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GENERALIZED Φ-EPI MAPS AND TOPOLOGICAL COINCIDENCE PRINCIPLES

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Abstract. In this paper we present the notion of a Φ-epi map for a general class of maps and we present coincidence and homotopy properties for these maps.
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1. Introduction

The notion of a 0-epi map was introduced by Furi, Martelli and Vignoli [4] and extended in a variety of settings in the literature by other authors, see for example [5, 7, 9, 11]. In this paper we present a generalization of Φ-epi maps motivated in part, for example, from continuation theorems of set valued maps which have continuous selections [1, 2]. In particular we present coincidence, homotopy and normalization properties of these Φ-epi maps.

2. Φ-epi maps

Let E be a normal topological vector space and U an open subset of E.

We will consider the classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ of maps.

Definition 2.1. We say $F \in D(\overline{U}, E)$ (respectively $F \in A(\overline{U}, E)$) if $F : \overline{U} \to 2^E$ and $F \in \mathbf{D}(\overline{U}, E)$ (respectively $F \in \mathbf{A}(\overline{U}, E)$); here $\overline{U}$ denotes the closure of U in E and 2^E denotes the family of nonempty subsets of E.

Definition 2.2. We say $F \in B(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{B}(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, E)$ of F.

Remark 2.3. Note Ψ is a selection of F (in Definition 2.2) if $\Psi(x) \subseteq F(x)$ for $x \in \overline{U}$.

Remark 2.4. We say $F \in D(E, E)$ (respectively $F \in A(E, E)$) if $F : E \to 2^E$ and $F \in \mathbf{D}(E, E)$ (respectively $F \in \mathbf{A}(E, E)$). We say $F \in B(E, E)$ if $F : E \to 2^E$ and $F \in \mathbf{B}(E, E)$ and there exists a selection $\Psi \in D(E, E)$ of F.

In this section we fix a $\Phi \in B(\overline{U}, E)$.

Definition 2.5. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 2.6. We say $F \in B_\Phi(\overline{U}, E)$ if $F \in B(\overline{U}, E)$ and $F(x) \subseteq \Phi(x)$ for $x \in \partial U$.

Definition 2.7. Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is Φ-epi if for any map $G \in B_\Phi(\overline{U}, E)$ and any selection $\Psi \in D(\overline{U}, E)$ of G there exists $x \in U$ with $F(x) \cap \Psi(x) \neq \emptyset$.

Remark 2.8. In Definition 2.7 note $\emptyset \neq F(x) \cap \Psi(x) \subseteq F(x) \cap G(x)$. We could take $G = \Phi$ in Definition 2.7 (note $\Phi \in B_\Phi(\overline{U}, E)$). Then if $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi then for any selection $\phi \in D(\overline{U}, E)$ of Φ there exists $x \in U$ with $F(x) \cap \phi(x) \neq \emptyset$ so $\emptyset \neq F(x) \cap \phi(x) \subseteq F(x) \cap \Phi(x)$.

Remark 2.9. In Definition 2.7 note $\Psi(x) \subseteq G(x) \subseteq \Phi(x)$ for $x \in \partial U$ i.e. $\Psi \in D_\Phi(\overline{U}, E)$ (we say $H \in D_\Phi(\overline{U}, E)$ if $H \in D(\overline{U}, E)$ and $H(x) \subseteq \Phi(x)$ for $x \in \partial U$).

Remark 2.10. In Definition 2.7 if $G \in B_\Phi(\overline{U}, E)$ and if $G \in D(\overline{U}, E)$ then an example of a selection Ψ of G is G itself. In applications we see by appropriately choosing U, E, D and B then automatically there exists a selection Ψ of F in Definition 2.2 (see for example Remark 2.20).

Our first result is a coincidence property for Φ-epi maps.

Theorem 2.11. *Let E be a normal topological vector space, U an open subset of E, $G \in B(\overline{U}, E)$, $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi and suppose*

$$\begin{cases} \mu(\,.\,)\, G(\,.\,) + (1 - \mu(\,.\,))\, \Phi(\,.\,) \in B(\overline{U}, E) \ \text{ for any} \\ \text{continuous map } \ \mu : \overline{U} \to [0,1] \ \text{ with } \ \mu(\partial U) = 0. \end{cases} \tag{2.1}$$

For any selection $\Lambda \in D(\overline{U}, E)$ of G and any selection $\phi \in D(\overline{U}, E)$ of Φ assume

$$\begin{cases} K = \{x \in \overline{U} : \ F(x) \cap [t\, \Lambda(x) + (1-t)\, \phi(x)] \neq \emptyset \ \text{ for some } \ t \in [0,1]\} \\ \text{is closed and } K \text{ does not intersect } \partial U \end{cases} \tag{2.2}$$

and

$$\begin{cases} \mu(\,.\,)\, \Lambda(\,.\,) + (1 - \mu(\,.\,))\, \phi(\,.\,) \in D(\overline{U}, E) \ \text{ for any continuous} \\ \text{map } \ \mu : \overline{U} \to [0,1] \ \text{ with } \ \mu(\partial U) = 0 \ \text{ and } \ \mu(K) = 1. \end{cases} \tag{2.3}$$

Then there exists $x \in U$ with $F(x) \cap \Lambda(x) \neq \emptyset$ (so $\emptyset \neq F(x) \cap \Lambda(x) \subseteq F(x) \cap G(x)$.

Proof. Let $\Lambda \in D(\overline{U}, E)$ be any selection of G and $\phi \in D(\overline{U}, E)$ be any selection of Φ and let

$$K = \left\{x \in \overline{U} : \ F(x) \cap [t\, \Lambda(x) + (1-t)\, \phi(x)] \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

Note $K \neq \emptyset$ since $F(x) \cap \phi(x) \neq \emptyset$ for some $x \in U$ (note $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi and note Remark 2.8). From (2.2) we note K is closed and $K \cap \partial U = \emptyset$. Then Urysohn's lemma guarantees that there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$

and $\mu(K) = 1$. Define the mappings J and Ψ by

$$J(x) = \mu(x)\, G(x) + (1 - \mu(x))\, \Phi(x) \quad \text{and} \quad \Psi(x) = \mu(x)\, \Lambda(x) + (1 - \mu(x))\, \phi(x).$$

Now (2.1) guarantees that $J \in B(\overline{U}, E)$ and also note for $x \in \partial U$ we have $J(x) = \Phi(x)$, so $J \in B_\Phi(\overline{U}, E)$. Note (2.3) guarantees that $\Psi \in D(\overline{U}, E)$ and Ψ is a selection of J. Now since F is Φ-epi there exists $x \in U$ with $F(x) \cap \Psi(x) \neq \emptyset$. Thus $x \in K$ and as a result $\mu(x) = 1$. Consequently $F(x) \cap \Lambda(x) \neq \emptyset$. □

Remark 2.12. We can remove the assumption that E is normal in the statement of Theorem 2.11 provided we put conditions on the maps so that K is compact (the existence of the map μ in the proof above is then guaranteed since topological vector spaces are completely regular). Note as well, for example, if the maps F, Λ and ϕ are upper semicontinuous with compact values then K in (2.2) is closed.

Remark 2.13. Let E be a complete metrizable locally convex topological vector space and U an open subset of E. Let $\mathbf{D} = D$ and $\mathbf{B} = B$. We say $Q \in D(\overline{U}, E)$ if $Q : \overline{U} \to E$ is a continuous "compact" map. We say $R \in B(\overline{U}, E)$ if $R : \overline{U} \to C(E)$ (here $C(E)$ denotes the family of nonempty closed convex subsets of E) is a lower semicontinuous "compact" map (the existence of a continuous selection Ψ of R is guaranteed from Michael's selection theorem [3, 6, 8] and note Ψ is compact since Ψ is a selection of R and R is compact, so $\Psi \in D(\overline{U}, E)$). Note (2.1) and (2.3) are immediate (from the properties of sums and products of lower semicontinuous maps [3, 6]; we include the map being "compact" here because of the normalization property (see Remark 2.20)). Also (2.2) reduces to showing

$$K = \{x \in \overline{U} :\ t\, \Lambda(x) + (1 - t)\, \phi(x) \in F(x) \ \text{ for some } \ t \in [0, 1]\}$$

is closed and does not intersect ∂U; for example if the map F is upper semicontinuous with closed values then K would be closed.

We now rewrite Theorem 2.11 as a Leray-Schauder type alternative.

Theorem 2.14. *Let E be a normal topological vector space, U an open subset of E, $G \in B(\overline{U}, E)$, $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi and suppose* (2.1) *holds. For any selection $\Lambda \in D(\overline{U}, E)$ of G and any selection $\phi \in D(\overline{U}, E)$ of Φ assume*

$$K = \left\{x \in \overline{U} :\ F(x) \cap [t\, \Lambda(x) + (1-t)\, \phi(x)] \neq \emptyset \ \textit{for some}\ t \in [0, 1]\right\} \ \textit{is closed} \tag{2.4}$$

and (2.3) *holds. Then either*

(A1) there exists $x \in \overline{U}$ with $F(x) \cap \Lambda(x) \neq \emptyset$,

or

(A2) there exists $x \in \partial U$ and $\lambda \in (0, 1)$ with $F(x) \cap [\lambda\, \Lambda(x) + (1 - \lambda)\, \phi(x)] \neq \emptyset$,

holds.

Proof. Suppose $(A2)$ does not hold and $F(x) \cap \Lambda(x) = \emptyset$ for $x \in \partial U$ (since otherwise $(A1)$ holds). Also note $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$ since $F \in A_{\partial U}(\overline{U}, E)$, so $F(x) \cap \phi(x) = \emptyset$ for $x \in \partial U$ (note $F(x) \cap \phi(x) \subseteq F(x) \cap \Phi(x)$). Thus

$$\text{there exists } x \in \partial U \ \text{ and } \ \lambda \in [0, 1] \ \text{ with } \ F(x) \cap [\lambda\, \Lambda(x) + (1 - \lambda)\, \phi(x)] \neq \emptyset$$

cannot occur, so (2.2) holds. Now Theorem 2.11 guarantees that there exists $x \in U$ with $F(x) \cap \Lambda(x) \neq \emptyset$. □

Our next result is a homotopy type property for Φ-epi maps.

Theorem 2.15. *Let E be a normal topological vector space, U an open subset of E, $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi, H is a map defined on $\overline{U} \times [0,1]$ with values in E with $H(x,0) = \{0\}$ for $x \in \partial U$ and $F(\,.\,) - H(\,.\,,1) \in A(\overline{U}, E)$. Suppose*

$$\{x \in \overline{U} :\ F(x) \cap [\Phi(x) + H(x,t)] \neq \emptyset \text{ for some } t \in [0,1]\} \text{ does not intersect } \partial U. \tag{2.5}$$

Assume there exists a selection h of H such that for any $G \in B_\Phi(\overline{U}, E)$ and any selection $\Psi \in D(\overline{U}, E)$ of G we have

$$G(\,.\,) + H(\,.\,,0) \in B(\overline{U}, E) \quad \text{and} \quad \Psi(\,.\,) + h(\,.\,,0) \in D(\overline{U}, E) \tag{2.6}$$

$$K = \{x \in \overline{U} :\ F(x) \cap [\Psi(x) + h(x,t)] \neq \emptyset \ \text{ for some } \ t \in [0,1]\} \ \text{ is closed} \tag{2.7}$$

and

$$\begin{cases} G(\,.\,) + H(\,.\,,\mu(\,.\,)) \in B(\overline{U}, E) \ \text{ and } \ \Psi(\,.\,) + h(\,.\,,\mu(\,.\,)) \in D(\overline{U}, E) \\ \text{for any continuous map } \ \mu : \overline{U} \to [0,1] \ \text{ with } \ \mu(\partial U) = 0 \ \text{ and } \ \mu(K) = 1. \end{cases} \tag{2.8}$$

Then $F(\,.\,) - H(\,.\,,1)$ is Φ-epi.

Proof. Let $G \in B_\Phi(\overline{U}, E)$ and let $\Psi \in D(\overline{U}, E)$ be any selection of G. We must show that there exists $x \in U$ with $[F(x) - H(x,1)] \cap \Psi(x) \neq \emptyset$. Let h, H be as in the statement of Theorem 2.15 and let

$$K = \left\{x \in \overline{U} :\ F(x) \cap [\Psi(x) + h(x,t)] \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

Consider $t = 0$. Note $G(\,.\,) + H(\,.\,,0) \in B_\Phi(\overline{U}, E)$ (note (2.6) and also for $x \in \partial U$ we have $G(x) + H(x,0) = G(x) \subseteq \Phi(x)$ since $G \in B_\Phi(\overline{U}, E)$), $\Psi(\,.\,) + h(\,.\,,0) \in D(\overline{U}, E)$ is a selection of $G(\,.\,) + H(\,.\,,0)$, and since F is Φ-epi we see that there exists $x \in U$ with $F(x) \cap [\Psi(x) + h(x,0)] \neq \emptyset$, so $K \neq \emptyset$. In addition K is closed (see (2.7)). Also note $K \cap \partial U = \emptyset$ (note if $x \in \partial U$ then $F(x) \cap [\Psi(x) + h(x,t)] \subseteq F(x) \cap [G(x) + H(x,t)] \subseteq F(x) \cap [\Phi(x) + H(x,t)]$ and note (2.5)). Now Urysohn's lemma guarantees that there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Define maps J and Λ by

$$J(x) = G(x) + H(x, \mu(x)) \quad \text{and} \quad \Lambda(x) = \Psi(x) + h(x, \mu(x)).$$

Note from (2.8) that $J \in B_\Phi(\overline{U}, E)$ (note if $x \in \partial U$ then $J(x) = G(x) + H(x, \mu(x)) = G(x) \subseteq \Phi(x)$ since $G \in B_\Phi(\overline{U}, E)$), and $\Lambda \in D(\overline{U}, E)$ is a selection of J. Now since F is Φ-epi then there exists $x \in U$ with $F(x) \cap \Lambda(x) \neq \emptyset$ i.e. $F(x) \cap [\Psi(x) + h(x, \mu(x))] \neq \emptyset$. Thus $x \in K$ and as a result $\mu(x) = 1$. Consequently $F(x) \cap [\Psi(x) + h(x,1)] \neq \emptyset$, so $F(x) \cap [\Psi(x) + H(x,1)] \neq \emptyset$ i.e. $[F(x) - H(x,1)] \cap \Psi(x) \neq \emptyset$. □

Remark 2.16. We can remove the assumption that E is normal in the statement of Theorem 2.15 provided we put conditions on the maps so that K is compact.

Remark 2.17. Note in Theorem 2.15 we could replace (2.5) with either (here h and Ψ are as described in the statement of Theorem 2.15)

$$\{x \in \overline{U} :\ F(x) \cap [\Psi(x) + h(x,t)] \neq \emptyset \ \text{ for some } \ t \in [0,1]\} \ \text{ does not intersect } \ \partial U$$

or

$$\{x \in \overline{U}:\ F(x) \cap [G(x) + H(x,t)] \neq \emptyset \ \text{ for some } \ t \in [0,1]\} \ \text{ does not intersect } \ \partial U.$$

Next we present three normalization type results.

Theorem 2.18. *Let E be a normal topological vector space, U an open subset of E and $0 \in U$. Suppose*

$$i \in A(\overline{U}, E) \quad \textit{where} \ \ i \ \ \textit{is the identity map} \tag{2.9}$$

$$x \notin \lambda\, \Phi(x) \ \ \textit{for} \ \ x \in \partial U \ \ \textit{and} \ \ \lambda \in (0,1] \tag{2.10}$$

and

$$\overline{U} \ \ \textit{is a retract of} \ \ E \ \ \textit{i.e. there exists a retraction} \ \ r : E \to \overline{U}. \tag{2.11}$$

For any $H \in B_\Phi(\overline{U}, E)$ and any selection $\Psi \in D(\overline{U}, E)$ of H assume

$$K = \left\{x \in \overline{U} :\ x \in \lambda\, \Psi(x) \ \ \textit{for some} \ \ \lambda \in [0,1]\right\} \quad \textit{is closed} \tag{2.12}$$

$$\left\{ \begin{array}{l} \textit{for any continuous map} \ \ \eta : E \to [0,1] \ \ \textit{with} \ \ \eta(E \setminus U) = 0 \ \ \textit{and} \\ \eta(K) = 1 \ \ \textit{the map} \ \ J \in D(E,E) \ \ \textit{where} \ \ J(x) = \eta(x)\, \Psi(r(x)) \end{array} \right. \tag{2.13}$$

and

$$\textit{any map} \ \ \Lambda \in D(E,E) \ \ \textit{has a fixed point} \tag{2.14}$$

hold. Then i is Φ-epi.

Proof. Let $H \in B_\Phi(\overline{U}, E)$ and let $\Psi \in D(\overline{U}, E)$ be a selection of H. We must show there exists $x \in U$ with $x \in \Psi(x)$ (i.e. $i(x) \cap \Psi(x) \neq \emptyset$). Let

$$K = \left\{x \in \overline{U} :\ x \in \lambda\, \Psi(x) \ \text{ for some } \ \lambda \in [0,1]\right\}.$$

Now $K \neq \emptyset$ (note $0 \in U$) and K is closed. Also note $K \subset U$ since if there exists $x \in \partial U$ and some $\lambda \in [0,1]$ with $x \in \lambda\Psi(x)$ then $x \in \lambda\Phi(x)$ since $\Psi(x) \subseteq H(x) \subseteq \Phi(x)$, a contradiction (see (2.10)). Urysohn's Lemma guarantees that there exists a continuous map $\eta : E \to [0,1]$ with $\eta(K) = 1$ and $\eta(E \setminus U) = 0$. Define a map J by $J(x) = \eta(x)\, \Psi(r(x))$ where r is given in (2.11). Now (2.13) and (2.14) guarantees that there exists $x \in E$ with $x \in \eta(x)\, \Psi(r(x))$. If $x \in E \setminus U$ then $\eta(x) = 0$, a contradiction since $0 \in U$. Thus $x \in U$ and so $x \in \eta(x)\, \Psi(x)$. As a result $x \in K$ so $\eta(x) = 1$. Thus $x \in \Psi(x)$. □

Remark 2.19. We can remove the assumption that E is normal in the statement of Theorem 2.18 provided we put conditions on the maps so that K is compact.

Remark 2.20. We first recall the PK maps from the literature. Let Z and W be subsets of topological vector spaces Y_1 and Y_2 and F a multifunction. We say $F \in PK(Z, W)$ if W is convex and there exists a map $S : Z \to W$ with

$$Z = \cup\{int\, S^{-1}(w) :\ w \in W\}, \ co\,(S(x)) \subseteq F(x)$$

for $x \in Z$ and $S(x) \neq \emptyset$ for each $x \in Z$; here $S^{-1}(w) = \{z :\ w \in S(z)\}$.

Let E be a locally convex topological vector space, U an open convex subset of E, $0 \in U$ and $\overline{U}$ paracompact. Let $\mathbf{D} = D$ and $\mathbf{B} = B$. We say $Q \in D(\overline{U}, E)$ if $Q : \overline{U} \to E$ is a continuous compact map. We say $R \in B(\overline{U}, E)$ if $R \in PK(\overline{U}, E)$ and R is a compact map (the existence of a continuous selection Ψ of R is guaranteed from

[10, Theorem 1.3] and note Ψ is compact since Ψ is a selection of R and R is compact, so $\Psi \in D(\overline{U}, E)$). Note (2.11), (2.12), (2.13) and (2.14) (Schauder-Tychonoff fixed point theorem) hold. We note that a "compact" map above could be replaced by a more general "compactness type" map; see [1, 2].

Theorem 2.21. *Let E be a locally convex topological vector space, U an open convex subset of E, $0 \in U$ and suppose* (2.9) *and* (2.10) *hold. Let $r : E \to \overline{U}$ be given by*

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for } \ x \in E,$$

where μ is the Minkowski functional on $\overline{U}$ (i.e. $\mu(x) = \inf\{\alpha > 0 : \ x \in \alpha\,\overline{U}\}$), and for any $H \in B_\Phi(\overline{U}, E)$ and any selection $\Psi \in D(\overline{U}, E)$ of H assume

$$r\,\Psi \in D(\overline{U}, \overline{U}) \tag{2.15}$$

and

$$\text{any map } \ \Lambda \in D(\overline{U}, \overline{U}) \ \text{ has a fixed point} \tag{2.16}$$

hold. Then i is Φ-epi.

Proof. Let $H \in B_\Phi(\overline{U}, E)$ and let $\Psi \in D(\overline{U}, E)$ be a selection of H. Let $\Lambda = r\,\Psi$. Now (2.15) and (2.16) guarantee that there exists $x \in \overline{U}$ with $x \in r\,\Psi(x)$. Then $x = r(y)$ where $y \in \Psi(x)$; here $x \in \overline{U} = U \cup \partial U$. If we show

$$x \in U \quad \text{and} \quad r(y) = y \tag{2.17}$$

then $x = y$ so $x \in \Psi(x)$ and we are finished. It remains to show (2.17). Let $x \in \partial U$. Then $\mu(x) = 1$ so

$$1 = \mu(x) = \mu(r(y)) = \frac{\mu(y)}{\max\{1, \mu(y)\}},$$

so $\mu(y) \geq 1$. Thus $x = r(y) = \frac{y}{\mu(y)}$ so with $\lambda = \frac{1}{\mu(y)}$ we have $x \in \lambda\,\Psi(x)$, so $x \in \lambda\,\Phi(x)$ since $x \in \partial U$ and $\Psi(x) \subseteq H(x) \subseteq \Phi(x)$, and this contradicts (2.10). Thus $x \in U$. Then $\mu(x) < 1$ so

$$1 > \mu(x) = \mu(r(y)) = \frac{\mu(y)}{\max\{1, \mu(y)\}},$$

and as a result $\mu(y) < 1$. Thus $r(y) = y$, so (2.17) holds. □

Theorem 2.22. *Let E be a topological space, U an open subset of E and suppose* (2.9) *and* (2.14) *hold. In addition assume*

$$\Phi \in B(E, E) \ \text{ and there is no } \ z \in E \setminus U \ \text{ with } \ z \in \Phi(z). \tag{2.18}$$

For any $H \in B_\Phi(\overline{U}, E)$ and any selection $\Psi \in D(\overline{U}, E)$ of H assume

$$J \in D(E, E) \ \text{ where } \ J(x) = \begin{cases} \Psi(x), & x \in U \\ \Phi(x), & x \in E \setminus U. \end{cases} \tag{2.19}$$

Then i is Φ-epi.

Proof. Let $H \in B_\Phi(\overline{U}, E)$ and let $\Psi \in D(\overline{U}, E)$ be a selection of H. Let

$$J(x) = \begin{cases} \Psi(x), & x \in U \\ \Phi(x), & x \in E \setminus U. \end{cases}$$

Now (2.14) and (2.19) guarantee that there exists $x \in E$ with $x \in J(x)$. If $x \in E \setminus U$ then $x \in \Phi(x)$, which contradicts (2.19). Thus $x \in U$ so $x \in \Psi(x)$. □

We now show that the ideas in this section can be applied to other natural situations. Let E be a normal topological vector space, Y a topological vector space, and U an open subset of E. Also let $L : dom\, L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear single valued map with $L + T : dom\, L \to Y$ a bijection; for convenience we say $T \in H_L(E, Y)$.

Definition 2.23. We say $F \in D(\overline{U}, Y; L, T)$ (respectively $F \in A(\overline{U}, Y; L, T)$) if $F : \overline{U} \to 2^Y$ with $(L+T)^{-1}(F+T) \in D(\overline{U}, E)$ (respectively $(L+T)^{-1}(F+T) \in A(\overline{U}, E)$).

Definition 2.24. We say $F \in B(\overline{U}, Y; L, T)$ if $F : \overline{U} \to 2^Y$ with $(L+T)^{-1}(F+T) \in B(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, Y; L, T)$ of F.

For the remainder of this section we fix a $\Phi \in B(\overline{U}, Y; L, T)$.

Definition 2.25. We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ if $F \in A(\overline{U}, Y; L, T)$ with

$$(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$$

for $x \in \partial U$.

Definition 2.26. We say $F \in B_{\Phi}(\overline{U}, Y; L, T)$ if $F \in B(\overline{U}, Y; L, T)$ and

$$(L+T)^{-1}(F+T)(x) \subseteq (L+T)^{-1}(\Phi+T)(x)$$

for $x \in \partial U$.

Definition 2.27. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$. We say F is $(L, T)\,\Phi$-epi if for any map $G \in B_{\Phi}(\overline{U}, Y; L, T)$ and any selection $\Psi \in D(\overline{U}, Y; L, T)$ of G there exists $x \in U$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Psi+T)(x) \neq \emptyset$.

Remark 2.28. If $F \in A_{\partial U}(\overline{U}, Y; L, T)$ is $(L, T)\,\Phi$-epi then for any selection $\phi \in D(\overline{U}, Y; L, T)$ of Φ (note $\Phi \in B(\overline{U}, Y; L, T)$) there exists $x \in U$ with

$$(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\phi+T)(x) \neq \emptyset.$$

Theorem 2.29. *Let E be a normal topological vector space, Y a topological vector space, U an open subset of E, $L : dom\, L \subseteq E \to Y$ a linear single valued map and $T \in H_L(E, Y)$. Suppose $G \in B(\overline{U}, Y; L, T)$, $F \in A_{\partial U}(\overline{U}, Y; L, T)$ is $(L, T)\,\Phi$-epi, and suppose*

$$\begin{cases} \mu(\,.\,)\, G(\,.\,) + (1 - \mu(\,.\,))\, \Phi(\,.\,) \in B(\overline{U}, Y; L, T) \ \textit{ for any} \\ \textit{continuous map } \ \mu : \overline{U} \to [0,1] \ \textit{ with } \ \mu(\partial U) = 0. \end{cases} \tag{2.20}$$

For any selection $\Lambda \in D(\overline{U}, Y; L, T)$ of G and any selection $\phi \in D(\overline{U}, Y; L, T)$ of Φ assume

$$\begin{cases} K = \{x \in \overline{U} : (L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}[t\Lambda(x) + (1-t)\phi(x) + T(x)] \neq \emptyset \\ \textit{for some } \ t \in [0,1]\} \ \textit{ is closed and } \ K \ \textit{ does not intersect } \ \partial U \end{cases} \tag{2.21}$$

and

$$\left\{\begin{array}{l} \mu(\,.\,)\,\Lambda(\,.\,) + (1-\mu(\,.\,))\,\phi(\,.\,) \in D(\overline{U},Y;L,T) \quad \textit{for any continuous} \\ \textit{map} \;\; \mu:\overline{U}\to[0,1] \quad \textit{with} \;\; \mu(\partial U)=0 \;\; \textit{and} \;\; \mu(K)=1. \end{array}\right. \tag{2.22}$$

Then there exists $x\in U$ *with* $(L+T)^{-1}\,(F+T)(x)\cap(L+T)^{-1}\,(\Lambda+T)(x)\neq\emptyset$.

Proof. Let $\Lambda\in D(\overline{U},Y;L,T)$ be any selection of G and $\phi\in D(\overline{U},Y;L,T)$ be any selection of Φ and let K be as in (2.21). Now $K\neq\emptyset$ (see Remark 2.28) is closed and $K\cap\partial U=\emptyset$ so there exists a continuous map $\mu:\overline{U}\to[0,1]$ with $\mu(\partial U)=0$ and $\mu(K)=1$. Define the mappings J and Ψ by

$$J(x)=\mu(x)\,G(x)+(1-\mu(x))\,\Phi(x) \quad\text{and}\quad \Psi(x)=\mu(x)\,\Lambda(x)+(1-\mu(x))\,\phi(x)$$

Now $J\in B_\Phi(\overline{U},Y;L,T)$ (note if $x\in\partial U$ then since $\mu(x)=0$ we have $(L+T)^{-1}\,(J+T)(x)=(L+T)^{-1}\,(\Phi+T)(x)$) and $\Psi\in D(\overline{U},Y;L,T)$ is a selection of J. Now since F is $(L,T)\,\Phi$-epi there exists $x\in U$ with $(L+T)^{-1}\,(F+T)(x)\cap(L+T)^{-1}\,(\Psi+T)(x)\neq\emptyset$. Thus $x\in K$ so $\mu(x)=1$, and as a result

$$(L+T)^{-1}\,(F+T)(x)\cap(L+T)^{-1}\,(\Lambda+T)(x)\neq\emptyset. \qquad \square$$

Theorem 2.30. *Let E be a normal topological vector space, Y a topological vector space, U an open subset of E, $L:dom\,L\subseteq E\to Y$ a linear single valued map, $T\in H_L(E,Y)$, $F\in A_{\partial U}(\overline{U},Y;L,T)$ is $(L,T)\,\Phi$-epi, H is a map defined on $\overline{U}\times[0,1]$ with values in Y with $(L+T)^{-1}\,H(x,0)=\{0\}$ for $x\in\partial U$, $F(\,.\,)-H(\,.\,,1)\in A(\overline{U},Y;L,T)$ and assume*

$$\left\{\begin{array}{l} \{x\in\overline{U}:\;(L+T)^{-1}\,(F+T)(x)\cap(L+T)^{-1}\,[\Phi(x)+H(x,t)+T(x)]\neq\emptyset \\ \textit{for some} \;\; t\in[0,1]\} \;\; \textit{does not intersect} \;\; \partial U. \end{array}\right. \tag{2.23}$$

Assume there exists a selection h of H such that for any $G\in B_\Phi(\overline{U},Y;L,T)$ and any selection $\Psi\in D(\overline{U},Y;L,T)$ of G we have

$$G(\,.\,)+H(\,.\,,0)\in B(\overline{U},Y;L,T) \quad \textit{and} \quad \Psi(\,.\,)+h(\,.\,,0)\in D(\overline{U},Y;L,T) \tag{2.24}$$

$$\left\{\begin{array}{l} K=\{x\in\overline{U}:\;(L+T)^{-1}\,(F+T)(x)\cap(L+T)^{-1}\,[\Psi(x)+h(x,t)+T(x)]\neq\emptyset \\ \textit{for some} \;\; t\in[0,1]\} \;\; \textit{is closed} \end{array}\right. \tag{2.25}$$

and

$$\left\{\begin{array}{l} G(\,.\,)+H(\,.\,,\mu(\,.\,))\in B(\overline{U},Y;L,T) \;\; \textit{and} \;\; \Psi(\,.\,)+h(\,.\,,\mu(\,.\,))\in D(\overline{U},Y;L,T) \\ \textit{for any continuous map} \;\; \mu:\overline{U}\to[0,1] \;\; \textit{with} \;\; \mu(\partial U)=0 \;\; \textit{and} \;\; \mu(K)=1. \end{array}\right. \tag{2.26}$$

Then $F(\,.\,)-H(\,.\,,1)$ is $(L,T)\,\Phi$-epi.

Proof. Let $G\in B_\Phi(\overline{U},Y;L,T)$ and let $\Psi\in D(\overline{U},Y;L,T)$ be any selection of G. Let h, H be as in the statement of Theorem 2.30 and let K be as in (2.25). Consider $t=0$. Now $G(\,.\,)+H(\,.\,,0)\in B_\Phi(\overline{U},Y;L,T)$ (note for $x\in\partial U$ we have

$$(L+T)^{-1}\,[G(x)+H(x,0)+T(x)]=(L+T)^{-1}\,(G+T)(x)\subseteq(L+T)^{-1}\,(\Phi+T)(x)$$

since $G \in B_\Phi(\overline{U}, Y; L, T))$, $\Psi(\,.\,) + h(\,.\,, 0) \in D(\overline{U}, Y; L, T)$ is a selection of $G(\,.\,) + H(\,.\,, 0)$, and since F is (L, T) Φ-epi we see there exists $x \in U$ with

$$(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}[\Psi(x) + h(x,0) + T(x)] \neq \emptyset,$$

so $K \neq \emptyset$. In addition K is closed and note $K \cap \partial U = \emptyset$ (note if $x \in \partial U$ then $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}[\Psi(x)+h(x,t)+T(x)] \subseteq (L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}[G(x)+H(x,t)+T(x)] \subseteq (L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}[\Phi(x)+H(x,t)+T(x)]$, and note (2.23)). Then there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Define maps J and Λ by

$$J(x) = G(x) + H(x, \mu(x)) \quad \text{and} \quad \Lambda(x) = \Psi(x) + h(x, \mu(x)).$$

Note from (2.26) that $J \in B_\Phi(\overline{U}, Y; L, T)$ (note if $x \in \partial U$ then

$$\begin{aligned}(L+T)^{-1}(J+T)(x) &= (L+T)^{-1}[G(x) + H(x,0) + T(x)] \\ &= (L+T)^{-1}(G+T)(x) \subseteq (L+T)^{-1}(\Phi+T)(x)\end{aligned}$$

since $G \in B_\Phi(\overline{U}, Y; L, T))$ and $\Lambda \in D(\overline{U}, Y; L, T)$ is a selection of J. Now since F is (L, T) Φ-epi there exists $x \in U$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Lambda+T)(x) \neq \emptyset$ i.e. $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Psi(x) + h(x, \mu(x)) + T(x)) \neq \emptyset$. Thus $x \in K$ so $\mu(x) = 1$ and as a result

$$(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Psi(x) + h(x,1) + T(x)) \neq \emptyset. \qquad \square$$

Remark 2.31. We can remove the assumption that E is normal in the statement of Theorem 2.29 and Theorem 2.30 provided we put conditions on the maps so that K is compact.

Remark 2.32. Note in Theorem 2.30 we could replace (2.23) with either (here Ψ and h are as described in the statement of Theorem 2.30)

$$\begin{cases} \{x \in \overline{U} : \ (L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}[\Psi(x) + h(x,t) + T(x)] \neq \emptyset \\ \text{for some} \ \ t \in [0,1]\} \ \text{ does not intersect } \ \partial U \end{cases}$$

or

$$\begin{cases} \{x \in \overline{U} : \ (L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}[G(x) + H(x,t) + T(x)] \neq \emptyset \\ \text{for some} \ \ t \in [0,1]\} \ \text{ does not intersect } \ \partial U. \end{cases}$$

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