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FIXED POINT RESULTS IN B-METRIC SPACE

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Abstract. We give another proof of Czerwik's fixed point theorem in the setting of b-metric spaces, improving a recent version of this theorem in b-metric spaces obtained in [M. Jovanović, Z. Kadelburg, S. Radenović, Common fixed point results in metric-type spaces, Fixed Point Theory Appl., (2010)]. An analogue of Reich contraction principle and Kannan's fixed point theorem is proved in this space. Our results generalize many known results in fixed point theory.

Key Words and Phrases: Fixed points, b-metric space.
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1. Introduction and preliminaries

In the papers of Bakhtin [2] and Czerwik [7, 8], the notion of b-metric space has been introduced and some fixed point theorems for single-valued and multi-valued mappings in b-metric spaces proved. Successively, this notion has been reintroduced by Khamsi [13] and Khamsi and Hussain [14], with the name of metric-type space. Several results have appeared in metric-type spaces, we refer to [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 18].

Definition 1.1. Let X be a nonempty set and let $b \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

A triplet (X, d, b), is called a b-metric space.

Note that a metric space is included in the class of b-metric spaces. In fact, the notions of convergent sequence, Cauchy sequence and complete space are defined as in metric spaces. Some examples of b-metric spaces can be seen in [1, 3, 4, 7, 8].

In the paper [10] (Theorem 3.3) M. Jovanović, Z. Kadelburg and S. Radenović obtain the following theorem (analogue of Banach contraction principle in b-metric space).

Theorem 1.1. *Let* (X, d, b) *be a complete b-metric space and* $T : X \to X$ *be a mapping satisfying:*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{1.1}$$

for all $x, y \in X$*, where* $0 < \lambda < \frac{1}{b}$*. Then* T *has a unique fixed point* x^**, and for every* $x_0 \in X$*, the sequence* $\{T^n x_0\}$ *converges to* x^**.*

In this paper, we prove that in Theorem 1.1 the condition $0 < \lambda < \frac{1}{b}$ can be relaxed to the following one $0 < \lambda < 1$. Also, we obtain versions of the Kannan and Reich contraction principle in b-metric space and improve some results from the literature.

2. Results

The next theorem is known, see for example Theorem 12.2 in Kirk and Shahzad [16]. We give another proof here.

Theorem 2.1. *Let* (X, d, b) *be a complete b-metric space and* $T : X \to X$ *be a mapping satisfying:*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{2.1}$$

for all $x, y \in X$*, where* $0 < \lambda < 1$*. Then* T *has a unique fixed point* x^**, and for every* $x_0 \in X$*, the sequence* $\{T^n x_0\}$ *converges to* x^**.*

Proof. Let $\lambda \in (0, 1)$. Since $\lim\limits_{n\to\infty} \lambda^n = 0$, there exists a natural number n_0 such that

$$0 < \lambda^k \cdot b^2 < 1, \tag{2.2}$$

for all $k \geq n_0$.

Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. Then (2.1) implies that

$$d(x_{n+n_0}, x_n) \leq \lambda^n d(x_{n_0}, x_0), \tag{2.3}$$

and

$$d(x_{m+n_0}, x_{n+n_0}) \leq \lambda^{n_0} d(x_m, x_n), \tag{2.4}$$

Applying the triangle-type inequality (3) to triples (x_m, x_{m+n_0}, x_n) we have

$$d(x_m, x_n) \leq b\left(d(x_m, x_{m+n_0}) + d(x_{m+n_0}, x_n)\right), \tag{2.5}$$

and applying the triangle-type inequality (3) to triples $(x_{m+n_0}, x_{n+n_0}, x_n)$ we obtain

$$d(x_m, x_n) \leq b\left[d(x_m, x_{m+n_0}) + b\left(d(x_{m+n_0}, x_{n+n_0}) + d(x_{n+n_0}, x_n)\right)\right]. \tag{2.6}$$

From (2.6), together with (2.3) and (2.4), we obtain

$$d(x_m, x_n) \leq b\left[\lambda^m d(x_0, x_{n_0}) + b\left(\lambda^{n_0} d(x_m, x_n) + \lambda^n d(x_{n_0}, x_0)\right)\right]. \tag{2.7}$$

So,

$$(1 - \lambda^{n_0} b^2) d(x_m, x_n) \leq b d(x_0, x_{n_0})\left[\lambda^m + b\lambda^n\right].$$

Now (2.2) implies that

$$d(x_m, x_n) \leq \frac{b d(x_0, x_{n_0})\left[\lambda^m + b\lambda^n\right]}{1 - \lambda^{n_0} b^2}. \tag{2.8}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X,d,b) there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{2.9}$$

Now we obtain that x^* is the unique fixed point of T. Namely, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(x^*,Tx^*) &\le b[d(x^*,x_{n+1}) + d(x_{n+1},Tx^*)] \\ &= b[d(x^*,x_{n+1}) + d(Tx_n,Tx^*)] \\ &\le b[d(x^*,x_{n+1}) + \lambda d(x_n,x^*)]. \end{aligned}$$

Since, $\lim_{n\to\infty} d(x^*,x_n) = 0$ and $\lim_{n\to\infty} d(x^*,x_{n+1}) = 0$, we have $d(x^*,Tx^*) = 0$ i. e., $Tx^* = x^*$.

For uniqueness, let y^* be another fixed point of T. Then it follows from (2.1) that $d(x^*,y^*) = d(Tx^*,Ty^*) \le \lambda d(x^*,y^*) < d(x^*,y^*)$, is a contradiction. Therefore, we must have $d(x^*,y^*) = 0$, i.e., $x^* = y^*$. □

Example 2.1. Let $X = \mathbb{R}$, $d(x,y) = (x-y)^2$ for all $x,y \in X$ and $T : X \to X$ be defined by $Tx = \sqrt{\frac{2}{3}}x$. Then $(X,d,2)$ is a complete b-metric space. Theorem 2.1 is applicable taking $\lambda = \frac{2}{3}$. On the other hand, Theorem 1.1 is not applicable since condition $0 < \lambda < \frac{1}{b}$ implies $\lambda < \frac{1}{2}$.

In the b-metric space X, let $B[x,r]$ denote the closed ball with centre x and radius r.

Corollary 2.1. *Let (X,d,b) be a complete b-metric space, $x_0 \in X$ and $T : B[x_0,r] \to X$ be a mapping satisfying:*

$$d(Tx,Ty) \le \lambda d(x,y) \tag{2.10}$$

for all $x,y \in B[x_0,r]$, where $0 < \lambda < \frac{1}{b}$ and $r = \frac{b}{1-\lambda b}d(x_0,Tx_0)$. Then T has a unique fixed point $x^* \in B[x_0,r]$ and the sequence $T^n x_0$ converges to x^*.

Proof. We show that $T : B[x_0,r] \to B[x_0,r]$. Let $x \in B[x_0,r]$. Then

$$\begin{aligned} d(Tx,x_0) &\le b[d(Tx,Tx_0) + d(Tx_0,x_0)] \\ &\le b[\lambda d(x,x_0) + d(Tx_0,x_0)] \\ &\le b\left[\frac{\lambda b}{1-\lambda b}d(x_0,Tx_0) + d(Tx_0,x_0)\right] \\ &= \frac{b}{1-\lambda b}d(x_0,Tx_0) = r. \end{aligned}$$

By Theorem 2.1, the result follows. □

Corollary 2.2. *If (X,d,b) is a complete b-metric space and $T : X \to X$ is a contraction mapping, then T has a unique fixed point x^*, and for any $x_0 \in X$ the sequence $T^n x_0$ converges to x^*. In fact,*

$$d(x^*,T^n x_0) \le \frac{b^3\lambda^n}{1-\lambda^p b^2} d(x_0,T^p x_0), \tag{2.11}$$

where $p \in \mathbb{N}$ such that $0 < \lambda^p \cdot b^2 < 1$.

Proof. We have

$$d(x^*, T^n x_0) \leq b[d(x^*, x_m) + d(x_m, T^n x_0)],$$

now from inequality (2.8) we obtain

$$d(x^*, T^n x_0) \leq b\left[d(x^*, x_m) + \frac{bd(x_0, T^p x_0)\left[\lambda^m + b\lambda^n\right]}{1 - \lambda^p b^2}\right] \tag{2.12}$$

Let m tend to infinity in the inequality (2.12), then from Theorem 2.1 we obtain the proof. □

Remark 2.1. Obviously Theorem 2.1 is a generalization of the results from [10] (Theorem 3.3) and [15] (Theorem 1).

The following result is a version of Kannan's fixed point theorem (see [12]) in b-metric spaces.

Theorem 2.2. *Let (X, d, b) be a complete b-metric space and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)] \tag{2.13}$$

for all $x, y \in X$, where $0 < \lambda < \frac{1}{2}$. Then T has a unique fixed point x^* and for any $x_0 \in X$ the sequence $T^n x_0$ converges to x^* if any of the following conditions are satisfied

(i) *T is continuous at a point $x^* \in X$,*
(ii) *$\lambda b < 1$,*
(iii) *$b < 2$.*

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. Then (2.13) implies that

$$d(x_{n+1}, x_n) \leq \lambda[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]. \tag{2.14}$$

From (2.14) we have

$$d(x_{n+1}, x_n) \leq \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n). \tag{2.15}$$

Now by the induction we show that for each $n \geq 1$

$$d(x_{n+1}, x_n) \leq \left(\frac{\lambda}{1-\lambda}\right)^n d(x_1, x_0). \tag{2.16}$$

Applying the condition (2.13) to a pair (x_m, x_n), we obtain

$$d(x_m, x_n) \leq \lambda\left[d(x_{m-1}, x_m) + d(x_{n-1}, x_n)\right], \tag{2.17}$$

From (2.17), together with (2.16), we obtain

$$d(x_m, x_n) \leq \lambda d(x_1, x_0)\left[\left(\frac{\lambda}{1-\lambda}\right)^{m-1} + \left(\frac{\lambda}{1-\lambda}\right)^{n-1}\right]. \tag{2.18}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d, b) there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{2.19}$$

Now we obtain that x^* is a unique fixed point of T. Namely, for any $n \in \mathbb{N}$ we have

$$d(x^*, Tx^*) \leq b[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)]. \tag{2.20}$$

(i) Suppose that T is a continuous map at a point $x^* \in X$.
Since $\lim\limits_{n\to\infty} d(x^*, x_{n+1}) = 0$ and T is a continuous at a point x^* we have

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = d(Tx^*, Tx^*) = 0$$

and from (2.20) we obtain $d(x^*, Tx^*) = 0$ i. e., $Tx^* = x^*$.
(ii) Let $\lambda b < 1$,

$$\begin{aligned} d(x^*, Tx^*) &\leq b[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] \\ &= b[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)] \\ &\leq b[d(x^*, x_{n+1}) + \lambda(d(x_n, Tx_n) + d(x^*, Tx^*))] \\ &= bd(x^*, x_{n+1}) + b\lambda d(x_n, x_{n+1}) + b\lambda d(x^*, Tx^*). \end{aligned}$$

So, we have

$$(1 - b\lambda)d(x^*, Tx^*) \leq bd(x^*, x_{n+1}) + b\lambda d(x_n, x_{n+1}). \tag{2.21}$$

Since $\lim\limits_{n\to\infty} d(x^*, x_{n+1}) = 0$, $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$ and $0 < b\lambda < 1$ we have $Tx^* = x^*$.
(iii) Clearly (iii) implies (ii).

For uniqueness, let y^* be another fixed point of T. Then it follows from (2.13) that

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \lambda[d(x^*, Tx^*) + d(y^*, Ty^*)] = 0.$$

Therefore, we must have $d(x^*, y^*) = 0$, i.e., $x^* = y^*$.

□

Reich [19], for mappings $T : X \to X$, generalized the fixed point theorems by Banach and Kannan, using contractive condition:

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty), \tag{2.22}$$

for all $x, y \in X$, where α, γ are nonnegative constants with $\alpha+\beta+\gamma < 1$. An example in [19] shows that the condition (2.22) is a proper generalization of (2.10) and (2.13).

The next is a fixed point theorem of Reich type in b-metric spaces.

Theorem 2.3. *Let (X, d, b) be a complete b-metric space and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \tag{2.23}$$

for all $x, y \in X$, where α, β, γ are nonnegative constants with $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point x^* and for any $x_0 \in X$ the sequence $T^n x_0$ converges to x^* if satisfies any one of the following conditions

(i) *T is continuous at a point $x^* \in X$,*
(ii) *$b\beta < 1$,*
(iii) *$b\gamma < 1$.*

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. From condition (2.23) we have that

$$d(x_{n+1}, x_n) \leq \alpha d(x_n, x_{n-1}) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n). \tag{2.24}$$

Therefore,

$$d(x_{n+1}, x_n) \leq \frac{\alpha + \gamma}{1 - \beta} d(x_n, x_{n-1}). \tag{2.25}$$

Put $\lambda = \frac{\alpha+\gamma}{1-\beta}$. We have that $\lambda \in [0, 1)$. It follows from (2.25) that

$$d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0) \text{ for all } n \geq 1. \tag{2.26}$$

From condition (2.23) we have

$$d(x_{n+p}, x_n) \leq \alpha d(x_{n+p-1}, x_{n-1}) + \beta d(x_{n+p-1}, x_{n+p}) + \gamma d(x_{n-1}, x_n). \tag{2.27}$$

From (2.26) and (2.27) we obtain

$$d(x_{n+p}, x_n) \leq \alpha d(x_{n+p-1}, x_{n-1}) + d(x_0, x_1) \left[\beta \lambda^{n+p-1} + \gamma \lambda^{n-1}\right]. \tag{2.28}$$

Let $r = \max\{\alpha, \lambda\}$. Then from (2.28) follows

$$d(x_{n+p}, x_n) \leq r d(x_{n+p-1}, x_{n-1}) + r^{n-1} d(x_0, x_1) \left[\beta r^p + \gamma\right]. \tag{2.29}$$

Applying the method of mathematical induction on inequality (2.29) we obtain

$$d(x_{n+p+k}, x_{n+k}) \leq r^k d(x_{n+p}, x_n) + k r^{n-1} d(x_0, x_1) \left[\beta r^p + \gamma\right]. \tag{2.30}$$

Let n_0 be a natural number such that

$$0 < r^{n_0} \cdot b^2 < 1. \tag{2.31}$$

Applying the triangle-type inequality (3) we obtain

$$d(x_{n+p}, x_n) \leq b\left[d(x_{n+p}, x_{n+p+n_0}) + b\left(d(x_{n+p+n_0}, x_{n+n_0}) + d(x_{n+n_0}, x_n)\right)\right]. \tag{2.32}$$

So, together with (2.25) and (2.30) we obtain

$$d(x_{n+p}, x_n) \leq b r^n (r^p + b) d(x_0, x_{n_0}) + b^2 \left[r^{n_0} d(x_{n+p}, x_n) + n_0 r^{n-1} d(x_0, x_1) \left(\beta r^p + \gamma\right)\right]$$

Now (2.31) implies that

$$d(x_{n+p}, x_n) \leq \frac{b r^n (r^p + b) d(x_0, x_{n_0}) + b^2 n_0 r^{n-1} (\beta r^p + \gamma) d(x_0, x_1)}{1 - b^2 r^{n_0}} \tag{2.33}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d, b) there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{2.34}$$

Now we obtain that x^* is the unique fixed point of T.
For any $n \in \mathbb{N}$ we have

$$d(x^*, Tx^*) \leq b[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)]. \tag{2.35}$$

(i) Suppose that T is a continuous map at a point $x^* \in X$.
Since, $\lim_{n\to\infty} d(x^*, x_{n+1}) = 0$ and T is a continuous at a point x^* we have

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = d(Tx^*, Tx^*) = 0$$

and from (2.35) we obtain $d(x^*, Tx^*) = 0$ i. e., $Tx^* = x^*$.
(ii) Suppose that $\beta b < 1$.
We have

$$\begin{aligned} d(Tx^*, x^*) &\leq b[d(Tx^*, x_{n+1}) + d(x_{n+1}, x^*)] \\ &= b[d(Tx^*, Tx_n) + d(x_{n+1}, x^*)] \\ &\leq b[\alpha d(x^*, x_n) + \beta d(x^*, Tx^*) + \gamma d(x_n, x_{n+1}) + d(x_{n+1}, x^*)]. \end{aligned}$$

Since $b\beta < 1$, we have $Tx^* = x^*$.
(iii) Suppose that $b\gamma < 1$.
Then we have,

$$\begin{aligned} d(x^*, Tx^*) &\leq b[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] \\ &= b[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)] \\ &\leq b[d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta d(x_n, x_{n+1}) + \gamma d(x^*, Tx^*)]. \end{aligned}$$

Since $b\gamma < 1$, we obtain $Tx^* = x^*$.

For uniqueness, let y^* be another fixed point of T. Then it follows from (2.23) that

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \leq \alpha d(x^*, y^*) + \beta d(x^*, Tx^*) + \gamma d(y^*, Ty^*) \\ &= \alpha d(x^*, y^*) < d(x^*, y^*) \end{aligned}$$

is a contradiction. Therefore, we must have $d(x^*, y^*) = 0$, i.e., $x^* = y^*$. □

Example 2.2. Let $X = [0, 2]$, $d(x, y) = (x - y)^2$ for all $x, y \in X$ and $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{x}{2} & x \in [0, 1], \\ \frac{x}{3} & x \in (1, 2]. \end{cases}$$

Then

(1) $(X, d, 2)$ is a complete b-metric space.
(2) There exist $\alpha, \beta, \gamma \geq 0, (\alpha = \beta = \gamma = \frac{2}{7})$ such that T satisfies the contraction condition (2.23) in Theorem 2.3 and 0 is unique fixed point of T.

Remark 2.2. Theorem 2.3 is an improvement of the results from [17] (Theorem 3.2) and [18] (Theorem 4).

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