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FIXED POINT THEOREMS FOR MULTIVALUED NONSELF KANNAN-BERINDE CONTRACTION MAPPINGS IN COMPLETE METRIC SPACES

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Abstract. In this paper, a new type of multivalued nonself Kannan-Berinde contraction mappings in metric spaces is introduced and studied. We establish the existence of fixed points of this type of mappings on a complete convex metric space. Our main results extend and generalize many well-known fixed point theorems of many other authors in the literature. We also give an example to illustrate our main results.
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1. Introduction

Fixed point theory plays an important role in the study of theory of equations in nonlinear analysis. They can be applied widely to solve importantly the existence of solutions of various equations. Further, it has various applications in many fields such as optimization, control theory and economics. A fundamental and well-known result is the Banach's contraction principle [10] that has been extended and generalized in many directions both single valued self-map version and multivalued self-map version, for instance, see [17, 34, 31, 16, 23, 11, 29] and for other associated results, see [28, 26, 27, 15, 32, 24, 20, 1, 2, 30].

Theorem 1.1. ***(Banach's Contraction Principle)*** *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction mapping, i.e., there exists $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

Then T has a unique fixed point in X.

In 1968, Kannan [22] extended the notions of Banach's contraction principle to a new type of mappings which is different from that of contraction as the following:

Definition 1.2. Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *Kannan mapping* if there exists $a \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X.$$

It is noted that a contraction mapping is continuous but a Kannan mapping is not.

On the other hand, Chatterjea [17] introduced a new concept of contraction mappings known as *Chatterjea contraction mapping* as follows:

$$d(Tx, Ty) \leq a[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X,$$

where $a \in \left[0, \frac{1}{2}\right)$.

Zamfirescu [34] proved a fixed point theorem for a new type of contraction mappings by combining the concept of Banach's contraction mapping, Kannan mapping and Chatterjea mapping. This mapping is known as *Zamfirescu operator*.

Theorem 1.3. ([34]) *Let (X, d) be a complete metric space and $T : X \to X$ a map for which there exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b, c < \frac{1}{2}$ such that for each pair $x, y \in X$, at least one of the following is true:*

(z_1) $d(Tx, Ty) \leq ad(x, y)$;
(z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;
(z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T is a Picard operator, that is, T has a unique fixed point $x_0 \in X$ and for each $x \in X$, $T^n x \to x_0$.

In 2004, Berinde [11] introduced and studied the fixed point theorems for weak contraction mapping or almost contraction mapping on a complete metric space which is more general than that of Kannan and Chatterjea mapping and Zamfirescu operator.

Definition 1.4. ([11]) Let (X, d) be a metric space. A map $T : X \to X$ is called *weak contraction* or *almost contraction* if there exist a constant $\delta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \text{ for all } x, y \in X.$$

Moreover, the Banach's contraction principle was extended to multivalued mappings in a complete metric space. The first well-known fixed point theorem for multivalued contraction mappings using the Pompeiu-Hausdorff metric was studied by Nadler [25].

Let (X, d) be a metric space and $CB(X)$ be the set of all nonempty closed bounded subsets of X. Let A be a subset of X. The *distance from x to A* is defined by

$$D(x, A) := \inf\{d(x, y) : y \in A\}.$$

For $A, B \in CB(X)$, we define

$$H(A, B) := \max\left\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\right\}.$$

It is called a *Pompeiu-Hausdorff distance from A to B*.

Let $T : X \to 2^X$ (collection of all nonempty subsets of X) be a multivalued mapping. A point $x \in X$ is said a *fixed point of* T if $x \in Tx$. We denote the set of all fixed points of T by $F(T)$, that is, $F(T) := \{x \in X : x \in Tx\}$.

In 1969, Nadler [25] extended the Banach's contraction principle for a multivalued mapping and proved the Banach's contraction principle in a complete metric space for multivalued version. He proved the following fixed point theorem.

Theorem 1.5. ***(Nadler's fixed point theorem)*** *Let (X, d) be a complete metric space and let T be a map from X into $CB(X)$. Suppose that T is a multivalued contraction mapping, i.e., there exists $k \in [0, 1)$ such that*

$$H(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

Then there exists $z \in X$ such that $z \in Tz$.

Nadler's fixed point theorem was extended and generalized in many directions. One of the well-known extensions is a fixed point theorem of multivalued almost contractions introduced by M. Berinde and V. Berinde [12]. They extended Nadler's fixed point theorem to a new class of multivalued self mappings, called multivalued almost contractions, defined as follows:

Let (X, d) be a metric space and let $T : X \to CB(X)$ be a multivalued mapping. Then T is said to be a *multivalued almost contraction* or *multivalued* (θ, L)*-almost contraction* if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + L \cdot D(y, Tx) \text{ for all } x, y \in X.$$

They proved that in a complete metric space, every multivalued almost contraction $T : X \to CB(X)$ has a fixed point.

In many real applicable existence problems, fixed point theorems of self-mappings may not be applied, but those of nonself mappings will be very useful and applicable.

Now, we will focus on the existence of fixed points for nonself multivalued contraction mappings which extended many important results, see [8, 21, 33, 4, 5, 6, 7, 18, 19, 3, 14], for example. In 1972, Assad and Kirk [8] obtained a new fixed point theorem for nonself multivalued mappings.

Theorem 1.6. ***(Assad and Kirk's fixed point theorem)*** *Let (X, d) be a complete and metrically convex metric space, K a nonempty closed subset of X, and let $T : K \to CB(X)$ a multivalued contraction mapping. If T satisfies Rothe's type condition, that is, $x \in \partial K$ implies $Tx \subset K$, then T has a fixed point in K.*

Recently, in 2013, Alghamdi, Berinde and Shahzad [3] considered multivalued nonself almost contractions on a convex metric space and proved the existence theorem of this mapping.

Theorem 1.7. ([3]) *Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Suppose that $T : K \to CB(X)$ is a multivalued almost contraction, that is,*

$$H(Tx, Ty) \leq \delta d(x, y) + L \cdot D(y, Tx) \text{ for all } x, y \in K,$$

with $\delta \in (0, 1)$ and some $L \geq 0$ such that $\delta(1 + L) < 1$. If T satisfies Rothe's type condition, then there exists $z \in K$ such that $z \in Tz$.

Motivated and inspired by all of these works mentioned above, we aim to study and prove the existence theorems for a new class of multivalued nonself contractions, which is more general than that of Berinde [12], in a complete metric space. We also give some examples to illustrate our main results.

2. Preliminaries

In this section, we recall some notations, definitions and known results that are useful for our main results.

Let (X,d) be a metric space and $CB(X)$ the set of all nonempty closed bounded subsets of X. Let A be a subset of X and any $x \in X$. The *distance from x to A* is defined by $D(x,A) := \inf\{d(x,y) : y \in A\}$. For $A, B \in CB(X)$, we define

$$H(A,B) := \max\left\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\right\}.$$

The mapping H is called a *Pompeiu-Hausdorff metric* on $CB(X)$ induced by d on X. It is known that $(CB(X),H)$ is a complete metric space whenever (X,d) is a complete metric space.

A metric space (X,d) is called *metrically convex* or *convex* if for each $x,y \in X$ with $x \neq y$ there exists $z \in X, x \neq z \neq y$, such that

$$d(x,y) = d(x,z) + d(z,y).$$

It is known that in a convex metric space each two points are the endpoint of at least one metric segment (see [8]). The following proposition and lemmas are useful for our main results.

Proposition 2.1. ([8]) *Let (X,d) be a complete and convex metric space, K a nonempty closed subset of X. If $x \in K$ and $y \notin K$, then there exists a point z in the boundary of K, denote by ∂K, such that*

$$d(x,y) = d(x,z) + d(z,y).$$

For convenience, we denote

$$P[x,y] = \{z \in \partial K : d(x,y) = d(x,z) + d(z,y)\}.$$

The following lemmas are direct consequences of the definition of Pompeiu-Hausdroff metric.

Lemma 2.2. *For $A, B \in CB(X)$ and $a \in A$, then $D(a,B) \leq H(A,B)$.*

Lemma 2.3. *Let $A, B \in CB(X)$ and $k > 1$. Then for $a \in A$, there exists $b \in B$ such that $d(a,b) \leq kH(A,B)$.*

3. Main results

In this section, we introduce and study a new type of nonself multivalued contraction, called Kannan-Berinde contraction mapping, which is more general than that of Berinde's contraction and prove its fixed point theorem under some conditions.

Definition 3.1. Let (X, d) be a metric space and K a nonempty subset of X. A mapping $T : K \to CB(X)$ is said to be a *multivalued Kannan-Berinde contraction* if there exist $\delta \in [0, 1)$, $a \in \left[0, \frac{1}{3}\right)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \delta d(x, y) + a[D(x, Tx) + D(y, Ty)] + L \cdot D(y, Tx)$$

for any $x, y \in K$.

Example 3.2. Let $X = \{0, 1, 2\}$ and $K = \{0, 1\}$. Define a map $T : K \to CB(X)$ by

$$Tx = \begin{cases} \{1, 2\} & \text{if } x = 0; \\ \{0, 2\} & \text{if } x = 1. \end{cases}$$

Then we see that

$$\begin{aligned} H(T(0), T(1)) &= \frac{1}{2} \cdot (1) + \frac{1}{4}[1 + 1] + L \cdot (0) \\ &= \frac{1}{2} d[0, 1) + \frac{1}{4}[D(0, T(0)) + D(1, T(1))] + L \cdot D(1, T(0)), \end{aligned}$$

and for any $0 \leq \delta < 1$ and $L \geq 0$,

$$H(T(0), T(1)) = 1 > \delta \cdot (1) + L \cdot (0) = \delta d[0, 1) + L \cdot D(1, T(0)).$$

Hence T is a multivalued Kannan-Berinde contraction for $\delta = \frac{1}{2}, a = \frac{1}{4}$ and $L \geq 0$ arbitrary but T is not a multivalued almost contraction.

We now prove our main result.

Theorem 3.3. *Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Suppose that a map $T : K \to CB(X)$ is a multivalued mapping satisfying the following properties:*

(i) T satisfies Rothe's type condition, that is, $x \in \partial K$ implies $Tx \subset K$;
(ii) T is a multivalued Kannan-Berinde contraction mapping with

$$\delta(1 + a + L) + a(3 + L) < 1.$$

Then T has a fixed point in K.

Proof. From the assumption (ii), $\delta(1+a+L)+a(3+L) < 1$, there exists $k > 1$ such that

$$\delta(1 + a + L) + a(3 + L) < \frac{1}{k^2} < 1.$$

Then we get

$$k^2[\delta(1 + a + L) + a(3 + L)] < 1.$$

We note that $ka, k\delta < 1$ and

$$\begin{aligned} k^2[\delta(1 + a + L) + a(3 + L)] &= k^2(\delta + 3a + \delta a + \delta L + aL) \\ &\geq k^2 \left(\frac{\delta + 3a}{k} + \delta a + \delta L + aL \right) \\ &= k\delta + 3ka + k^2\delta a + k^2\delta L + k^2 aL. \end{aligned}$$

So we have

$$k\delta + 3ka + k^2\delta a + k^2\delta L + k^2 aL \quad < \quad 1.$$

Hence

$$\frac{(1+ka+kL)(k\delta+ka)}{(1-ka)^2} < 1.$$

Now, we construct two sequences $\{x_n\}$ and $\{y_n\}$ as the following. Let $x_0 \in K$ and $y_1 \in Tx_0$. If $y_1 \in K$, we denote $x_1 = y_1$. Consider in case $y_1 \notin K$, by Proposition 2.1, there exists $x_1 \in P[x_0, y_1]$ such that $d(x_0, y_1) = d(x_0, x_1) + d(x_1, y_1)$. So we have $x_1 \in K$, and, by Lemma 2.3, there exists $y_2 \in Tx_1$ such that $d(y_1, y_2) \leq kH(Tx_0, Tx_1)$. If $y_2 \in K$, we denote $x_2 = y_2$. Otherwise, $y_2 \notin K$, then there exists $x_2 \in P[x_1, y_2]$ such that $d(x_1, y_2) = d(x_1, x_2) + d(x_2, y_2)$. Thus $x_2 \in K$, by Lemma 2.3, there exists $y_3 \in Tx_2$ such that $d(y_2, y_3) \leq kH(Tx_1, Tx_2)$.

Continuing the argument, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

(1) $y_{n+1} \in Tx_n$;
(2) $d(y_n, y_{n+1}) \leq kH(Tx_{n-1}, Tx_n)$, where
  (a) $x_n = y_n$ if and only if $y_n \in K$;
  (b) $x_n \in P[x_{n-1}, y_n]$ if and only if $y_n \notin K$, i.e., $x_n \neq y_n$ and

$$x_n \in \partial K \text{ such that } d(x_{n-1}, y_n) = d(x_{n-1}, x_n) + d(x_n, y_n).$$

Next, we show that the sequence $\{x_n\}$ is a Cauchy sequence.
Now, we put

$$P_1 := \{x_i \in \{x_n\} : x_i = y_i, i = 1, 2, ...\};$$

$$P_2 := \{x_i \in \{x_n\} : x_i \neq y_i, i = 1, 2, ...\}.$$

Note that $\{x_n\} \subset K$. Moreover, if $x_i \in P_2$, then x_{i-1} and x_{i+1} belong to the set P_1. By virtue of (i), we cannot have two consecutive terms of $\{x_n\}$ in the set P_2. For $n \geq 2$, we consider the three possibilities as the following.

Case 1. If $x_n, x_{n+1} \in P_1$, then $x_n = y_n$ and $x_{n+1} = y_{n+1}$.
Then we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &= d(y_n, y_{n+1}) \\ &\leq kH(Tx_{n-1}, Tx_n) \\ &\leq k\delta d(x_{n-1}, x_n) + ka[D(x_{n-1}, Tx_{n-1}) + D(x_n, Tx_n)] \\ &\quad + kL \cdot D(x_n, Tx_{n-1}). \\ &\leq k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, x_n) + kad(x_n, x_{n+1}), \end{aligned}$$

which implies

$$d(x_n, x_{n+1}) \leq \left(\frac{k\delta + ka}{1-ka}\right) d(x_{n-1}, x_n).$$

Case 2. If $x_n \in P_1$ and $x_{n+1} \in P_2$, then $x_n = y_n$ and $x_{n+1} \in P[x_n, y_{n+1}]$, i.e.,

$$d(x_n, y_{n+1}) = d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}).$$

From (ii), we have

$$\begin{aligned} d(y_n, y_{n+1}) &\leq kH(Tx_{n-1}, Tx_n) \\ &\leq k\delta d(x_{n-1}, x_n) + ka[D(x_{n-1}, Tx_{n-1}) + D(x_n, Tx_n)] \\ &\quad + kL \cdot D(x_n, Tx_{n-1}). \\ &\leq k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, x_n) + kad(y_n, y_{n+1}), \end{aligned}$$

which follows that

$$d(y_n, y_{n+1}) \leq \left(\frac{k\delta + ka}{1 - ka}\right) d(x_{n-1}, x_n).$$

So, we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, y_{n+1}) - d(x_{n+1}, y_{n+1}) \\ &\leq d(x_n, y_{n+1}) \\ &= d(y_n, y_{n+1}) \\ &\leq \left(\frac{k\delta + ka}{1 - ka}\right) d(x_{n-1}, x_n). \end{aligned}$$

Case 3. If $x_n \in P_2$ and $x_{n+1} \in P_1$, then $x_{n-1} \in P_1$, that is, $x_{n-1} = y_{n-1}$, $x_{n+1} = y_{n+1}$, and $x_n \in P[x_{n-1}, y_n]$, that is,

$$d(x_{n-1}, y_n) = d(x_{n-1}, x_n) + d(x_n, y_n).$$

Since $y_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$ and $k\delta < 1$, by (ii), we have

$$\begin{aligned} d(y_n, y_{n+1}) &\leq kH(Tx_{n-1}, Tx_n) \\ &\leq k\delta d(x_{n-1}, x_n) + ka[D(x_{n-1}, Tx_{n-1}) + D(x_n, Tx_n)] \\ &\quad + kL \cdot D(x_n, Tx_{n-1}). \\ &\leq k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, y_n) + kad(x_n, y_{n+1}) + kLd(x_n, y_n) \\ &= k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, y_n) + kad(x_n, x_{n+1}) + kLd(x_n, y_n). \end{aligned}$$

Then, we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, y_n) + d(y_n, x_{n+1}) \\ &= d(x_n, y_n) + d(y_n, y_{n+1}) \\ &\leq (1 + kL)d(x_n, y_n) + k\delta d(x_{n-1}, x_n) + kad(x_{n-1}, y_n) + kad(x_n, x_{n+1}) \\ &\leq (1 + kL)d(x_n, y_n) + (1 + kL)d(x_{n-1}, x_n) + kad(x_{n-1}, y_n) \\ &\quad + kad(x_n, x_{n+1}) \\ &= (1 + kL)d(x_{n-1}, y_n) + kad(x_{n-1}, y_n) + kad(x_n, x_{n+1}), \end{aligned}$$

which implies that

$$d(x_n, x_{n+1}) \leq \left(\frac{1 + ka + kL}{1 - ka}\right) d(x_{n-1}, y_n) = \left(\frac{1 + ka + kL}{1 - ka}\right) d(y_{n-1}, y_n).$$

Since $x_{n-1} \in P_1$ and $x_n \in P_2$, it follows from Case 2 that

$$d(y_{n-1}, y_n) \leq \left(\frac{k\delta + ka}{1 - ka}\right) d(x_{n-2}, x_{n-1}).$$

Thus

$$d(x_n, x_{n+1}) \leq \frac{(1+ka+kL)(k\delta+ka)}{(1-ka)^2} \cdot d(x_{n-2}, x_{n-1}).$$

Since

$$h := \frac{(1+ka+kL)(k\delta+ka)}{(1-ka)^2} < 1,$$

we obtain that

$$d(x_n, x_{n+1}) \leq hd(x_{n-2}, x_{n-1}).$$

We note that

$$\frac{k\delta+ka}{1-ka} \leq \frac{(1+ka+kL)(k\delta+ka)}{(1-ka)} \leq \frac{(1+ka+kL)(k\delta+ka)}{(1-ka)^2} = h.$$

From Case 1, Case 2 and Case 3, we can conclude that for $n \geq 2$,

$$d(x_n, x_{n+1}) = \begin{cases} hd(x_{n-1}, x_n) & \text{if } x_n, x_{n+1} \in P_1 \text{ or } x_n \in P_1, x_{n+1} \in P_2; \\ hd(x_{n-2}, x_{n-1}) & \text{if } x_n \in P_2, x_{n+1} \in P_1. \end{cases}$$

Following Assad and Kirk [8], by induction, we get that for $n \geq 2$,

$$d(x_n, x_{n+1}) \leq r \cdot h^{n/2},$$

where $r := h^{-1/2} \cdot \max\{d(x_0, x_1), d(x_1, x_2)\}$.
For $m > n$, we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq r \cdot h^{n/2} + r \cdot h^{(n+1)/2} + ... + r \cdot h^{(m-1)/2} \\ &= r \cdot (h^{n/2} + h^{(n+1)/2} + .. + h^{(m-1)/2}). \end{aligned}$$

Since $h < 1$, it follows that $\{x_n\}$ is a Cauchy sequence in K. Since X is complete and K is closed, there exists $x \in K$ such that

$$\lim_{n \to \infty} x_n = x.$$

From the construction of $\{x_n\}$, there is a subsequence $\{x_{n_j}\}$ such that $\{x_{n_j}\} \subset P_1$. So, $x_{n_j} = y_{n_j} \in Tx_{n_j-1}$. Finally, we show that x is a fixed point.
From $0 \leq D(x, Tx_{n_j-1}) \leq d(x, x_{n_j})$ for each $j \in \mathbb{N}$, it follows that $D(x, Tx_{n_j-1}) \to 0$ as $j \to \infty$.
For each $j \in \mathbb{N}$, we have

$$\begin{aligned} D(x, Tx) &\leq d(x, x_{n_j}) + H(Tx_{n_j-1}, Tx) \\ &\leq d(x, x_{n_j}) + \delta d(x_{n_j-1}, x) + a[D(x_{n_j-1}, Tx_{n_j-1}) + D(x, Tx)] \\ &\quad + L \cdot D(x, Tx_{n_j-1}) \\ &\leq d(x, x_{n_j}) + \delta d(x_{n_j-1}, x) + ad(x_{n_j-1}, x_{n_j}) + aD(x, Tx) \\ &\quad + L \cdot D(x, Tx_{n_j-1}). \end{aligned}$$

So, we obtain

$$(1-a)D(x, Tx) \leq d(x, x_{n_j}) + \delta d(x_{n_j-1}, x) + ad(x_{n_j-1}, x_{n_j}) + L \cdot D(x, Tx_{n_j-1}).$$

Letting $j \to \infty$, we get $(1-a)D(x,Tx) = 0$.
Since $0 \le a < \frac{1}{3}$, we get that $D(x,Tx) = 0$, hence $x \in Tx$, that is, T has a fixed point in K. This completes the proof. □

As a consequence of Theorem 3.3, when we put $a = 0$, we obtain Theorem 9 of [3] as our special case as follows.

Corollary 3.4. (*Theorem 9 of* [3]) *Let (X,d) be a complete convex metric space and K a nonempty closed subset of X. Suppose that a map $T : K \to CB(X)$ is a multivalued mapping satisfying the following properties:*

(i) *T has the Rothe's boundary condition;*
(ii) *there exist $\delta \in [0,1)$ and $L \ge 0$ with $\delta(1+L) < 1$ such that*

$$H(Tx,Ty) \le \delta d(x,y) + L \cdot D(y,Tx), \text{ for any } x,y \in K.$$

Then T has a fixed point in K.

If we put $a = 0$ and $L = 0$ in Theorem 3.3, then we also obtain Theorem 1 of [8].

Corollary 3.5. (*Theorem 1 of* [8]) *Let (X,d) be a complete convex metric space and K a nonempty closed subset of X. Suppose that a map $T : K \to CB(X)$ is a multivalued mapping satisfying the following properties:*

(i) *T has the Rothe's boundary condition;*
(ii) *there exists $\delta \in [0,1)$ such that*

$$H(Tx,Ty) \le \delta d(x,y), \text{ for any } x,y \in K.$$

Then T has a fixed point in K.

Remark 3.6. In Theorem 3.3, if we put $a = 0$ and T is a nonself single-valued mapping, we obtain Theorem 3.3 of Berinde and Pacurar [13] without the property (M). We also note that in Theorem 3.3, if we put $\delta = L = 0$, $a \in \left[0, \frac{1}{3}\right)$ and T is a nonself single-valued Kannan mapping, then we obtain Theorem 2.1 of Balog and Berinde [9] in the case of complete graph.

Next, we give an example to illustrate Theorem 3.3.

Example 3.7. Let $X = \mathbb{R}$ and $K = \left[0, \frac{1}{2}\right]$ endowed with usual metric, that is, $d(x,y) = |x-y|$ for all $x,y \in X$. Define a mapping $T : K \to CB(X)$ by

$$Tx = \begin{cases} \left[0, \frac{x}{10}\right] & \text{if } x \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]; \\ \{-\frac{1}{8}\} & \text{if } x = \frac{1}{5}; \\ \{\frac{1}{2}\} & \text{if } x \in \left(\frac{1}{4}, \frac{1}{2}\right]. \end{cases}$$

We see that

$$\partial K = \left\{0, \frac{1}{2}\right\} \Rightarrow T(0) \text{ and } T\left(\frac{1}{2}\right) \text{ are subset of } K,$$

which implies T satisfies Rothe's boundary condition. Now, we show that T is a multivalued Kannan-Berinde contraction satisfying all conditions of Theorem 3.3. We consider the following six cases:

Case 1. If $x = y = \frac{1}{5}$ or $x, y \in \left(\frac{1}{4}, \frac{1}{2}\right]$, then

$$H(Tx, Ty) = 0 \le \delta d(x, y) + a[D(x, Tx) + D(y, Ty)] + L \cdot D(y, Tx)$$

for any $\delta \in [0, 1), a \in \left[0, \frac{1}{3}\right)$ and $L \ge 0$.

Case 2. If $x \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]$ and $y = \frac{1}{5}$, we note that

$$\left|\frac{x}{10} + \frac{1}{8}\right| \le \frac{3}{20} \text{ and } \left|\frac{1}{5} - \frac{x}{10}\right| \ge \frac{7}{40}.$$

Then, we have

$$\begin{aligned} H(Tx, Ty) &= H\left(\left[0, \frac{x}{10}\right], \left\{-\frac{1}{8}\right\}\right) \\ &= \left|\frac{x}{10} + \frac{1}{8}\right| \le \frac{3}{20} \\ &\le \delta\left|x - \frac{1}{5}\right| + a\left[\frac{9}{10}x + \frac{13}{40}\right] + L \cdot \left|\frac{1}{5} - \frac{x}{10}\right| \\ &= \delta d(x, y) + a[D(x, Tx) + D(y, Ty)] + L \cdot D(y, Tx), \end{aligned}$$

when $L \ge \frac{6}{7}, 0 \le \delta < 1$ and $0 \le a < \frac{1}{3}$ such that $\delta(1 + a + L) + a(3 + L) < 1$.

Case 3. If $x = \frac{1}{5}$ and $y \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]$, then

$$\left|\frac{y}{10} + \frac{1}{8}\right| \le \frac{3}{20} \text{ and } \left|y + \frac{1}{8}\right| \ge \frac{1}{8}.$$

So, we have

$$H(Tx, Ty) = \left|\frac{y}{10} + \frac{1}{8}\right| \le \frac{3}{20} \le \delta\left|\frac{1}{5} - y\right| + a\left[\frac{13}{40} + \frac{9}{10}y\right] + L \cdot \left|y + \frac{1}{8}\right|,$$

when $L \ge \frac{6}{5}, 0 \le \delta < 1$ and $0 \le a < \frac{1}{3}$ such that $\delta(1 + a + L) + a(3 + L) < 1$.

Case 4. If $x \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]$ and $y \in \left(\frac{1}{4}, \frac{1}{2}\right]$, then

$$\left|\frac{x}{10} - \frac{1}{2}\right| \le \frac{1}{2} \text{ and } \left|y - \frac{x}{10}\right| > \frac{9}{40}.$$

So, we have

$$H(Tx, Ty) = \left|\frac{x}{10} - \frac{1}{2}\right| \le \frac{1}{2} \le \delta|x - y| + a\left[\frac{9}{10}x + \frac{1}{2} - y\right] + L \cdot \left|y - \frac{x}{10}\right|.$$

when $L \ge \frac{20}{9}, 0 \le \delta < 1$ and $0 \le a < \frac{1}{3}$ such that $\delta(1 + a + L) + a(3 + L) < 1$.

Case 5. If $x \in \left(\frac{1}{4}, \frac{1}{2}\right]$ and $y \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]$, then

$$\left|\frac{1}{2} - \frac{y}{10}\right| \le \frac{1}{2} \text{ and } \left|y - \frac{1}{2}\right| \ge \frac{1}{4}.$$

So, we have

$$H(Tx, Ty) = \left|\frac{1}{2} - \frac{y}{10}\right| \le \frac{1}{2} \le \delta\left|x - y\right| + a\left[\frac{1}{2} - x + \frac{9}{10}y\right] + L \cdot \left|y - \frac{1}{2}\right|,$$

when $L \ge 2, 0 \le \delta < 1$ and $0 \le a < \frac{1}{3}$ such that $\delta(1 + a + L) + a(3 + L) < 1$.

Case 6. If $x, y \in \left[0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{4}\right]$, then we have

$$\begin{aligned} H(Tx, Ty) &= H\left(\left[0, \frac{x}{10}\right], \left[0, \frac{y}{10}\right]\right) \\ &= \left|\frac{x}{10} - \frac{y}{10}\right| \\ &\leq \frac{1}{9}\left(\frac{9}{10}x + \frac{9}{10}y\right) \\ &= \frac{1}{9}[D(x, Tx) + D(y, Ty)] \\ &\leq \delta d(x, y) + a[D(x, Tx) + D(y, Ty)] + L \cdot D(y, Tx). \end{aligned}$$

We choose that $a = \frac{1}{9}$, $0 \leq \delta < 1$ and $L \geq 0$ such that $\delta(1 + a + L) + a(3 + L) < 1$. Now, by summarizing all cases, we conclude that T is a multivalued Kannan-Berinde contraction with $a = \frac{1}{9}$, $L = \frac{20}{9}$ and

$$0 \leq \delta < \frac{1 - \frac{1}{9}(3 + \frac{20}{9})}{1 + \frac{1}{9} + \frac{20}{9}} = \frac{17}{135},$$

which the condition $\delta(1 + a + L) + a(3 + L) < 1$ is also satisfied. Therefore, T is a multivalued Kannan-Berinde contraction that satisfies all assumptions in Theorem 3.3, and there exist $z \in K$ such that $z \in Tz$. Notice that $F(T) = \left\{0, \dfrac{1}{2}\right\}$. However, we see that T is not multivalued contraction mapping. If we put $x = \frac{1}{2}$ and $y = \frac{1}{5}$, then

$$H\left(T\left(\frac{1}{2}\right), T\left(\frac{1}{5}\right)\right) = H\left(\left\{\frac{1}{2}\right\}, \left\{-\frac{1}{8}\right\}\right) = \frac{5}{8} > k \cdot \frac{3}{10} = kd\left(\frac{1}{2}, \frac{1}{5}\right),$$

for all $0 \leq k < 1$.

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