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STABILITY OF A CLASS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION

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Abstract. The aim of the present paper is to investigate the Hyers-Ulam stability and generalized Hyers-Ulam stability of certain class of fractional integro-differential equation with boundary conditions using fixed point approach.

Key Words and Phrases: Hyers-Ulam stability, fractional integro-differential equation, boundary condition.
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1. Introduction

This paper is concerned with the stability of the following fractional integro-differential equation with the given boundary condition

$$^{c}D^{\alpha}y(t) = F\left(t, y(t), \int_0^t k(t,s,y(s))ds\right) \tag{1.1}$$

$$ay(0) + by(T) = c \tag{1.2}$$

where $^{c}D^{\alpha}$ is Caputo derivative of order α, $F : I := [0,T] \times X \times X \to X$, where X is a Banach space and a, b, c are real constants with $a + b \neq 0$. Define

$$(Sy)(t) = \int_0^t k(t,s,y(s))ds,$$

where S is a nonlinear operator.

"Under what conditions does there exists an additive mapping near an approximately additive mapping?", this is the problem proposed by Ulam [15] in 1940. In the next year, the first positive answer was given by Hyers [7] for additive functions defined on Banach spaces. The generalization of Hyers result was given by Rassias [14] in the year 1978. By this pioneering result, the stability concept had been rapidly devoloped and become one of the central subjects in mathematical analysis.

Motivated by this result, S.M. Jung [8] initiated the application of these concepts in differential equation and integral equation via fixed point method by using some ideas of Cadariu and Radu [2]. Following this, many authors proved the stability of differential equations, integral equations and integro-differential equations (see [1], [3], [5], [6] etc..) using fixed point approach in Banach spaces.

On the otherhand, fractional differential equations arise as a major field of research in recent years, mainly finding the existence and uniqueness results of linear, nonlinear and integro-differential equations of fractional order. As well as there are few works devoted to the stability concepts of fractional order differential equations (see [11], [16], [12] etc..). Here we note that, the study of stability of fractional order integro-differential equations is new in the research area.

In this paper, we prove the Hyers-Ulam stability of a class of fractional order integro-differential equation (1.1) with the given boundary condition (1.2) by applying the fixed point method.

This paper is organized as follows: In Section 2, the Hyers-Ulam stability of fractional integro-differential equation (1.1) with boundary condition (1.2) is proved. In Section 3, the generalized Hyers-Ulam stability of fractional integro-differential equation (1.1) with boundary condition (1.2) is proved.

Definition 2.1. [16] For a nonempty set X, a function $d : X \times X \to [0, \infty]$ is called generalized metric on X if and only if d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

This concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such space a generalized complete metric space.

Theorem 2.2. [4] *Let (X,d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$, If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the following are true:*

(a) The sequence $\{\Lambda^n x\}$ converges to a fixed end point x^* of Λ

(b) x^* is the unique fixed point of Λ in $X^* = \{y \in X | d(\Lambda^k x, y) < \infty\}$;

(c) If $y \in X^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$.

Theorem 2.3. [10] *Let $0 < \alpha < 1$ and let $f : [0, T] \to \mathbb{R}$ be continuous. A function $y \in C(J, \mathbb{R})$ is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), (Sy)(s)) ds \\ &\quad - \frac{1}{a+b} \left[\frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s), (Sy)(s)) ds - c \right] \end{aligned}$$

if and only if y is a solution of the fractional integro-differential equation

$$\begin{aligned} {}^c D^\alpha y(t) &= f(t, y(t), (Sy)(s)), t \in [0, T] \\ ay(0) + by(T) &= c. \end{aligned}$$

2. Hyers-Ulam stability

In this section, authors investigate the Hyers-Ulam stability of fractional integro-differential equation (1.1) with the boundary condition (1.2).

Theorem 2.4. *Set* $M := (L_1 + L_1L_2)\left(1 + \frac{b}{a+b}\right) < 1$. *Let* L_1 *and* L_2 *be positive constants with* $0 < \frac{MT^\alpha}{\Gamma(\alpha+1)} < 1$. *Suppose that* $F : I \times \mathbb{R} \to \mathbb{R}$ *is a continuous function which satisfies a Lipschitz condition*

$$|F(t,x,\bar{x}) - F(t,y,\bar{y})| \leq L_1\left[|x-y| + |\bar{x}-\bar{y}|\right] \quad \forall t \in I, \quad x,y,\bar{x},\bar{y} \in \mathbb{R} \tag{2.1}$$

and $k : I \times I \times \mathbb{R} \to \mathbb{R}$ *is a continous function which satisfies a Lipschitz condition*

$$|k(t,s,f) - k(t,s,g)| \leq L_2\left[|f-g|\right] \quad \forall t,s \in I, \quad \forall f,g \in \mathbb{R} \tag{2.2}$$

If for $\varepsilon \geq 0$, *in a continuously differential function* $y : I \to \mathbb{R}$ *satisfies*

$$|{}^cD^\alpha y(t) - F(t, y(t), (Sy)t)| \leq \varepsilon \tag{2.3}$$

for all $t \in I$, *then there exists a unique continuous function* $y_0 : I \to \mathbb{R}$ *such that*

$$y_0(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, y_0(s), (Sy_0)(s))ds - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s, y_0(s), (Sy_0)(s))ds + \frac{c}{a+b} \tag{2.4}$$

and

$$|y(t) - y_0(t)| \leq \frac{T^\alpha \varepsilon}{\Gamma(\alpha+1) - MT^\alpha} \tag{2.5}$$

Proof. Let X denote the set of all real valued continuous functions on I. We define a generalized complete metric (see [8]) on X as follows

$$d(f,g) = \inf\{C \in [0,\infty] \,|\, |f(t) - g(t)| \leq C \quad \forall t \in I\} \tag{2.6}$$

Now, define an operator $\Lambda : X \to X$ by

$$(\Lambda f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, f(s), (Sf)(s))ds - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s, f(s), (Sf)(s))ds + \frac{c}{a+b} \tag{2.7}$$

for all $f \in X$.
Next we check that Λ is strictly contractive on X.
Let $f, g \in X$ and let $C_{fg} \in [0,\infty]$ be an arbitrary constant such that $d(f,g) \leq C_{fg}$. Then, by (2.6) we get,

$$|f(t) - g(t)| \leq C_{fg} \tag{2.8}$$

for any $t \in I$.
Using (2.1), (2.2), (2.7), and (2.8), we have

$$|(\Lambda f)t - (\Lambda g)t| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|F(s, f(s), (Sf)(s)) - F(s, g(s), (Sg)(S))\right| ds$$

$$+\frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|F(s,f(s),(Sf)(s))-F(s,g(s),(Sg)(S))\right|ds$$
$$\leq \frac{L_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[|f(s)-g(s)|+|(Sf)(s)-(Sg)(s)|\right]ds$$
$$+\frac{bL_1}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left[|f(s)-g(s)|+|(Sf)(s)-(Sg)(s)|\right]ds$$
$$\leq \frac{(L_1+L_1L_2)}{\Gamma(\alpha)}C_{fg}\int_0^t (t-s)^{\alpha-1}ds+\frac{b(L_1+L_1L_2)}{(a+b)\Gamma(\alpha)}C_{fg}\int_0^T (T-s)^{\alpha-1}ds$$
$$\leq \frac{(L_1+L_1L_2)C_{fg}}{\Gamma(\alpha+1)}\left[t^\alpha+\frac{bT^\alpha}{a+b}\right]\leq \frac{T^\alpha M}{\Gamma(\alpha+1)}C_{fg}$$

for all $t \in I$. That is

$$d\left(\Lambda f,\Lambda g\right)\leq \frac{T^\alpha M}{\Gamma(\alpha+1)}C_{fg}.$$

Hence we can conclude that

$$d\left(\Lambda f,\Lambda g\right)\leq \frac{T^\alpha M}{\Gamma(\alpha+1)}C_{fg}\leq \frac{T^\alpha M}{\Gamma(\alpha+1)}d(f,g)$$

for all $f, g \in X$. Let g_0, be any arbitrary element in X. Then there exists a constant $0 < C < \infty$ with

$$|(\Lambda g_0)(t)-g_0(t)| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,f(s),(Sf)(s))ds\right.$$
$$\left.-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s,f(s),(Sf)(s))ds+\frac{c}{a+b}-g_0(t)\right|$$
$$\leq C$$

for all $t \in I$, since $F(t,(g_0)(t),(Sg_0)(t))$ and $(g_0)(t)$ are bounded on I. Thus, (2.6) implies that

$$d(\Lambda g_0, g_0) < \infty$$

Therefore according to theorem (2.2), there exists a continuous function $y_0 : I \to \mathbb{R}$ such that the sequence $\{\Lambda^n g_0\}$ converges to y_0 and $\Lambda y_0 = y_0$, that is, y_0 is a solution of (1.1).

we will now verify that

$$\{g \in X | d(g_0, g) < \infty\} = X$$

Since g and g_0 are bounded on I, for any $g \in X$, there exists a constant $0 < C_g < \infty$ such that

$$|g_0(t) - g(t)| \leq C_g$$

Hence, we have $d(g_0, g) < \infty$ for all $g \in X$. That is $\{g \in X|d(g_0, g) < \infty\} = X$.
Therefore, in view of theorem (2.2), we conclude that y_0 given by (2.4) is the unique continuous function.

From (2.3) we have

$$-\varepsilon \leq^c D^\alpha_{a+}y(t) - F(t, y(t),(Sy)(t)) \leq \varepsilon \text{ for all } t \in I.$$

If we integrate each term in the above inequality from 0 to t and substitute the boundary conditions we obtain

$$\left| y(t) - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s), (Sy)(s))ds \right.$$
$$\left. + \frac{b}{\Gamma(\alpha)(a+b)} \int_0^t (t-s)^{\alpha-1} F(s, y(s), (Sy)(s))ds - \frac{c}{a+b} \right| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \varepsilon$$

for any $t \in I$.
That is, it holds that

$$|y(t) - (\Lambda y)(t)| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \varepsilon$$
$$i.e \quad d(y, \Lambda y) \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \varepsilon \tag{2.9}$$

for each $t \in I$. Finally theorem (2.2), together with (2.9) implies that

$$d(y, y_0) \leq \frac{1}{1 - \frac{MT^\alpha}{\Gamma(\alpha+1)}} d(y, \Lambda y) \leq \frac{T^\alpha}{\Gamma(\alpha+1) - MT^\alpha} \varepsilon$$

that is, the inequality (2.5) be true for all $t \in I$.

3. Generalized Hyers-Ulam stability

In this section, authors established generalized Hyers-Ulam stability of the fractional integro-differential equation (1.1) with boundary condition (1.2).

Theorem 2.5. *Set* $M := (L_1 + L_1 L_2)\left(1 + \frac{b}{a+b}\right) < 1$. *Let* K, L_1 *and* L_2 *be positive constants with* $0 < KM < 1$. *Assume that* $F : I \times \mathbb{R} \to \mathbb{R}$ *is a continuous function which satisfies the Lipschitz condition (2.1) and* $k : I \times I \times \mathbb{R} \to \mathbb{R}$ *is a continuous function which satisfies a Lipschitz condition (2.2). If a continuously differential function* $y : I \to \mathbb{R}$ *satisfies*

$$|{}^c D^\alpha y(t) - F(t, y(t), (Sy)t)| \leq \varphi(t) \tag{3.1}$$

for all $t \in I$, *where* $\varphi : I \to (0, \infty)$ *is a continuous function with*

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds \right| \leq K\varphi(t) \tag{3.2}$$

for all $t \in I$, *then there exists a unique continuous function* $y_0 : I \to \mathbb{R}$ *such that*

$$y_0(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y_0(s), (Sy_0)(s))ds$$
$$- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y_0(s), (Sy_0)(s))ds + \frac{c}{a+b} \tag{3.3}$$

and

$$|y(t) - y_0(t)| \leq \frac{K}{1-KM} \varphi(t) \quad \forall t \in I \tag{3.4}$$

Proof. Let X denote the set of all real valued continuous functions on I. We set a generalised complete metric (see [8]) on X as follows

$$d(f,g) = \inf\{C \in [0,\infty] \,|\, |f(t)-g(t)| \leq C\varphi(t) \quad \forall t \in I\} \tag{3.5}$$

Define an operator $\Lambda : X \to X$ by

$$\begin{aligned}(\Lambda f)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F(s,f(s),(Sf)(s))ds \\ &- \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} F(s,f(s),(Sf)(s))ds + \frac{c}{a+b}\end{aligned} \tag{3.6}$$

for all $t \in I$ and $f \in X$.

Now we check that Λ is strictly contractive on X.

For any $f,g \in X$, let $C_{fg} \in [0,\infty]$ be an arbitrary constant with $d(f,g) \leq C_{fg}$, that is, by (3.5) we have

$$|f(t)-g(t)| \leq C_{fg}\varphi(t) \tag{3.7}$$

for any $t \in I$.

Then it follows from (2.1), (2.2), (3.2), (3.6) and (3.7) that

$$\begin{aligned}|(\Lambda f)t - (\Lambda g)t| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|F(s,f(s),(Sf)(s)) - F(s,g(s),(Sg)(s))\right| ds \\ &+ \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|F(s,f(s),(Sf)(s)) - F(s,g(s),(Sg)(s))\right| ds \\ &\leq \frac{L_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[|f(s)-g(s)| + |(Sf)(s)-(Sg)(s)|\right] ds \\ &+ \frac{bL_1}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left[|f(s)-g(s)| + |(Sf)(s)-(Sg)(s)|\right] ds \\ &\leq \frac{L_1+L_1L_2}{\Gamma(\alpha)}C_{fg}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds + \frac{b(L_1+L_1L_2)}{(a+b)\Gamma(\alpha)}C_{fg}\int_0^T (T-s)^{\alpha-1}\varphi(s)ds \\ &\leq KMC_{fg}\varphi(t)\end{aligned}$$

for all $t \in I$. That is

$$d(\Lambda f, \Lambda g) \leq KMC_{fg}\varphi(t).$$

Hence we can conclude that

$$d(\Lambda f, \Lambda g) \leq KMd(f,g)$$

for any $f,g \in X$, where we note that $0 < KM < 1$.

It follows from (3.6) that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < \infty$ with

$$\begin{aligned}|(\Lambda g_0)(t) - g_0(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F(s,g_0(s),(Sg_0)(s))ds \right. \\ &\left. - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} F(s,g_0(s),(Sg_0)(s))ds + \frac{c}{a+b} - g_0(t)\right| \\ &\leq C\varphi(t)\end{aligned}$$

for all $t \in I$, since $F(t, g_0(t), (Sg_0)(t))$ and $g_0(t)$ are bounded on I and $\min_{t \in I} \varphi(t) > 0$. Thus (3.5) implies that

$$d(\Lambda g_0, g_0) < \infty$$

Therefore, according to theorem (2.2), there exists a continuous function $y_0 : I \to \mathbb{R}$ such that the sequence $\{\Lambda^n g_0\}$ converges to y_0 in (X, d) and $\Lambda y_0 = y_0$, that is, y_0 is a solution of (1.1) for every $t \in I$.
We will now verify that

$$\{g \in X | d(g_0, g) < \infty\} = X$$

Since g and g_0 are bounded on I, for any $g \in X$, and $\min_{t \in I} \varphi(t) > 0$, there exists a constant $0 < C_g < \infty$ such that

$$|g_0(t) - g(t)| \leq C_g$$

Hence, we have $d(g_0, g) < \infty$ for all $g \in X$. That is $\{g \in X | d(g_0, g) < \infty\} = X$. Therefore from theorem (2.2), we conclude that y_0 is the unique continuous function with the property (2.4).
From (3.1) we have

$$-\varphi(t) \leq {}^c D^{\alpha}_{a+} y(t) - F(t, y(t), (Sy)(t)) \leq \varphi(t) \tag{3.8}$$

for all $t \in I$.
If we integrate each term in the above inequality and substitute the boundary conditions we obtain

$$\begin{aligned} &\left| y(t) - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s), (Sy)(s)) ds \right. \\ &\left. + \frac{b}{\Gamma(\alpha)(a+b)} \int_0^t (t-s)^{\alpha-1} F(s, y(s), (Sy)(s)) ds - \frac{c}{a+b} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds \end{aligned}$$

for any $t \in I$.
Thus, by (3.2) and (3.6), we get

$$|y(t) - (\Lambda y)(t)| \leq K\varphi(t)$$

for each $t \in I$, which implies that

$$d(y, \Lambda y) \leq K\varphi(t). \tag{3.9}$$

Finally using theorem (2.2), together with (3.9), we concludde that

$$d(y, y_0) \leq \frac{1}{1-KM} d(y, \Lambda y) \leq \frac{K}{1-KM} \varphi(t) \tag{3.10}$$

Consequently, this yields the inequality (3.4) for all $t \in I$.

Remark 2.6. In theorem (2.5), we have examined the generalized Hyers-Ulam stability of the fractional integro-differential equation (1.1) defined on a bounded and closed interval. We will now show that theorem (2.5) is also valid for the case of unbounded intervals.

Theorem 2.7. *For given nonnegative real number T, let I denote either $(-\infty, T]$ or $\mathbb{R}$ or $[0, \infty)$. Let K, M be positive constants with $0 < KM < 1$. Suppose that $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition (2.1) for all $t \in I$ and $x, y \in \mathbb{R}$. If a continuously differential function $y : I \to \mathbb{R}$ satisfies the differential inequality (3.1) for all $t \in I$, where $\varphi : I \to (0, \infty)$ is a continuous function satisfying (3.2) for each $t \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ which satisfies (2.4) and (3.4) for all $t \in I$.*

Proof. Let $I = \mathbb{R}$. We first show that y is a unique continuous function. For any $n \in \mathbb{N}$, we define $I_n = [-n, n]$. In accordence with theorem (2.5), there exists a unique continuous function $y_n : I_n \to \mathbb{R}$ such that

$$\begin{aligned} y_n(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, f(s), (Sf)(s)) ds \\ &- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, f(s), (Sf)(s)) ds + \frac{c}{a+b} \end{aligned} \tag{3.11}$$

and

$$|y(t) - y_n(t)| \leq \frac{K}{1-KM} \varphi(t) \tag{3.12}$$

for all $t \in I$.

The uniqueness of y_n implies that if $t \in I_n$, then

$$y_n(t) = y_{n+1}(t) = y_{n+2}(t) = \dots \tag{3.13}$$

For any $t \in \mathbb{R}$, we define $n(t) \in \mathbb{N}$ as

$$n(t) = \min\{n \in \mathbb{N} \quad | \quad t \in I_n\}. \tag{3.14}$$

Moreover, let us define a function $y_0 : \mathbb{R} \to \mathbb{R}$ by

$$y_0(t) = y_{n(t)}(t) \tag{3.15}$$

and we claim that y_0 is continuous. We take the integer $n_1 = n(t_1)$ for an arbitrary $t_1 \in \mathbb{R}$. Then, t_1 belongs to the interior of I_{n_1+1} and there exists an $\varepsilon > 0$ such that $y_0(t) = y_{n_1+1}(t)$ for all t with $t_1 - \varepsilon < t < t_1 + \varepsilon$. Since y_{n_1+1} is continuous at t_1, y_0 is continuous at t_1 for any $t_1 \in \mathbb{R}$. Now, we will prove that u_0 satisfies (2.4) and (3.5) for all $t \in \mathbb{R}$. Let $n(t)$ be an integer for an arbitrary $t \in \mathbb{R}$. Then, from (3.11) and (3.15), we have $t \in I_{n(t)}$ and

$$\begin{aligned} y_0(t) = y_n(t) = \frac{1}{\Gamma(\alpha)} &\int_0^t (t-s)^{\alpha-1} F(s, y_{n(t)}(s), (Sy_{n(t)})(s)) ds \\ - \frac{b}{(a+b)\Gamma(\alpha)} &\int_0^T (T-s)^{\alpha-1} F(s, y_{n(t)}(s), (Sy_{n(t)})(s)) ds + \frac{c}{a+b} \end{aligned}$$

Since $n(s) \leq n(t)$ for any $s \in I_{n(t)}$, the last equality is correct and we have

$$y_{n(t)}(s) = y_{n(s)}(s) = y_0(s)$$

by (3.13) and (3.15).

Since $t \in I_{n(t)}$ for all $t \in \mathbb{R}$, by (3.12) and (3.15), we have

$$|y(t) - y_0(t)| \leq \left|y(t) - y_{n(t)}(t)\right| \leq \frac{K}{1-KM} \varphi(t)$$

for all $t \in \mathbb{R}$. Finally we prove that y_0 is unique. Assume that $x_0 : \mathbb{R} \to \mathbb{R}$ is another continuous function which satisfies (2.4) and (3.5), with x_0 in place of y_0, for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$ be a discretionary number. Since the resrictions $x_0|I_{n(t)}$ and $y_0|I_{n(t)}$ both satisfy (2.4) and (3.5) for all $t \in I_{n(t)}$, the uniqueness of $y_{n(t)} = y_0|_{I_{n(t)}}$ suggests that,

$$y_0(t) = y_0|_{I_{n(t)}}(t) = x_0|_{I_{n(t)}}(t) = x_0(t)$$

Similarly the proof can be done for the classes $I = (-\infty, T]$ and $I = [0, \infty)$.

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