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NEW EXTENSION OF SOME COMMON FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

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Abstract. In the current paper, some common fixed point theorems are presented for generalized φ-weak contraction mappings and A_φ-contraction mappings. Also, we examine the existence and uniqueness of common fixed points for single-valued mappings satisfying the notion of weak compatibility in a complete metric space. Our results generalize and extend many results existing in literature.
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1. Introduction

At present fixed point theory is an immensely active area of research due to its applications in multiple fields. It concerns the results which state that under certain conditions a self map on a set admits a fixed point. Among all the results in fixed point theory 'Banach Contraction Principle' in metric fixed point theory is the most celebrated one due to its simplicity and ease of applicability in major areas of mathematics. Following Banach Contraction Principle, Boyd and Wong [5] investigated the fixed point results in nonlinear contraction mappings.

The study of fixed point results in partially ordered sets was initiated by Ran and Reurings [22]. Their results are hybrid of two classical theorems: Banach fixed point theorem and Knaster-Tarski fixed point theorem. Neito & Rodríguez-López ([15], [16]) extended the main results of Ran and Reurings showing that monotonicity and continuity are not necessary for uniqueness of fixed point. Subsequently, many

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authors extended and generalized this fixed point theorem from different points of view.

There have been many exciting developments in the field of existence and uniqueness of fixed points in various directions [1, 4, 6, 11, 12, 13, 14, 18, 26, 27].

Another important direction of generalization of Banach Contraction Principle concerns the coincidence points and common fixed points of pair of mappings satisfying contractive type conditions.

Jungck and Rhoades [9] generalized the notion of weakly commuting mappings by introducing the concept of compatible maps, that is, the class of mappings such that they commute at their coincidence points.

Recently, Parvaneh [17] proved some common fixed point theorems for weakly compatible pair of mapping in the set up of complete metric space. Again Zhang and Song [30] proved some common fixed point theorems for two single valued generalized φ-weak contraction mappings. Inspired by their work, in the present paper, we prove some common fixed point theorems for generalized φ-weak contraction mappings using the notion of weak compatibility. Also, we prove a common fixed point theorem for A_φ contraction mapping in the setting of complete metric space.

2. Preliminaries

First, we list some important definitions and theorems, which are useful for our main results. Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and set of real numbers, respectively.

Definition 2.1. [3] A self mapping $T : X \to X$ on a metric space (X, d) is said to be a φ-weak contraction if there exists a map $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)).$$

In 1997, Alber and Guerre-Delabriere [3] defined the concept of φ-weak contraction. Also, Rhoades [23] proved the following fixed point theorem for φ-weak contraction mapping, which is one of the generalizations of Banach contraction principle.

Theorem 2.2. [23] *Let (X, d) be a metric space and $T : X \to X$ be a mapping on X such that for all $x, y \in X$, we have*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and non decreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

The following concept of generalized φ-weak contraction was introduced by Zhang and Song in 2009.

Definition 2.3. [30] Let (X, d) be a metric space. Two self mappings $S, T : X \to X$ are said to be generalized φ-weak contractions if there exists a map $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that for all $x, y \in X$,

$$d(Tx, Sy) \leq N(x, y) - \varphi(N(x, y)),$$

where $N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}(d(x, Sy) + d(y, Tx))\}$.

Zhang and Song proved the following theorem for two single valued generalized φ-weak contraction mappings.

Theorem 2.4. [30] *Let (X, d) be a metric space and $S, T : X \to X$ be two mappings such that for all $x, y \in X$,*

$$d(Tx, Sy) \leq N(x, y) - \varphi(N(x, y)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi continuous function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then T and S have a unique common fixed point.

Definition 2.5. [24] Consider the class of functions $\Phi = \{\varphi | \varphi : \mathbb{R}_+ \to \mathbb{R}_+\}$, which satisfies the following assertions:

(i) $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$,
(ii) $(\varphi^n(t))_{n\in N}$ converges to 0 for all $t > 0$,
(iii) $\sum \varphi^n(t)$ converges for all $t > 0$.

If conditions $(i - ii)$ hold then φ is called a comparison function, and, if the comparison function satisfies (iii), then φ is called a strong comparison function.

Remark 2.6. [24] Any strong comparison function is a comparison function.

Remark 2.7. [24] If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, then $\varphi(t) < t$, for all $t > 0$, $\varphi(0) = 0$ and φ is right continuous at 0.

Definition 2.8. [2] Suppose $\mathbb{R}_+$ is the set of all non negative real numbers and A be the collection of all functions $\alpha : \mathbb{R}^3_+ \to \mathbb{R}_+$ which satisfies the following conditions:

(i) α is continuous on $\mathbb{R}^3_+$ (with respect to the Euclidean metric on $\mathbb{R}^3_+$),
(ii) $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all a, b.

Definition 2.9. [2] Let (X, d) be a metric space and T a self map on X. T is said to be a A-contraction if

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty))$$

for all $x, y \in X$ and some $\alpha \in A$.

Definition 2.10. [9] Let (X, d) be a metric space and T, S are two self maps on X. T and S are said to be weakly compatible if for all $x \in X$ the equality $Tx = Sx$ implies $TSx = STx$.

3. Main results

Now we discuss our main results.

Definition 3.1. Let $\mathbb{R}_+$ be the set of all non-negative real numbers and A_φ be the collection of all functions $\alpha : \mathbb{R}^3_+ \to \mathbb{R}_+$ which satisfies the following conditions:

(i) α is continuous on $\mathbb{R}^3_+$ (with respect to the Euclidean metric on $\mathbb{R}^3_+$).
(ii) for all $u, v \in \mathbb{R}_+$, $u \leq \alpha(u, v, v)$ or $u \leq \alpha(v, u, v)$ or $u \leq \alpha(v, v, u)$, then $u \leq \varphi(v)$, where φ is a strong comparison function.

When $\varphi(t) = kt$ as $k \in (0, 1)$ for all $t > 0$, then we have $\alpha \in A$.

Definition 3.2. Let (X, d) be a metric space and E be a nonempty closed subset of X. Suppose $T, S : E \to E$ are two self maps and $A : E \to X$ is a mapping on X. We define

$$M(x, y) = \max\{d(Tx, Sy), d(Ax, Tx), d(Ay, Sy), \frac{1}{2}(d(Ax, Sy) + d(Ay, Tx))\}.$$

Theorem 3.3. *Let (X, d) be a complete metric space and E be a nonempty closed subset of X. Let $T, S : E \to E$ be self maps such that for all $x, y \in E$*

$$d(Tx, Sy) \leq M(x, y) - \varphi(M(x, y)), \tag{3.1}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is lower semi continuous with $0 < \varphi(t) < t$ for $t \in (0, \infty)$ and $\varphi(0) = 0$ and $A : E \to X$ satisfying the following assertions:

(i) $T(E) \subseteq A(E)$ and $S(E) \subseteq A(E)$;
(ii) the pair (T, A) and (S, A) are weakly compatible.

Also, assume that $A(E)$ is a closed subset of X. Then T, A and S have a unique common fixed point.

Proof. Let $x_0 \in E$. Using (3.1) there exists two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ such that $x_{n+1} = Ax_n$ and $y_0 = Tx_0 = Ax_1, y_1 = Sx_1 = Ax_2, y_2 = Tx_2 = Ax_3, \ldots y_{2n} = Tx_{2n} = Ax_{2n+1}, y_{2n+1} = Sx_{2n+1} = Ax_{2n+2}$ for all $n \geq 0$.

We complete the proof in three steps.

Step I. We will prove that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

From 3.1 we have,

$$\begin{aligned} d(y_{2k+1}, y_{2k}) &= d(Sx_{2k+1}, Tx_{2k}) \\ &= d(Tx_{2k}, Sx_{2k+1}) \end{aligned}$$

$$\Rightarrow d(y_{2k+1}, y_{2k}) \leq M(x_{2k}, x_{2k+1}) - \varphi(M(x_{2k}, x_{2k+1})) \tag{3.2}$$

where

$$\begin{aligned} M(x_{2k}, x_{2k+1}) &= \max\{d(Tx_{2k}, Sx_{2k+1}), d(Ax_{2k}, Tx_{2k}), d(Ax_{2k+1}, Sx_{2k+1}), \\ &\quad \frac{1}{2}(d(Ax_{2k}, Sx_{2k+1}) + d(Ax_{2k+1}, Tx_{2k}))\} \\ &= \max\{d(y_{2k}, y_{2k+1}), d(y_{2k-1}, y_{2k}), d(y_{2k}, y_{2k+1}), \\ &\quad \frac{1}{2}(d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k}))\} \\ &= \max\{d(y_{2k}, y_{2k+1}), d(y_{2k-1}, y_{2k}), \frac{1}{2}(d(y_{2k-1}, y_{2k+1}))\} \\ &\leq \max\{d(y_{2k}, y_{2k+1}), d(y_{2k-1}, y_{2k}), \frac{1}{2}(d(y_{2k-1}, y_{2k}) + d(y_{2k}, y_{2k+1}))\} \\ &= \max\{d(y_{2k}, y_{2k+1}), d(y_{2k-1}, y_{2k})\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} d(y_{2k+1}, y_{2k}) &\leq \max\{d(y_{2k}, y_{2k+1}), d(y_{2k-1}, y_{2k})\} \\ &\quad - \varphi[\max\{d(y_{2k}, y_{2k+1}), d(y_{2k-1}, y_{2k})\}] \\ &= d(y_{2k-1}, y_{2k}). \end{aligned}$$

That is,

$$d(y_{2k+1}, y_{2k}) \leq d(y_{2k-1}, y_{2k}).$$

If $n = 2k+1$, similarly we can prove that $d(y_{2k+2}, y_{2k+1}) \leq d(y_{2k+1}, y_{2k})$. Therefore, for all $n \geq 0$, $d(y_{n+1}, y_n) \leq d(y_n, y_{n-1})$. Thus $\{d(y_n, y_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers and hence it is convergent.
Also,

$$\begin{aligned} M(x_{2k}, x_{2k+1}) &= M(Ax_{2k-1}, Ax_{2k}) \\ &= M(y_{2k-2}, y_{2k-1}). \end{aligned}$$

Thus

$$M(y_{2k-2}, y_{2k-1}) = \max\{d(y_{2k}, y_{2k+1}), d(y_{2k-1}, y_{2k})\}.$$

Assume that $\lim_{n\to\infty} d(y_{n+1}, y_n) = \lim_{n\to\infty} M(y_{n-2}, y_{n-1}) = r$.
Now, by the lower semi continuity of φ, we have

$$\varphi(r) \leq \lim_{n\to\infty} inf \varphi(M(y_{n-2}, y_{n-1})).$$

We claim that $r = 0$.
Now from 3.2,

$$\begin{aligned} d(y_{n+1}, y_n) &\leq M(x_n, x_{n+1}) - \varphi(M(x_n, x_{n+1})) \\ &= M(Ax_{n-1}, Ax_n) - \varphi(M(Ax_{n-1}, Ax_n)) \\ &= M(y_{n-1}, y_n) - \varphi(M(y_{n-1}, y_n)). \end{aligned}$$

Taking limit as $n \to \infty$ on the above inequality, we have

$$r \leq r - \varphi(r) \Rightarrow \varphi(r) \leq 0.$$

Thus $\varphi(r) = 0$, by the property of the function φ. Hence

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = r = 0. \tag{3.3}$$

Step II. $\{y_n\}$ is Cauchy.
Since $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$, it is sufficient to show that the subsequence $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequence $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of $\{y_{2n}\}$ such that $d(y_{2m(k)}, y_{2n(k)}) \geq \varepsilon$ for $n(k) > m(k) > k$. This means that $d(y_{2m(k)}, y_{2n(k)-1}) < \varepsilon$.
From triangle inequality,

$$\begin{aligned} \varepsilon &\leq d(y_{2m(k)}, y_{2n(k)}) \\ &\leq d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \\ &\leq \varepsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}). \end{aligned}$$

Letting $k \to \infty$ and 3.3, we can conclude that

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \varepsilon. \tag{3.4}$$

Moreover, we have

$$|d(y_{2m(k)}, y_{2n(k)+1}) - d(y_{2m(k)}, y_{2n(k)})| \leq d(y_{2n(k)}, y_{2n(k)+1}) \tag{3.5}$$

and

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)}, y_{2m(k)-1}) \tag{3.6}$$

and

$$|d(y_{2n(k)}, y_{2m(k)-2}) - d(y_{2n(k)}, y_{2m(k)-1})| \leq d(y_{2m(k)-2}, y_{2m(k)-1}). \tag{3.7}$$

Using 3.3, 3.4, 3.5, 3.6 and 3.7 we get

$$\begin{aligned}\lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}) &= \lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)-1}) \\ &= \lim_{k\to\infty} d(y_{2m(k)-2}, y_{2n(k)}) = \varepsilon.\end{aligned} \tag{3.8}$$

Now from 3.1 we have

$$\begin{aligned}d(y_{2m(k)-1}, y_{2n(k)}) &= d(Sx_{2m(k)-1}, Tx_{2n(k)}) \\ &= d(Tx_{2n(k)}, Sx_{2m(k)-1}) \\ &\leq M(x_{2n(k)}, x_{2m(k)-1}) - \varphi(M(x_{2n(k)}, x_{2m(k)-1})).\end{aligned} \tag{3.9}$$

where,

$$\begin{aligned}M(x_{2n(k)}, x_{2m(k)-1}) &= \max\{d(Tx_{2n(k)}, Sx_{2m(k)-1}), d(Ax_{2n(k)}, Tx_{2n(k)}), \\ &d(Ax_{2m(k)-1}, Sx_{2m(k-1)}), \\ &\frac{1}{2}(d(Ax_{2n(k)}, Sx_{2m(k)-1}) + d(Ax_{2m(k)-1}, Tx_{2n(k)}))\} \\ &= \max\{d(y_{2n(k)}, y_{2m(k)-1}), d(y_{2n(k)-1}, y_{2n(k)}), \\ &d(y_{2m(k)-2}, y_{2m(k)-1}), \\ &\frac{1}{2}(d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2n(k)}))\}.\end{aligned}$$

Now, we consider the following cases:
If $M(x_{2n(k)}, x_{2m(k)-1}) = d(y_{2n(k)}, y_{2m(k)-1})$, then taking limit as $k \to \infty$ in 3.9, we get

$$\varepsilon \leq \varepsilon - \varphi(\varepsilon) \Rightarrow \varphi(\varepsilon) = 0.$$

By our assumption about φ, we have $\varepsilon = 0$, which is a contradiction.
If $M(x_{2n(k)}, x_{2m(k)-1}) = d(y_{2n(k)-1}, y_{2n(k)})$, then taking limit as $k \to \infty$ in 3.9, we get

$$\varepsilon \leq 0 - \varphi(0),$$

gives a contradiction.
If $M(x_{2n(k)}, x_{2m(k)-1}) = d(y_{2m(k)-2}, y_{2m(k)-1})$, then taking limit as $k \to \infty$ in 3.9, we get

$$\varepsilon \leq 0 - \varphi(0),$$

gives a contradiction.
Finally, if $M(x_{2n(k)}, x_{2m(k)-1}) = \frac{1}{2}(d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2n(k)}))$, then taking limit as $k \to \infty$ in 3.9, we get

$$\varepsilon \leq \frac{1}{2}(\varepsilon + \varepsilon) - \varphi(\frac{1}{2}(\varepsilon + \varepsilon)).$$

i. e., $\varepsilon \leq \varepsilon - \varphi(\varepsilon)$, which is a contradiction.
Hence $\{y_n\}$ must be a Cauchy sequence.

Step III. T, S and A have a common fixed point.

Since (X, d) is complete and $\{y_n\}$ is a Cauchy sequence in X, so there is a $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Also, being a closed subset of X, E is complete and $\{y_n\} \subseteq E$. Thus, we have $z \in E$. By assumption $A(E)$ is closed, so there exist $u \in E$ such that $z = Au$. For all $n \in \mathbb{N}$,

$$d(Tu, y_{2n+1}) = d(Tu, Sx_{2n+1}) \leq M(u, x_{2n+1}) - \varphi(M(u, x_{2n+1})). \tag{3.10}$$

Now,

$$\begin{aligned} M(u, x_{2n+1}) &= \max\{d(Tu, Sx_{2n+1}), d(Au, Tu), d(Ax_{2n+1}, Sx_{2n+1}), \\ &\frac{1}{2}(d(Au, Sx_{2n+1}) + d(Ax_{2n+1}, Tu))\} \\ &= \max\{d(Tu, y_{2n+1}), d(z, Tu), d(y_{2n}, y_{2n+1}), \\ &\frac{1}{2}(d(z, y_{2n+1}) + d(y_{2n}, Tu))\}. \end{aligned}$$

Taking limit as $n \to \infty$ in 3.10 and apply the same procedure done in 3.9, we get $d(Tu, z) = 0$. So $Tu = z$. Similarly we can show that $Su = z$.
Therefore $Tu = Su = Au = z$. Since the pair (A, T) and (A, S) are weakly compatible, we have $Tz = Sz = Az$.
Now, we can have

$$d(Tz, y_{2n+1}) = d(Tz, Sx_{2n+1}) \leq M(z, x_{2n+1}) - \varphi(M(z, x_{2n+1})) \tag{3.11}$$

where,

$$\begin{aligned} M(z, x_{2n+1}) &= \max\{d(Tz, Sx_{2n+1}), d(Az, Tz), d(Ax_{2n+1}, Sx_{2n+1}), \\ &\frac{1}{2}(d(Az, Sx_{2n+1}) + d(Ax_{2n+1}, Tz))\} \\ &= \max\{d(Tz, y_{2n+1}), d(Tz, Tz), d(y_{2n}, y_{2n+1}), \\ &\frac{1}{2}(d(Tz, y_{2n+1}) + d(y_{2n}, Tz))\}. \end{aligned}$$

Taking limit as $n \to \infty$ in 3.11 and apply the same procedure in 3.9, we get $d(Tz, z) = 0 \Rightarrow Tz = z$. From $Tz = Sz = Az$, we conclude that $Tz = Sz = Az = z$. Thus, z is a common fixed point of T, S and A.
If there exists another point $v \in E$ such that $v = Tv = Sv = Av$, then using similar argument, we get

$$\begin{aligned} d(z, v) &= d(Tz, Sv) \\ &\leq M(z, v) - \varphi(M(z, v)) \\ &= d(z, v) - \varphi(d(z, v))). \end{aligned}$$

Hence $z = v$. Therefore T, S and A have a unique common fixed point. □

The following examples demonstrate the use of Theorem 3.3.

Example 3.4. (1) Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is a complete metric space. Let $E = [0, 1]$. The self maps $T, S : E \to E$ are defined as

$$T(0) = 0,\ T(x) = \frac{1}{x+5}(x \neq 0) \text{ and } S(x) = \frac{1}{3}x^2, \text{ for all } x \in E.$$

Let the mappings $A : E \to X$ and $\varphi : [0, \infty] \to [0, \infty]$ be defined by $A(x) = x$, for all $x \in E$, and $\varphi(t) = \frac{t^2}{6}$. Here, we see that all conditions of the Theorem 3.3 are satisfied. This implies that T, S and A have a unique common fixed point $(x = 0)$.

(2) Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is a complete metric space. Let $E = [0, 1]$. The self maps $T, S : E \to E$ are defined as

$$T(x) = \frac{1}{[x]+2} \text{ and } S(x) = \frac{1}{2}, \text{ for all } x \in E.$$

Let the mapping $A : E \to X$ be defined by

$$Ax = \begin{cases} \frac{1}{3} + 2x, & \text{if } x \in [0, \frac{1}{4}] \\ 1/2, & \text{if } x \in (\frac{1}{4}, 1) \\ 0, & \text{if } x = 1, \end{cases}$$

and $\varphi : [0, \infty) \to [0, \infty)$ be defined by $\varphi(t) = \frac{t}{6}$. The conditions of Theorem 3.3 are satisfied. Thus we conclude that T, S and A have a unique common fixed point $(x = \frac{1}{2})$.

For some more details on this subject as well as for related examples which support our theoretical approach see [7, 8, 10, 19, 20, 21, 25, 28, 29, 30].

Our next result illustrates an application of Theorem 3.3 in determining the existence and uniqueness of a common fixed point.

Theorem 3.5. *Let (X, d) be a complete metric space and T, S, φ and A be mappings satisfying the conditions of Theorem 3.3. Assume that A is a continuous function on X and for all $x \in X$,*

$$d(ATx, TAx) \leq d(Ax, Tx)$$

and

$$d(ASx, SAx) \leq d(Ax, Sx).$$

Then T, S and A have a unique common fixed point.

Proof. From the previous Theorem 3.3, we obtain that $\{y_n\}$ is a Cauchy sequence converging to some $z \in X$. Being a closed subset of X, E is complete and $\{y_n\} \subseteq E$. Thus $z \in E$.
Applying the continuity on A, we have $Ay_n \to Az$.
We know that

$$z = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Ax_{2n+1}$$

and

$$z = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Ax_{2n+2}.$$

Also,

$$\begin{aligned} d(Ty_{2n+1}, Az) &\le d(Ty_{2n+1}, Ay_{2n+2}) + d(Ay_{2n+2}, Az) \\ &= d(TAx_{2n+2}, ATx_{2n+2}) + d(Ay_{2n+2}, Az) \\ &\le d(Tx_{2n+2}, Ax_{2n+2}) + d(Ay_{2n+2}, Az) \\ &= d(y_{2n+2}, y_{2n+1}) + d(Ay_{2n+2}, Az). \end{aligned}$$

Therefore, $\lim_{n\to\infty} d(Ty_{2n+1}, Az) = 0$ and we can have

$$d(Ty_{2n+1}, Sz) \le M(y_{2n+1}, z) - \varphi(M(y_{2n+1}, z)) \tag{3.12}$$

where,

$$\begin{aligned} M(y_{2n+1}, z) = \max\{&d(Ty_{2n+1}, Sz), d(Ay_{2n+1}, Ty_{2n+1}), d(Az, Sz), \\ &\frac{1}{2}(d(Ay_{2n+1}, Sz) + d(Az, Ty_{2n+1}))\}. \end{aligned}$$

Now we consider the following cases:

If $M(y_{2n+1}, z) = d(Ty_{2n+1}, Sz)$, then taking limit as $n \to \infty$ in 3.12, we get

$$d(Az, Sz) \le d(Az, Sz) - \varphi(d(Az, Sz)),$$

which is a contradiction.

If $M(y_{2n+1}, z) = d(Ay_{2n+1}, Ty_{2n+1})$, then taking limit as $n \to \infty$ in 3.12, we get

$$d(Az, Sz) \le d(Az, Az) - \varphi(d(Az, Az)) = 0 - \varphi(0) = 0.$$

If $M(y_{2n+1}, z) = d(Az, Sz)$, then taking limit as $n \to \infty$ in 3.12, we get

$$d(Az, Sz) \le d(Az, Sz) - \varphi(d(Az, Sz)),$$

gives a contradiction.

Finally, if $M(y_{2n+1}, z) = \frac{1}{2}(d(Ay_{2n+1}, Sz) + d(Az, Ty_{2n+1}))$, then taking limit as $n \to \infty$ in 3.12, we get

$$\begin{aligned} & d(Az, Sz) \le \frac{1}{2}(d(Az, Sz) + d(Az, Az) - \varphi(\frac{1}{2}(d(Az, Sz) + d(Az, Az))) \\ \Rightarrow\ & d(Az, Sz) \le \frac{1}{2}(d(Az, Sz)) \\ \Rightarrow\ & d(Az, Sz) \le 0. \end{aligned}$$

Hence we must have $d(Az, sz)) = 0$. Thus $Az = Sz$. Similarly, we can prove that $Tz = Az$.

By our assumption $A(E)$ is closed, so $z \in A(E)$. Also, $z \in E$.

If $Az = z$, then we have $Tz = Sz = Az = z$.

Suppose $Tz = Sz = Az = t$ for some $t \in E$. Using weakly compatibility of the pair (T, A) and (S, A), we have $At = Tt = St$.

Now,

$$d(Tt, t) = d(Tt, Sz) \le M(t, z) - \varphi(M(t, z)) \tag{3.13}$$

where,

$$\begin{aligned} M(t,z) &= \max\{d(Tt,Sz), d(At,Tt), d(Az,Sz), \frac{1}{2}(d(At,Sz)+d(Az,Tt))\} \\ &= \max\{d(Tt,t), d(At,At), d(t,t), \frac{1}{2}(d(Tt,t)+d(t,Tt))\} \\ &= d(Tt,t). \end{aligned}$$

Therefore from 3.13 we have

$$d(Tt,t) \le d(Tt,t) - \varphi(d(Tt,t)).$$

Hence $\varphi(d(Tt,t)) = 0$. So, $d(Tt,t) = 0$. That is $Tt = t$. Thus, $At = Tt = St = t$. Now

$$d(z,t) = d(Tz,St) \le M(z,t) - \varphi(M(z,t)) \tag{3.14}$$

where,

$$\begin{aligned} M(z,t) &= \max\{d(Tz,St), d(Az,Tz), d(At,St), \frac{1}{2}(d(Az,St)+d(At,Tz))\} \\ &= \max\{d(z,t), 0, 0, \frac{1}{2}(d(z,t)+d(t,z))\} \\ &= d(z,t). \end{aligned}$$

Therefore from 3.14 we get $d(z,t) = 0$. i. e., $z = t$. Hence A, T and S have a unique common fixed point. □

Theorem 3.6. *Let (X,d) be a complete metric space and E be a nonempty closed subset of X. Let $T,S : E \to E$ be self maps. If there exists some $\alpha \in A_\varphi$ such that for all $x,y \in X$,*

$$d(Tx,Sy) \le \alpha(d(Ax,Ay), d(Ax,Tx), d(Ay,Sy)), \tag{3.15}$$

where $A : E \to X$ satisfying the following assertions:

(i) $T(E) \subseteq A(E)$ and $S(E) \subseteq A(E)$;
(ii) the pairs (T,A) and (S,A) are weakly compatible.

Also, assume that $A(E)$ is a closed subset of X. Then T, A and S have a unique common fixed point.

Proof. Let $x_0 \in E$. Define two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ such that

$$y_{2n} = Tx_{2n} = Ax_{2n-1} \text{ and } y_{2n+1} = Sx_{2n+1} = Ax_{2n},$$

for all $n \ge 0$. Now, from equation 3.15 we have

$$\begin{aligned} d(y_{2n+2}, y_{2n+1}) &= d(Tx_{2n+2}, Sx_{2n+1}) \\ &\le \alpha(d(Ax_{2n+2}, Ax_{2n+1}), d(Ax_{2n+2}, Tx_{2n+2}), d(Ax_{2n+1}, Sx_{2n+1})) \\ &= \alpha(d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1})). \end{aligned}$$

By the definition of α,

$$d(y_{2n+2}, y_{2n+1}) \le \varphi(d(y_{2n+1}, y_{2n})).$$

Similarly,

$$\begin{aligned} d(y_{2n+2}, y_{2n+3}) &= d(Tx_{2n+2}, Sx_{2n+3}) \\ &\leq \alpha(d(Ax_{2n+2}, Ax_{2n+3}), d(Ax_{2n+2}, Tx_{2n+2}), d(Ax_{2n+3}, Sx_{2n+3})) \\ &= \alpha(d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3})). \end{aligned}$$

By the definition of α,

$$d(y_{2n+2}, y_{2n+3}) \leq \varphi(d(y_{2n+1}, y_{2n+2})).$$

Continuing this way, we get

$$\begin{aligned} d(y_{2n+2}, y_{2n+3}) &\leq \varphi(d(y_{2n+1}, y_{2n+2})) \\ &\leq \varphi(\varphi(d(y_{2n}, y_{2n+1})) \\ &= \varphi^2(d(y_{2n}, y_{2n+1})) \\ &\dots \\ &\leq \varphi^{2n+2}(d(y_0, y_1)). \end{aligned}$$

Thus,

$$d(y_n, y_{n+1}) \leq \varphi^n(d(y_0, y_1))$$

for all $n \in \mathbb{N}$.
Since $d(y_0, y_1) \geq 0$. So, from the definition 2.5(ii), we have $\lim_{n\to\infty} \varphi^n(d(y_0, y_1)) = 0$. Now, for a given $\varepsilon > 0$, there is a positive integer n_0 such that for all $n \geq n_0$,

$$\varphi^n(d(y_0, y_1) < \varepsilon - \varphi(\varepsilon).$$

Hence

$$d(y_n, y_{n+1}) < \varepsilon - \varphi(\varepsilon). \tag{3.16}$$

Now, for any $m, n \in \mathbb{N}$ with $m > n \geq n_0$, we claim that

$$d(y_n, y_m) < \varepsilon. \tag{3.17}$$

We prove the claim by induction on m. The inequality holds for $m = n + 1$ by using equation 3.16. Assume that inequality 3.17 holds for $m = k$. i.e., $d(y_n, y_k) < \varepsilon$. Now if $m = k + 1$, we have

$$\begin{aligned} d(y_n, y_{k+1}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{k+1}) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(d(y_n, y_k)) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) \\ &= \varepsilon. \end{aligned}$$

By induction on m, we conclude that the inequality 3.17 holds for $m > n \geq n_0$. Thus $\{y_n\}$ is a Cauchy sequence. Since (X, d) is complete and $\{y_n\}$ is Cauchy in X, so there is a $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Also, E is closed and $\{y_n\} \subseteq E$, we have $z \in E$. By assumption $A(E)$ is closed, so there exist $u \in E$ such that $Au = z$.

Now

$$\begin{aligned} d(Tu, y_{n+1}) &= d(Tu, Sx_{2n+1}) \\ &\leq \alpha(d(Au, Ax_{2n+1}, d(Au, Tu), d(Ax_{2n+1}, Sx_{2n+1})) \\ &= \alpha(d(z, y_{2n+2}, d(z, Tu), d(y_{2n+2}, y_{2n+1})). \end{aligned}$$

If $n \to \infty$

$$d(Tu, z) \leq \alpha(0, d(z, Tu), 0).$$

Hence $d(Tu, z) \leq \varphi(0) = 0$. Thus $Tu = Sz$. Similarly, we can show that $Su = z$. Therefore $Tu = Su = Au = z$. Since the pair (A, T) and (A, S) are weakly compatible, we have $Tz = Sz = Az$.
Now, we can have

$$\begin{aligned} d(Tz, y_{2n+1}) &= d(Tz, Sx_{2n+1}) \\ &\leq \alpha(d(Az, Ax_{2n+1}, d(Az, Tz), d(Ax_{2n+1}, Sx_{2n+1})) \\ &= \alpha(d(Tz, y_{2n+2}, d(Tz, Tz), d(y_{2n+2}, y_{2n+1})). \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$d(Tz, z) \leq \alpha(d(Tz, z), 0, 0).$$

Thus $d(Tz, z) \leq \varphi(0) = 0$, which gives $Tz = z$. From $Tz = Sz = Az$, we conclude that $Tz = Sz = Az = z$. So, z is a common fixed point of T, S and A.
If there exist another fixed point $v \in E$ such that $v = Tv = Sv = Av$, then we get

$$\begin{aligned} d(z, v) &= d(Tz, Sv) \\ &= \alpha(d(z, v), d(Az, Tz), d(Av, Sv)) \\ &= \alpha(d(z, v), 0, 0). \end{aligned}$$

Thus $d(z, v) \leq \varphi(0) = 0$. i.e., $z = v$. Hence T, S and A have a unique common fixed point. ☐

Corollary 3.7. *Suppose (X, d) be a complete metric space and E be a nonempty closed subset of X. Let $T, S : E \to E$ be self maps. If there exist some $\alpha \in A$ such that for all $x, y \in X$*

$$d(Tx, Sy) \leq \alpha(d(Bx, By), d(Bx, Tx), d(By, Sy)) \tag{3.18}$$

where $B : E \to X$ satisfies the following assertions:

(i) $T(E) \subseteq B(E)$ and $S(E) \subseteq B(E)$;
(ii) the pair (T, B) and (S, B) are weakly compatible.

Also, assume that $B(E)$ is a closed subset of X. Then T, B and S have a unique common fixed point.

Conclusion. In this paper, using the notion of weak compatibility, we have extended some common fixed point theorems for generalized φ-weak contractions and A_φ contractions defined on a complete metric space. The results discussed in this paper are mainly concerned with the existence and uniqueness of common fixed point. Study of coincidence points and coupled coincidence points for these maps would also be interesting topics for future study.

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