$$

and

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - z\| = 0.$$

Therefore,

$$\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$), because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence, $\lim \Gamma_n = 0$, that is, $\lim_{n\to\infty}\|x_n - z\| = 0$. Hence, $\{x_n\}$ converges strongly to z. Similarly, $y_n \to z$. This completes the proof.

Remark 3.4. For example, our iterative Algorithm 3.1 complements the scheme of Kraikaew and Saejung [21] (so also [28, 26, 25]), where the Lipschitz constant of A has to be known apriori while in our results, we obtain strong convergence results when the Lipschitz constant of A is unknown and the Lipschitz constant is not used in our scheme as an input parameter in an infinite dimensional Hilbert space.

If $S := I$, the identity mapping, then our Theorem 3.3 reduces to the following corollary.

Corollary 3.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : H \to H$ be a Lipschitz continuous monotone mapping and $\Gamma \neq \emptyset$. Let*

$u \in H$ be fixed but arbitrary. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences generated by the following manner:

Algorithm 3.2. *Given $\rho \in (0,1)$, $\mu \in (0,1)$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be real sequences in (0,1). Let $x_1 \in H$ be arbitrary.*

Step 1. *Compute*

$$y_n = P_C(x_n - \lambda_n Ax_n), \ \forall n \geq 1,$$

where $\lambda_n = \rho^{l_n}$ and l_n is the smallest non-negative integer l such that

$$\lambda_n \|Ax_n - Ay_n\| \leq \mu \|r_{\rho^{l_n}}(x_n)\| = \mu \|x_n - y_n\|$$

Step 2. *Compute*

$$\begin{cases} z_n = \alpha_n u + (1-\alpha_n) P_{T_n}(x_n - \lambda_n Ay_n) \\ x_{n+1} = \beta_n x_n + (1-\beta_n) z_n, \ n \geq 1, \end{cases}$$

where $T_n := \{z \in H : \langle x_n - \lambda_n Ax_n - y_n, z - y_n\rangle \leq 0\}$.
Set $n \leftarrow n+1$ and go to Step 1.

Assume further that

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < a \leq \beta_n \leq b < 1$.

Then the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ generated by Algorithm 3.2 strongly converge to $z \in \Gamma$, where $z = P_\Gamma u$.

3.2. **The second Halpern type extragradient method.** In this subsection we present another Halpern type subgradient extragradient method which finds a solution of the Variational inequality for a Lipschitz continuous monotone operator whose Lipschitz constant is unknown, which is also a fixed point of a given quasi-nonexpansive mapping. Then, we establish a strong convergence theorem of the sequence generated by our scheme.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $S : H \to H$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at the origin and denote by $F(S)$ its fixed point set. Let $A : H \to H$ be a Lipschitz continuous monotone mapping but the Lipschitz constant is unknown and $F(S) \cap \Gamma \neq \emptyset$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences generated by the following manner:

Algorithm 4.1. Given $\rho \in (0,1)$, $\mu \in (0,1)$. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty}$ and $\{\omega_n\}_{n=1}^{\infty}$ be real sequences in (0,1) such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $x_1 \in H$ be arbitrary and given a fixed $u \in H$.

Step 1. Compute

$$y_n = P_C(x_n - \lambda_n Ax_n), \ \forall n \geq 1,$$

where $\lambda_n = \rho^{l_n}$ and l_n is the smallest non-negative integer l such that

$$\lambda_n \|Ax_n - Ay_n\| \leq \mu \|r_{\rho^{l_n}}(x_n)\| = \mu \|x_n - y_n\| \tag{3.15}$$

Step 2. Compute

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(\omega_n Sx_n + (1-\omega_n) P_{T_n}(x_n - \lambda_n Ay_n)), \ n \geq 1, \tag{3.16}$$

where $T_n := \{z \in H : \langle x_n - \lambda_n Ax_n - y_n, z - y_n\rangle \leq 0\}$.
Set $n \leftarrow n+1$ and go to Step 1.

Remark 3.6. By Lemma 3.1, Algorithm 4.1 is well defined and implementable.

Theorem 3.7. *Assume that*

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\beta_n \geq \epsilon_1 > 0,\ \gamma_n \geq \epsilon_2 > 0$;

(d) $0 < c \leq \omega_n \leq d < 1$.

Then the sequences $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *generated by Algorithm 4.1 strongly converge to* $z \in \Gamma$, *where* $z = P_{F(S)\cap\Gamma}u$.

Proof. Let $z = P_{F(S)\cap\Gamma}u$ and $t_n = P_{T_n}(u_n)$ with $u_n = (x_n - \lambda_n Ay_n),\ \forall n \geq 1$. Then following the method of proof in Theorem 3.3, we can show that

$$\|t_n - z\|^2 \leq \|x_n - z\|^2 - (1-\mu)\|x_n - y_n\|^2 - (1-\mu)\|y_n - t_n\|^2.$$

Let $z_n := \omega_n Sx_n + (1-\omega_n)t_n,\ \forall n \geq 1$. Then

$$\begin{aligned}\|z_n - z\| &\leq \omega_n\|Sx_n - z\| + (1-\omega_n)\|t_n - z\|\\ &\leq \omega_n\|x_n - z\| + (1-\omega_n)\|x_n - z\|\\ &= \|x_n - z\|.\end{aligned}$$

Furthermore, by (3.16), we have

$$\begin{aligned}\|x_{n+1} - z\| &\leq \alpha_n\|u - z\| + \beta_n\|x_n - z\| + \gamma_n\|z_n - z\|\\ &\leq \alpha_n\|u - z\| + \beta_n\|x_n - z\| + \gamma_n\|x_n - z\|\\ &= \alpha_n\|u - z\| + (1-\alpha_n)\|x_n - z\|\\ &\leq \max\Big\{\|x_n - z\|, \|u - z\|\Big\},\end{aligned}$$

which by induction implies that $\{x_n\}$ is bounded. So also is $\{z_n\}$. By Lemma 2.1 (ii) and (iii), we get

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n(u - z) + \beta_n(x_n - z) + \gamma_n(z_n - z)\|^2\\ &\leq \|\beta_n(x_n - z) + \gamma_n(z_n - z)\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle\\ &= \beta_n(\beta_n + \gamma_n)\|x_n - z\|^2 + \gamma_n(\beta_n + \gamma_n)\|z_n - z\|^2\\ &\quad -\beta_n\gamma_n\|z_n - x_n\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle\\ &\leq \beta_n(\beta_n + \gamma_n)\|x_n - z\|^2 + \gamma_n(\beta_n + \gamma_n)\|x_n - z\|^2\\ &\quad -\beta_n\gamma_n\|z_n - x_n\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle\\ &= (\beta_n + \gamma_n)^2\|x_n - z\|^2 - \beta_n\gamma_n\|z_n - x_n\|^2\\ &\quad +2\alpha_n\langle u - z, x_{n+1} - z\rangle\\ &\leq (1-\alpha_n)\|x_n - z\|^2 - \beta_n\gamma_n\|z_n - x_n\|^2\\ &\quad +2\alpha_n\langle u - z, x_{n+1} - z\rangle. \qquad (3.17)\end{aligned}$$

We now distinguish two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{||x_n - z||\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{||x_n - z||\}_{n=1}^{\infty}$ converges and $||x_n - z||^2 - ||x_{n+1} - z||^2 \to 0,\ n \to \infty$. By the boundedness of $\{x_n\}$, we have from (3.17) that

$$\beta_n\gamma_n\|z_n - x_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M, \tag{3.18}$$

for some $M > 0$. By Condition (c), we have that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0.$$

Observe that

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n u + \beta_n x_n + \gamma_n z_n - (\alpha_n x_n + \beta_n x_n + \gamma_n x_n) \\ &= \alpha_n(u - x_n) + \gamma_n(z_n - x_n). \end{aligned}$$

This implies that

$$\|x_{n+1} - x_n\| \le \alpha_n\|u - x_n\| + \gamma_n\|z_n - x_n\| \to 0,\ n \to \infty.$$

Also,

$$\|x_{n+1} - z_n\| \le \|x_{n+1} - x_n\| + \|x_n - z_n\| \to 0,\ n \to \infty.$$

By (3.16), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\|z_n - z\|^2 \\ &\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\Big(\omega_n\|Sx_n - z\|^2 \\ &\quad +(1-\omega_n)\|t_n - z\|^2\Big) \\ &\le \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \omega_n\gamma_n\|x_n - z\|^2 \\ &\quad +\gamma_n(1-\omega_n)\|t_n - z\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} -\|t_n - z\|^2 \le\ & \frac{1}{\gamma_n(1-\omega_n)}\Big[\alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 \\ & +\omega_n\gamma_n\|x_n - z\|^2 - \|x_{n+1} - z\|^2\Big]. \end{aligned} \tag{3.19}$$

Using (3.19) in (3.10), we have

$$\begin{aligned} (1-\mu)\|x_n - y_n\|^2 &\le \|x_n - z\|^2 - \|t_n - z\|^2 \\ &\le \|x_n - z\|^2 - \frac{1}{\gamma_n(1-\omega_n)}\|x_{n+1} - z\|^2 + \frac{\alpha_n}{\gamma_n(1-\omega_n)}\|u - z\|^2 \\ &\quad + \frac{\beta_n + \omega_n\gamma_n}{\gamma_n(1-\omega_n)}\|x_n - z\|^2 \end{aligned}$$

$$\begin{aligned}&= \frac{1-\alpha_n}{\gamma_n(1-\omega_n)}\|x_n - z\|^2 - \frac{1}{\gamma_n(1-\omega_n)}\|x_{n+1} - z\|^2\\&+ \frac{\alpha_n}{\gamma_n(1-\omega_n)}\|u - z\|^2\\&= \frac{1}{\gamma_n(1-\omega_n)}\Big[\|x_n - z\|^2 - \|x_{n+1} - z\|^2\Big]\\&+ \frac{\alpha_n}{\gamma_n(1-\omega_n)}\Big[\|u - z\|^2 - \|x_n - z\|^2\Big].\end{aligned}$$

This implies that

$$\|x_n - y_n\| \to 0, \ n \to \infty.$$

Similarly, by (3.19) and (3.10), we can show that

$$\|y_n - t_n\| \to 0, \ n \to \infty.$$

Hence,

$$\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\| \to 0, \ n \to \infty.$$

Now,

$$\|z_n - t_n\| \le \|x_n - t_n\| + \|x_n - z_n\| \to 0, \ n \to \infty.$$

From $z_n = \omega_n S x_n + (1-\omega_n)t_n$, we get

$$\|Sx_n - t_n\| = \frac{1}{\omega_n}\|z_n - t_n\| \to 0, \ n \to \infty.$$

Furthermore,

$$\|x_n - Sx_n\| \le \|Sx_n - t_n\| + \|x_n - t_n\| \to 0, \ n \to \infty. \tag{3.20}$$

Since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_j}\}$ such that $\{x_{n_j}\}$ converges weakly to some $w \in H$ and $\limsup_{n\to\infty}\langle u-z, z-x_n\rangle = \lim_{j\to\infty}\langle u-z, z-x_{n_j}\rangle$. Following the method of proof in Theorem 3.3, we can show that $w \in \Gamma$. Also, by the demiclosedness principle of $I - S$ and (3.20), we have that $w \in F(S)$. Hence, $w \in F(S) \cap \Gamma$. Consequently,

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z, z - x_n\rangle &= \lim_{j\to\infty}\langle u - z, z - x_{n_j}\rangle\\&= \langle u - z, z - w\rangle\\&\ge 0.\end{aligned}$$

Since $\|x_{n+1} - x_n\| \to 0, \ n \to \infty$, we have that

$$\limsup_{n\to\infty}\langle u - z, x_{n+1} - z\rangle \le 0.$$

From (3.17) we have

$$\|x_{n+1} - z\|^2 \le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \tag{3.21}$$

Using Lemma 2.2 in (3.21), we obtain $\lim_{n\to\infty}\|x_n - z\| = 0$. Thus, $x_n \to z, n \to \infty$.

Case 2. Assume that $\{\|x_n - z\|\}$ is not monotonically decreasing sequence. Set $\Gamma_n = \|x_n - z\|^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough)by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\}.$$

Clearly, τ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \le \Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \forall n \ge n_0.$$

This implies that $\|x_{\tau(n)} - z\| \le \|x_{\tau(n)+1} - z\|, \forall n \ge n_0$. Thus $\lim_{n\to\infty} \|x_{\tau(n)} - z\|$ exists. By using similar arguments as in Case 1, we obtain

$$\|x_{\tau(n)} - y_{\tau(n)}\| \to 0,\ n \to \infty, \|y_{\tau(n)} - t_{\tau(n)}\| \to 0,\ n \to \infty.$$

and

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0,\ n \to \infty.$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$ which converges weakly to w. Observe that since $\lim_{n\to\infty} \|x_{\tau(n)} - y_{\tau(n)}\| = 0$, we also have $y_{\tau(n)} \rightharpoonup w$. By similar argument in Case 1, we can show that $w \in F(S) \cap \Gamma$ and

$$\limsup_{n\to\infty} \langle u - z, z - x_{\tau(n)}\rangle \ge 0.$$

Since $\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0,\ n \to \infty$ and $\limsup_{n\to\infty} \langle u - z, z - x_{\tau(n)}\rangle \ge 0$, we can show that

$$\limsup_{n\to\infty} \langle u - z, x_{\tau(n)+1} - z\rangle \le 0.$$

By (3.17), we have

$$\|x_{\tau(n)+1} - z\|^2 \le (1 - \alpha_{\tau(n)})\|x_{\tau(n)} - z\|^2 + 2\alpha_{\tau(n)}\langle u - z, x_{\tau(n)+1} - z\rangle,$$

which implies that (noting that $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)} > 0$)

$$\|x_{\tau(n)} - z\|^2 \le \langle u - z, x_{\tau(n)+1} - z\rangle.$$

This implies that

$$\limsup_{n\to\infty} \|x_{\tau(n)} - z\| \le 0.$$

Thus,

$$\lim_{n\to\infty} \|x_{\tau(n)} - z\| = 0.$$

and

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - z\| = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \ge n_0$, it is easy to see that $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ if $n \ne \tau(n)$ (that is $\tau(n) < n$), because $\Gamma_j \ge \Gamma_{j+1}$ for $\tau(n) + 1 \le j \le n$. As a consequence, we obtain for all $n \ge n_0$,

$$0 \le \Gamma_n \le \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence, $\lim \Gamma_n = 0$, that is, $\lim_{n\to\infty} \|x_n - z\| = 0$. Hence, $\{x_n\}$ converges strongly to z. This completes the proof.

4. Applications

Let A be monotone operator on a real Hilbert space H. Define

$$A^{-1}(0) := \{x \in H : Ax = 0\}.$$

Then we have that $A^{-1}(0) \subset \Gamma$, where Γ is the set of solution of the variational inequality (1.2) and $P_H = I$, where I is the identity mapping. Using Algorithm 4.1 and Theorem 3.7, we give the following application in a real Hilbert space.

Theorem 4.1. *Let H be a real Hilbert space. Let A be a Lipschitz continuous monotone mapping of H into itself and let S be a quasi-nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}(0) \neq \emptyset$ and $I - S$ is demiclosed. Given $\rho \in (0,1),\ \mu \in (0,1)$. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty}$ and $\{\omega_n\}_{n=1}^{\infty}$ be real sequences in (0,1) such that $\alpha_n + \beta_n + \gamma_n = 1$. For an arbitrary but fixed $u \in H$ and $x_1 \in H$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

Algorithm 5.1.

Step 1. *Compute*

$$y_n = x_n - \lambda_n A x_n,\ \forall n \geq 1,$$

where $\lambda_n = \rho^{l_n}$ and l_n is the smallest non-negative integer l such that

$$\lambda_n \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|$$

Step 2. *Compute*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(\omega_n S x_n + (1-\omega_n)(x_n - \lambda_n A y_n)),\ n \geq 1. \tag{4.1}$$

Set $n \leftarrow n+1$ and go to Step 1.
Assume that

(a) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(b) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\beta_n \geq \epsilon_1 > 0,\ \gamma_n \geq \epsilon_2 > 0$;

Then the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ generated by Algorithm 5.1 strongly converge to $z \in \Gamma$, where $z = P_{F(S)\cap A^{-1}(0)} u$.
Let $B : H \to 2^H$ be a maximal monotone mapping and let J_r^B be the resolvent of B for each $r > 0$. We know that $F(J_r^B) = B^{-1}(0) := \{x \in H : 0 \in Bx\}$.

Theorem 4.2. *Let H be a real Hilbert space. Let A be a Lipschitz continuous monotone mapping of H into itself and let $B : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}(0) \cap A^{-1}(0) \neq \emptyset$. Given $\rho \in (0,1),\ \mu, \omega \in (0,1)$ and let J_r^B be the resolvent of B for each $r > 0$. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty}$ and $\{\omega_n\}_{n=1}^{\infty}$ be real sequences in (0,1) such that $\alpha_n + \beta_n + \gamma_n = 1$. For an arbitrary but fixed $u \in H$ and $x_1 \in H$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

Algorithm 6.1.

Step 1. *Compute*

$$y_n = x_n - \lambda_n A x_n,\ \forall n \geq 1,$$

where $\lambda_n = \rho^{l_n}$ and l_n is the smallest non-negative integer l such that

$$\lambda_n \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|$$

Step 2. *Compute*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(\omega_n J_r^B x_n + (1-\omega_n)(x_n - \lambda_n A y_n)), \ n \geq 1. \tag{4.2}$$

Set $n \leftarrow n+1$ and go to Step 1.
Assume that

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\beta_n \geq \epsilon_1 > 0, \ \gamma_n \geq \epsilon_2 > 0$;

Then the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ generated by Algorithm 6.1 strongly converge to $z \in \Gamma$, where $z = P_{B^{-1}(0)\cap A^{-1}(0)}u$.

Remark 4.3. (i) We remark here that even though the operator $A : H \to H$ is Lipschitz continuous monotone in this paper, the Lipschitz constant of A is not needed as an input parameter in our algorithms.

(ii) Our results carry over for the case when S is a β-demicontractive mapping on a real Hilbert space H with $F(S) \neq \emptyset$ (i.e., there exists $\beta \in [0,1)$ such that

$$\|Sx - q\|^2 \leq \|x - q\|^2 + \beta\|x - Sx\|^2, \forall x \in H, \ q \in F(S)).$$

It is known that if S is a β-demicontractive mapping on a real Hilbert space H with $F(S) \neq \emptyset$ and $S_\omega := (1-\omega)I + \omega S$ for $\omega \in (0,1]$, then S_ω is quasi-nonexpansive mapping and $F(S) = F(S_\omega)$, where $\omega \in (0, 1-\beta)$ (e.g., see [23]).

5. Final remarks

In this paper, we proposed two Halpern type subgradient extragradient methods for solving variational inequality and fixed point problem for quasi-nonexpansive mapping and the underline monotone operator is Lipschitz continuous but the Lipschitz constant is unknown. Furthermore, we established strong convergence results for the two methods and we do not need the Lipschitz constant of A as an input parameter. Our results in this paper complement the result in [30]. Also, we note that in order to find λ_n you have to update right-hand side in (3.1) and (3.15) which is $r_{\rho^{l_n}}(x_n) = x_n - P_C(x_n - \lambda_n A x_n)$. Every update requires one more projection onto C and you have to perform exactly l_n updates. This is a drawback of our result. In our future research, we shall propose an algorithm that converges strongly to a solution of a continuous monotone variational inequality in which the update of the inner loop requires only computation of A in infinite dimensional Hilbert spaces. Moreover, in the future, we shall designing new algorithms including inexact or perturbed methods as well as inertial-type extrapolation for the problems considered in this paper.

Acknowledgment. The research was carried out when the author was an Alexander von Humboldt Postdoctoral Fellow at the Institute of Mathematics, University of Wurzburg, Germany. He is grateful to the Alexander von Humboldt Foundation, Bonn for the fellowship and the Institute of Mathematics, University of Wurzburg, Germany for the hospitality and facilities.

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Received: February 1st, 2017; Accepted: July 6, 2017.