*Fixed Point Theory*, 20(2019), No. 2, 541-558
DOI: 10.24193/fpt-ro.2019.2.35
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

# FIXED POINT THEOREMS INVOLVING 1-SET-WEAKLY CONTRACTIVE OPERATORS IN WC-BANACH ALGEBRAS

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**Abstract.** In this paper, we establish some fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators acting on a WC-Banach algebra and involving 1-set-weakly contractive maps. The obtained results are applied to a nonlinear functional integral equation in a suitable Banach algebra. Our results are extensions of several earlier results.
**Key Words and Phrases**: WC-Banach algebra, fixed point theorems, weak topology, measure of weak noncompactness, 1-set-weakly contractive.
**2010 Mathematics Subject Classification**: 47A10, 47A55, 47A53, 47H10.

## 1. Introduction

Many problems arising in diverse areas of natural science consider the existence of solutions of the nonlinear equation

$$Ax + Bx = x, \quad x \in M,$$

where $M$ is a nonempty, closed, and convex subset of a Banach space $X$, and where $A$, $B : M \to X$ are nonlinear mappings. M.A. Krasnosel'skii noted that the inversion of a perturbed differential operator could lead to the sum of a contraction and a compact operator and in 1958 he established a fixed point theorem (called the Krasnosel'skii fixed point theorem). The proof involves combining the Banach contraction mapping principle and the Schauder fixed point theorem and its statement reads: if $S$ is a nonempty, closed, and convex subset of a Banach space $X$, and if $A$ and $B$ are two mappings from $S$ into $X$ such that $(i)$ $A$ is compact, $(ii)$ $B$ is a contraction, and *(iii)* $AS + BS \subset S$, then $A + B$ has, at least, one fixed point in $S$. In 1977, R.W. Legget [25] considered existence for the particular equation

$$x = x_0 + x \cdot Bx, \quad (x_0, x) \in X \times M, \tag{1.1}$$

where $M$ is a nonempty, bounded, closed, and convex subset of a Banach algebra $X$ and where $B : M \to X$ is a compact operator.
A natural generalization of (1.1) is

$$x = Ax \cdot Bx + Cx, \quad x \in M, \tag{1.2}$$

where $M$ is a nonempty, bounded, closed, and convex subset of a Banach algebra $X$ and $A, B, C : M \to M$ are operators. Now (1.2) was investigated by J. Banas in [4, 5], J. Caballero, B. Lopez, and K. Sadarangani in [12], and B.C. Dhage in [14, 15, 16] where the authors put conditions on the operators to guarantee the compactness of the operator $\left(\frac{I-C}{A}\right)^{-1} B$ and then they used the Schauder fixed point theorem. One result of B.C. Dhage is the following theorem:

**Theorem 1.1.** *Let $M$ be a nonempty, bounded, closed, and convex subset of a Banach algebra $X$ and let $A$, $B$, $C : M \to X$ be three operators, satisfying the following conditions:*

*(i) $A$ and $C$ are Lipschitzian with Lipschitz constants $\alpha$ and $\beta$ respectively,*

*(ii) $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(M)$, $I$ being the identity operator on $X$, and the operator $\left(\frac{I-C}{A}\right) : X \to X$ is defined by $\left(\frac{I-C}{A}\right) x := \frac{x-Cx}{Ax}$,*

*(iii) $B$ is completely continuous, and*

*(iv) $Ax \cdot By + Cx \in M, \ \forall \ x, y \in M$.*

*Then, the operator equation $x = Ax \cdot Bx + Cx$ has, at least, one solution in $M$, whenever $\alpha Q + \beta < 1$, where $Q = \|B(M)\| = \sup\{\|Bx\| : \ x \in M\}$.*

Attempts in the literature were made to prove an analogue of Theorem 1.1 in the weak topology setting. In [7], A. Ben Amar, S. Chouayekh and A. Jeribi established a weak variant of Theorem 1.1 in Banach algebras, satisfying the following sequential condition:

$$(\mathcal{P}) \qquad \begin{cases} \text{for any sequences } (x_n)_{n\in\mathbb{N}} \text{ and } (y_n)_{n\in\mathbb{N}} \text{ of } X \text{ such that } x_n \rightharpoonup x \\ \text{and } y_n \rightharpoonup y, \text{ then } x_n \cdot y_n \rightharpoonup x \cdot y \text{ (here } X \text{ is a Banach algebra).} \end{cases}$$

Their result requires both the weak sequential continuity and the weak compactness of the operators $A$, $B$ and $C$ and their proof is based on the Arino, Gautier and Penot fixed point theorem [3] and the weak sequential continuity of $\left(\frac{I-C}{A}\right)^{-1} B$. In [8], these authors established some other fixed point theorems in Banach algebras satisfying condition $(\mathcal{P})$ in the weak topology setting and their results used weakly condensing operators; see also a recent paper of M. Benjemaa, B. Krichen and M. Meslameni [10]. Also J. Banas and M.A. Taoudi [6] gave a generalization of some results established in [7] in the weak topology setting where the authors used the notion of a $WC$-Banach algebra (cf. Definition 3.1), which is weaker than property $(\mathcal{P})$ and the concept of the

De Blasi measure of weak noncompactness [13] (there are some inaccuracies in the proofs of Theorems 3.1 and 3.3 there, where the authors used invalid arguments using condition $(\mathcal{P})$ in a Banach algebra). A. Jeribi, B. Krichen and B. Mefteh corrected and improved in [23] some results of [6] and extended several earlier works using condition $(\mathcal{P})$. In particular they presented some fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators acting on a WC-Banach algebra and their assumptions were based on $\alpha$-set-weakly contractive operators, $\alpha \in (0, 1)$ and mappings satisfying assumptions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ (see Section 2). The study of weakly condensing operators and 1-set-weakly contractive maps is important in nonlinear functional analysis (see, for example [9, 25]). As a result it is of interest to continue the analysis initiated in [23], and we will consider 1-set-weakly contractive and establish new variants of Theorem 1.1 for three operators acting on WC-Banach algebras. Our result can be considered as extensions of those in [8, 18, 22].

## 2. Preliminaries

Throughout this section, $X$ denotes a Banach space. For any $r > 0$, $B_r$ denotes the closed ball in $X$ centered at $0_X$ with radius $r$ and $\mathcal{D}(A)$ denotes the domain of an operator $A$. Also $\Omega_X$ is the collection of all nonempty bounded subsets of $X$ and $\mathcal{K}^w$ is the subfamily of $\Omega_X$ consisting of all weakly compact subsets of $X$. Now, $\rightharpoonup$ denotes the weak convergence and $\rightarrow$ denotes the strong convergence in $X$, respectively. The measure of weak noncompactness, introduced by De Blasi [13], is the map $\omega : \Omega_X \rightarrow [0, +\infty)$ defined by

$$\omega(M) = \inf\big\{r > 0 : \text{ there exits } K \in \mathcal{K}^w \text{ such that } M \subseteq K + B_r\big\},$$

for all $M \in \Omega_X$. For convenience we recall some basic properties of $\omega(\cdot)$ needed below [2, 13].

**Lemma 2.1.** *Let $M_1$, $M_2$ be two elements of $\Omega_X$. Then, the following conditions are satisfied:*

(1) $M_1 \subseteq M_2$ *implies* $\omega(M_1) \leq \omega(M_2)$.
(2) $\omega(M_1) = 0$ *if, and only if,* $\overline{M_1}^w \in \mathcal{K}^w$, *i.e.* $\overline{M_1}^w$ *is the weak closure of* $M_1$.
(3) $\omega(\overline{M_1}^w) = \omega(M_1)$.
(4) $\omega(M_1 \cup M_2) = \max\{\omega(M_1), \omega(M_2)\}$.
(5) $\omega(\lambda M_1) = |\lambda|\omega(M_1)$ *for all* $\lambda \in \mathbb{R}$.
(6) $\omega(co(M_1)) = \omega(M_1)$, *where* $co(M_1)$ *denotes the convex hull of* $M_1$.
(7) $\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$.
(8) *if* $(M_n)_{n\geq 1}$ *is a decreasing sequence of nonempty bounded and weakly closed subsets of* $X$ *with* $\lim_{n\to\infty} \omega(M_n) = 0$, *then* $M_\infty := \cap_{n=1}^{\infty} M_n$ *is nonempty and* $\omega(M_\infty) = 0$ *i.e.,* $M_\infty$ *is relatively weakly compact.*

**Definition 2.1.** An operator $A : \mathcal{D}(A) \subseteq X \rightarrow X$ is said to be weakly sequentially continuous on $\mathcal{D}(A)$ if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$, $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$.

**Definition 2.2.** An operator $A : \mathcal{D}(A) \subseteq X \to X$ is said to be $\alpha$-set-weakly contractive if it maps bounded sets into bounded sets, and there exists some $\alpha \in [0, 1)$ such that $\omega(A(S)) \leq \alpha\omega(S)$ for all bounded subsets $S \subseteq \mathcal{D}(A)$.
An operator $A : \mathcal{D}(A) \subseteq X \to X$ is said to be $\omega$-condensing if it maps bounded sets into bounded sets, and $\omega(A(S)) < \omega(S)$ for all bounded sets $S \subseteq \mathcal{D}(A)$ with $\omega(S) > 0$.

**Remark 2.1.** Obviously, every $\alpha$-set-weakly contractive with $0 \leq \alpha < 1$ is $\omega$-condensing. The converse is not true. Suppose that the Banach space $X$ is not reflexive and take $\varphi : [0, 1] \longrightarrow \mathbb{R}_+$ a strictly decreasing and continuous function with $\varphi(0) = 1$. Consider the operator $T$ (here $T : B_X \longrightarrow B_X$) defined by $T(x) = \varphi(\|x\|)x$, $x \in B_X$. Note $T$ is a $\omega$-condensing mapping but is not $\alpha$-set-weakly contractive for any $\alpha \in [0, 1)$.

Let $A : \mathcal{D}(A) \subseteq X \to X$ be an operator. Recall the following conditions:

$$(\mathcal{H}1) \qquad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A) \text{ is a weakly convergent sequence in } X, \text{ then} \\ (Ax_n)_{n\in\mathbb{N}} \text{ has a strongly convergent subsequence in } X. \end{cases}$$

$$(\mathcal{H}2) \qquad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A) \text{ is a weakly convergent sequence in } X, \text{ then} \\ (Ax_n)_{n\in\mathbb{N}} \text{ has a weakly convergent subsequence in } X. \end{cases}$$

Note conditions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ were considered in [1, 18, 19, 21] and for some properties on maps satisfying $(\mathcal{H}1)$ and $(\mathcal{H}2)$ we refer the reader to the monograph of B. Krichen and A. Jeribi [20].

**Definition 2.3.** An operator $A : \mathcal{D}(A) \subseteq X \to X$ is said to be nonexpansive if

$$\|Ax - Ay\| \leq \|x - y\|,$$

for all $x, y \in \mathcal{D}(A)$.

**Definition 2.4.** An operator $A : \mathcal{D}(A) \subseteq X \to X$ is called $\mathcal{D}$-Lipschitzian if there exists a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|Ax - Ay\| \leq \phi(\|x - y\|),$$

for all $x, y \in \mathcal{D}(A)$, where $\phi(0) = 0$. The function $\phi$ is called the $\mathcal{D}$-function of $A$. Obviously every Lipschitzian mapping is $\mathcal{D}$-Lipschitzian, but the converse may not be true. Moreover, if $\phi(r) < r$ for $r > 0$, then the operator $A$ is called a nonlinear contraction with a contraction function $\phi$.

The following lemma will be useful.

**Lemma 2.2.** [1] *Let $A : \mathcal{D}(A) \subseteq X \to X$ be a $\mathcal{D}$-Lipschitzian operator (with a $\mathcal{D}$-function $\phi$) on a Banach space $X$ satisfying $(\mathcal{H}2)$. Then, for each bounded subset $S$ of $\mathcal{D}(A)$ one has*

$$\omega(A(S)) \leq \phi(\omega(S)).$$

## 3. Fixed point theorems in WC-Banach algebras

First, let us recall the following definition used in [6].

**Definition 3.1.** Let $X$ be a Banach algebra. We say that $X$ is a $WC$-Banach algebra, if the product $W \cdot W'$ of arbitrary weakly compact subsets $W$, $W'$ of $X$ is weakly compact.

**Example 3.1.** Clearly, every finite dimensional Banach algebra is a $WC$-Banach algebra. If $X$ is a Banach algebra satisfying condition $(\mathcal{P})$ then $X$ is a $WC$-Banach algebra [6]. If $X$ is a $WC$-Banach algebra then the set $\mathcal{C}(\mathcal{K}, X)$ (here $\mathcal{K}$ is a compact Hausdorff space) of all continuous functions from $\mathcal{K}$ to $X$ is also a $WC$-Banach algebra. The proof is based on Dobrakov's Theorem [17].

**Theorem 3.1.** [17] *Let $K$ be a compact Hausdorff space and $X$ be a Banach space. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{C}(\mathcal{K}, X)$, and $f \in \mathcal{C}(\mathcal{K}, X)$. Then $(f_n)_{n\in\mathbb{N}}$ is weakly convergent to $f$ if and only if $(f_n(t))_{n\in\mathbb{N}}$is weakly convergent to $f(t)$ for each $t \in K$.*

To present the main fixed point results of this section we need the following theorems.

**Theorem 3.2.** [9] *Let $S$ be a nonempty, bounded, closed, and convex subset of a Banach space $X$ and let $A : S \to S$ be a weakly sequentially continuous mapping. If $A$ is $\omega$-condensing, then it has, at least, a fixed point in $S$.*

**Lemma 3.1.** [6] *Let $M$ and $M'$ be two bounded subsets of a $WC$-Banach algebra $X$. Then, we have the following inequality*

$$\omega(M \cdot M') \leq \|M'\|\omega(M) + \|M\|\omega(M') + \omega(M)\omega(M').$$

Our first result (motivated in part by [23]) involves 1-set-weakly contractive operators in WC-Banach spaces.

**Theorem 3.3.** *Let $S$ be a nonempty, bounded, closed, and convex subset of a $WC$-Banach algebra $X$ and let $A$, $C : X \to X$ and $B : S \to X$ be operators, satisfying the following conditions:*

*(i) $F_\lambda := \left(\frac{I-\lambda C}{A}\right)^{-1}$ exists on $B(S)$ for all $\lambda \in (0,1)$,*

*(ii) $A(S)$ and $B(S)$ are relatively weakly compact,*

*(iii) $C$ is a 1-set-weakly contractive,*

*(iv) $A \cdot B$ and $C$ are weakly sequentially continuous,*

*(v) if $\lambda \in (0,1)$ and $(x_n)_{n\in\mathbb{N}}$ is a sequence of $S$ such that $x_n \rightharpoonup x$, $x \in S$, and $F_\lambda(Bx_n) \rightharpoonup y$, then there exists a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ with $A(F_\lambda(Bx_{n_j})) \cdot Bx_{n_j} + \lambda C(F_\lambda(Bx_{n_j})) \rightharpoonup A(y) \cdot Bx + \lambda C(y)$, and*

*(vi) if $\lambda \in (0,1)$ and $(x = Ax \cdot By + \lambda Cx,\ y \in S)$, then $x \in S$.*

*Then, the operator equation $x = Ax \cdot Bx + Cx$ has, at least, one solution in $S$.*

*Proof.* Consider $\lambda \in (0,1)$. From assumption $(i)$, it follows that, for each $y \in S$, there is a unique $x_{\lambda,y} \in X$ such that

$$\left(\frac{I-\lambda C}{A}\right) x_{\lambda,y} = By,$$

or, equivalently

$$Ax_{\lambda,y} \cdot By + \lambda C x_{\lambda,y} = x_{\lambda,y}.$$

Since hypothesis $(vi)$ holds, then $x_{\lambda,y} \in S$. Hence, the operator

$$T_\lambda := F_\lambda B := \left(\frac{I-\lambda C}{A}\right)^{-1} B : S \to S$$

is well defined.

Now we show $T_\lambda$ is $\omega$-condensing. Let $M$ be a subset of $S$ with $\omega(M) > 0$. Note

$$T_\lambda = AT_\lambda \cdot B + \lambda C T_\lambda, \tag{3.1}$$

so

$$\omega(T_\lambda(M)) \leq \omega(A(T_\lambda(M)) \cdot B(M) + \lambda C(T_\lambda(M))).$$

The properties of $\omega$ in Lemma's 2.1 and 3.1, and assumptions $(ii)$ and $(iii)$ on $A$, $B$ and $C$ yield

$$\begin{aligned} \omega(T_\lambda(M)) &\leq \omega(A(T_\lambda(M)) \cdot B(M)) + \omega(\lambda C(T_\lambda(M))) \\ &\leq \lambda\omega(T_\lambda(M)) \\ &< \omega(T_\lambda(M)). \end{aligned}$$

Hence, $T_\lambda(M)$ is relatively weakly compact, and in particular, $T_\lambda$ is $\omega$-condensing.

Next we show $T_\lambda$ is weakly sequentially continuous. Consider $(x_n)_{n\in\mathbb{N}}$, a sequence in $S$ which is weakly convergent to $x$. Using an argument similar to that above, we see that $\{T_\lambda x_n : n \in \mathbb{N}\}$ is relatively weakly compact. Consequently, there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $T_\lambda x_{n_j} \rightharpoonup y$. From (3.1) and assumption $(v)$, we see that there exists a subsequence $(x_{n_{i_j}})_{j\in\mathbb{N}}$ of $(x_{n_i})_{i\in\mathbb{N}}$ such that

$$T_\lambda x_{n_{i_j}} = A(T_\lambda x_{n_{i_j}}) \cdot Bx_{n_{i_j}} + \lambda C(T_\lambda x_{n_{i_j}}) \rightharpoonup A(y) \cdot Bx + \lambda C(y).$$

Then $y = A(y) \cdot Bx + \lambda C(y)$, so $y = F_\lambda B(x) = T_\lambda x$. Consequently, $T_\lambda x_{n_{i_j}} \rightharpoonup T_\lambda x$. Now, we claim $T_\lambda x_n \rightharpoonup T_\lambda x$. Suppose the contrary. Then there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a weak neighborhood $V^w$ of $T_\lambda x$, such that $T_\lambda x_{n_i} \notin V^w$ for all $i \in \mathbb{N}$. Since $(x_{n_i})_{i\in\mathbb{N}}$ converges weakly to $x$ then, arguing as before, we may extract a subsequence $(x_{n_{i_{j_k}}})_{k\in\mathbb{N}}$ of $(x_{n_i})_{i\in\mathbb{N}}$, such that $T_\lambda x_{n_{i_{j_k}}} \rightharpoonup T_\lambda x$, which is a contradiction, since $T_\lambda x_{n_{i_{j_k}}} \notin V^w$, for all $k \in \mathbb{N}$. As a result, $T_\lambda$ is weakly sequentially continuous.

Consequently, $T_\lambda$ satisfies the hypothesis of Theorem 3.2. Thus, $T_\lambda$ has a fixed point in $S$, say $x_\lambda$. Now, choose a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(0,1)$ such that $\lambda_n \to 1$, and consider the corresponding sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $S$ satisfying

$$Ax_n \cdot Bx_n + \lambda_n C x_n = x_n, \ n \in \mathbb{N}. \tag{3.2}$$

Note $T_\lambda$ is $\omega$-condensing, and note (3.1), so we obtain that $\{x_n : n \in \mathbb{N}\}$ is relatively weakly compact. Consequently, the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges weakly to some $x \in S$. Using the fact that $A \cdot B$ and $C$ are weakly sequentially continuous, we deduce that $x = Ax \cdot Bx + Cx$. □

Note that the invertibility of $(\frac{I-\lambda C}{A})^{-1}$ is considered in the case where the operator $A$ is regular.

**Remark 3.1.** $(a)$ Let $S$ be a nonempty, bounded, closed, and convex subset of a Banach algebra $X$ satisfying condition $(\mathcal{P})$. If $A$, $C : X \to X$ and $B : S \to X$ are weakly sequentially continuous operators and satisfy assumptions $(i)$, $(ii)$, $(iii)$, and $(vi)$ of Theorem 3.3, then $(iv)$ and $(v)$ hold.
$(b)$ Let $S$ be a nonempty, bounded, closed, and convex subset of a WC-Banach algebra $X$. If $A$, $C : X \to X$ and $B : S \to X$ are weakly sequentially continuous operators satisfying assumptions $(i)$, $(ii)$, $(iii)$, and $(vi)$ of Theorem 3.3 and if $A$ satisfies $(\mathcal{H}1)$, then assumptions $(iv)$ and $(v)$ are verified.

*Proof.* $(a)$. Let $(x_n)_{n\in\mathbb{N}}$, be a sequence in $S$ which converges weakly to $x \in S$. From the weak sequential continuity of $A$ and $B$, and condition $(\mathcal{P})$, we have $Ax_n \cdot Bx_n \rightharpoonup Ax \cdot Bx$. Thus $(iv)$ holds. Now let $\lambda \in (0, 1)$ and $(x_n)_{n\in\mathbb{N}}$ a sequence of $S$ such that $x_n \rightharpoonup x$ and $F_\lambda(Bx_n) \rightharpoonup y$. Now (3.1), the weak sequential continuity of $A$, $B$, and $C$ and condition $(\mathcal{P})$, guarantees that $A(F_\lambda(Bx_n))\cdot Bx_n+\lambda C(F_\lambda(Bx_n)) \rightharpoonup Ay\cdot Bx+\lambda Cy$. Thus $(v)$ holds.

$(b)$. Let $(x_n)_{n\in\mathbb{N}}$, be a sequence in $S$ which converges weakly to $x \in S$. The weak sequential continuity of $A$, and the fact that $A$ satisfies $(\mathcal{H}1)$, guarantees that there exists a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, such that $Ax_{n_j} \to Ax$. Now $B$ is weakly sequentially continuous so $Bx_{n_j} \rightharpoonup Bx$ and $Ax_{n_j} \cdot Bx_{n_j} \rightharpoonup Ax \cdot Bx$. Now, we claim $Ax_n \cdot Bx_n \rightharpoonup Ax\cdot Bx$. Suppose the contrary. Then there exists a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a weak neighborhood $V^w$ of $Ax\cdot Bx$, such that $Ax_{n_j}\cdot Bx_{n_j} \notin V^w$ for all $j \in \mathbb{N}$. Since $(x_{n_j})_{j\in\mathbb{N}}$ converges weakly to $x$ then, arguing as before, we may extract a subsequence $(x_{n_{j_k}})_{k\in\mathbb{N}}$ of $(x_{n_j})_{j\in\mathbb{N}}$, such that $Ax_{n_{j_k}} \cdot Bx_{n_{j_k}} \rightharpoonup Ax \cdot Bx$, which is a contradiction. Thus $(iv)$ holds. Now let $\lambda \in (0, 1)$ and $(x_n)_{n\in\mathbb{N}}$ a sequence of $S$ such that $x_n \rightharpoonup x$ and $F_\lambda(Bx_n) \rightharpoonup y$. Now (3.1), the weak sequential continuity of $A$, $B$, and $C$ and the fact that $A$ satisfies $(\mathcal{H}1)$ guarantees that there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $A(F_\lambda(Bx_{n_i}))Bx_{n_i} + C(F_\lambda(Bx_{n_i})) \rightharpoonup Ay \cdot Bx + \lambda Cy$. Thus $(v)$ holds. □

**Corollary 3.1.** *Let $S$ be a nonempty, bounded, closed, and convex subset of a WC-Banach algebra $X$ and let $A$, $C : X \to X$ and $B : S \to X$ be weakly sequentially continuous operators, satisfying the following conditions:*

*(i) $\left(\frac{I-\lambda C}{A}\right)^{-1}$ exists on $B(S)$ for all $\lambda \in (0, 1)$,*

*(ii) $A$ satisfies $(\mathcal{H}1)$, and $A(S)$ is relatively weakly compact,*

*(iii) $B(S)$ is relatively weakly compact,*

*(iv) $C$ is a 1-set-weakly contractive, and*

*(v) if* $\lambda \in (0,1)$ *and* $(x = Ax \cdot By + \lambda Cx,\ y \in S)$*, then* $x \in S$*.*

*Then, the operator equation* $x = Ax \cdot Bx + Cx$ *has, at least, one solution in* $S$*.*

In the following result, we present an extension of Theorem 3.1 in [22] in WC-Banach algebras.

**Theorem 3.4.** *Let* $S$ *be a nonempty, bounded, closed, and convex subset of a WC-Banach algebra* $X$ *and let* $A, C : X \to X$ *and* $B : S \to X$ *be three operators, satisfying the following conditions:*

*(i)* $F_1 := \left(\frac{I-C}{A}\right)^{-1}$ *exists on* $B(S)$*,*

*(ii) there exists* $\alpha \in [0,1)$ *such that* $\omega(A(M)\cdot B(M)+C(M)) \leq \alpha\omega(M)$ *for all* $M \subseteq S$*,*

*(iii) if* $(x_n)_{n\in\mathbb{N}}$ *is a sequence of* $S$ *such that* $x_n \rightharpoonup x$*,* $x \in S$ *and* $F_1(Bx_n) \rightharpoonup y$*, then there exists a subsequence* $(x_{n_j})_{j\in\mathbb{N}}$ *of* $(x_n)_{n\in\mathbb{N}}$ *with*

$$A(F_1(Bx_{n_j})) \cdot Bx_{n_j} + C(F_1(Bx_{n_j})) \rightharpoonup A(y) \cdot Bx + C(y),$$

*and*

*(iv)* $(x = Ax \cdot By + Cx,\ y \in S) \Longrightarrow x \in S$*.*

*Then, the operator equation* $x = Ax \cdot Bx + Cx$ *has, at least, one solution in* $S$*.*

*Proof.* Now $x \in S$ is a solution of the equation $x = Ax \cdot Bx + Cx$, if and only if, $x$ is a fixed point of the operator $T := \left(\frac{I-C}{A}\right)^{-1} B$. From assumption $(i)$, it follows that, for each $y \in S$, there is a unique $x_y \in X$ such that

$$\left(\frac{I-C}{A}\right) x_y = By,$$

or, equivalently,

$$Ax_y \cdot By + Cx_y = x_y.$$

Since hypothesis $(iv)$ holds, then $x_y \in S$. Hence, the map $T : S \to S$ is well defined. Now, define a sequence $(S_n)_{n\geq 1}$ of subsets of $S$ by

$$S_1 = S \quad \text{and} \quad S_{n+1} = \overline{co}(T(S_n)). \tag{3.3}$$

The sequence $(S_n)_{n\geq 1}$ consists of nonempty closed convex subsets of $S$. Using (3.3) and

$$T = AT \cdot B + CT, \tag{3.4}$$

and the fact that $T(S) \subseteq S$ gives

$$T(S_n) \subseteq A(T(S_n)) \cdot B(S_n) + C(T(S_n)) \subseteq A(S_n) \cdot B(S_n) + C(S_n). \tag{3.5}$$

Thus,

$$\omega(S_{n+1}) = \omega(\overline{co}(T(S_n))) = \omega(T(S_n)) \leq \omega(A(S_n) \cdot B(S_n) + C(S_n)),$$

and assumption $(ii)$ gives

$$\omega(S_{n+1}) \leq \alpha\omega(S_n).$$

Proceeding by induction we get

$$\omega(S_n) \leq \alpha^{n-1}\omega(S),$$

and therefore $\lim_{n\to\infty} \omega(S_n) = 0$, since $\alpha \in [0,1)$. Now from Property (8) of Lemma 2.1 we infer that $S_\infty := \cap_{n=1}^{\infty} S_n$ is a nonempty closed convex weakly compact subset of $S$. Also it is easy to see that $T(S_\infty) \subseteq S_\infty$. Consequently, $T(S_\infty)$ is relatively weakly compact. Next, using an argument similar to that used in the proof of Theorem 3.3, we deduce that the operator $T : S_\infty \to S_\infty$ is weakly sequentially continuous and the result follows from Theorem 3.2. □

**Corollary 3.2.** *Let $S$ be a nonempty, bounded, closed, and convex subset of a $WC$-Banach algebra $X$ and let $A$, $C : X \to X$ and $B : S \to X$ be weakly sequentially continuous operators, satisfying the following conditions:*

*(i)* $\left(\frac{I-C}{A}\right)^{-1}$ *exists on $B(S)$,*

*(ii) there exists $\alpha \in [0,1)$ such that $\omega(A(M)\cdot B(M)+C(M)) \leq \alpha\omega(M)$ for all $M \subseteq S$,*

*(iii) $A$ satisfies $(\mathcal{H}1)$, and*

*(iv)* $(x = Ax \cdot By + Cx,\ y \in S) \Longrightarrow x \in S$.

*Then, the operator equation $x = Ax \cdot Bx + Cx$ has, at least, one solution in $S$.*

**Remark 3.2.** Consider $A = 1_X$, where $1_X$ represents the unit element in the $WC$-Banach algebra $X$, and we have the following particular cases of Krasnoselskii-type theorems (see [20, 24, 28]).

($i$) If we take $A = 1_X$ in the above corollary, we obtain Corollary 3.3 below (this was established in [22]).
($ii$) If we take $A = 1_X$ and $C = 0$ in the above corollary we obtain the following well known result of D. O'Regan [27].

**Corollary 3.3.** *Let $S$ be a nonempty, bounded, closed, and convex subset of a Banach algebra $X$, and let $C : X \to X$ and $B : S \to X$ be weakly sequentially continuous operators satisfying the following conditions:*

*(i)* $(I - C)^{-1}$ *exists on $B(S)$,*

*(ii) there exists $\alpha \in [0,1)$ such that $\omega(B(M) + C(M)) \leq \alpha\omega(M)$ for all $M \subseteq S$, and*

*(iii)* $(x = By + Cx,\ y \in S) \Longrightarrow x \in S$.

*Then, $B + C$ has, at least, a fixed point in $S$.*

**Corollary 3.4.** *Let $S$ be a nonempty, bounded, closed, and convex subset of a Banach algebra $X$. Assume that $A : S \to S$ is weakly sequentially continuous and $\alpha$-set-weakly contractive. Then $A$ has, at least, a fixed point in $S$.*

**Corollary 3.5.** *Let $S$ be a nonempty, bounded, closed, and convex subset of a $WC$-Banach algebra $X$ and let $A$, $C : X \to X$ and $B : S \to X$ be weakly sequentially continuous operators satisfying the following conditions:*

*(i)* $\left(\frac{I-C}{A}\right)^{-1}$ *exists on* $B(S)$*,*

*(ii)* $A$ *satisfies* $(\mathcal{H}1)$*,*

*(iii)* $A(S)$*,* $B(S)$*, and* $C(S)$ *are relatively weakly compact, and*

*(iv)* $(x = Ax \cdot By + Cx,\ y \in S) \Longrightarrow x \in S$.

*Then, the operator equation* $x = Ax \cdot Bx + Cx$ *has, at least, one solution in* $S$*.*

*Proof.* The result follows from Corollary 3.2 since from Lemma 2.1 and Lemma 3.1 we have (for $M \subseteq S$),

$$\omega(A(M) \cdot B(M) + C(M)) \leq \omega(A(M) \cdot B(M)) + \omega(C(M))$$

$$\leq \|A(M)\|\omega(B(M)) + \|B(M)\|\omega(A(M)) + \omega(B(M))\omega(A(M)) = 0. \qquad \square$$

In our next result we consider $\mathcal{D}$-Lipschitzian operators.

**Theorem 3.5.** *Let* $S$ *be a nonempty, bounded, closed, and convex subset of a* $WC$*-Banach algebra* $X$ *and let* $A$*,* $C : X \to X$ *and* $B : S \to X$ *be weakly sequentially continuous operators satisfying the following conditions:*

*(i)* $A$ *and* $C$ *are* $\mathcal{D}$*-Lipschitzian with the* $\mathcal{D}$*-functions* $\phi_A$ *and* $\phi_C$ *respectively,*

*(ii)* $B(S)$ *is relatively weakly compact,*

*(iii)* $A$ *is regular on* $X$*, (i.e.* $A$ *maps into the set of all invertible elements of* $X$*) and satisfies* $(\mathcal{H}1)$*, and*

*(iv)* $(x = Ax \cdot By + Cx,\ y \in S) \Longrightarrow x \in S$.

*Then, the operator equation* $x = Ax \cdot Bx + Cx$ *has, at least, one solution in* $S$*, whenever* $L\phi_A(r) + \phi_C(r) < r$ *for* $r > 0$*, where* $L = \|B(S)\|$*.*

*Proof.* Let $y$ be fixed in $S$ and define the mapping

$$\begin{cases} \varphi_y : X \longrightarrow X, \\ x \longrightarrow \varphi_y(x) = Ax \cdot By + Cx. \end{cases}$$

Let $x_1, x_2 \in X$. From assumption $(i)$ we have

$$\begin{aligned} \|\varphi_y(x_1) - \varphi_y(x_2)\| &\leq \|Ax_1 \cdot By - Ax_2 \cdot By\| + \|Cx_1 - Cx_2\| \\ &\leq \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\| \\ &\leq L\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|). \end{aligned}$$

The Boyd-Wong fixed point theorem [11] guarantees the existence of a unique point $x_y \in X$, such that

$$\varphi_y(x_y) = x_y.$$

Hence, the operator $T := \left(\frac{I-C}{A}\right)^{-1} B : S \to X$ is well defined. In order to achieve the proof, we will apply Theorem 3.2, hence, we only have to prove that the operator

$T : S \to S$ is weakly sequentially continuous and $\omega$-condensing.
Indeed, Let $M$ be a subset of $S$ with $\omega(M) > 0$. Using (3.1), we have

$$T(M) \subset A(T(M)) \cdot B(M) + C(T(M)).$$

Now Lemma's 2.1, 2.2, and 3.1, together with the assumptions on $A$, $B$, and $C$, yield

$$\begin{aligned} \omega(T(M)) &\leq \omega(A(T(M)) \cdot B(M)) + \omega(C(T(M))) \\ &\leq L\phi_A(\omega(T(M))) + \phi_C(\omega(T(M))). \end{aligned} \tag{3.6}$$

If $\omega(T(M)) > 0$, then

$$\omega(T(M)) < \omega(T(M)).$$

Hence, $T(M)$ is relatively weakly compact, so in particular, $T$ is $\omega$-condensing. Next, we prove $T$ is weakly sequentially continuous. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $S$ which is weakly convergent to $x$. Using an argument similar to that above, we see that $\{Tx_n : n \in \mathbb{N}\}$ is relatively weakly compact. Consequently, there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $Tx_{n_i} \rightharpoonup y$. Now (3.1), the weak sequential continuity of $A$, $B$, and $C$ and the fact that $A$ satisfies $(\mathcal{H}1)$ guarantees that there exists a subsequence $(x_{n_{i_j}})_{j\in\mathbb{N}}$ of $(x_{n_i})_{i\in\mathbb{N}}$ such that

$$Tx_{n_{i_j}} = A(Tx_{n_{i_j}}) \cdot Bx_{n_{i_j}} + C(Tx_{n_{i_j}}) \rightharpoonup Ay \cdot Bx + Cy.$$

Then $y = Ay \cdot Bx + Cy$, so $y = Tx$. Consequently, $Tx_{n_{i_j}} \rightharpoonup Tx$. Now, we claim $Tx_n \rightharpoonup Tx$. Suppose the contrary. Then there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a weak neighborhood $V^w$ of $Tx$, such that $Tx_{n_i} \notin V^w$ for all $i \in \mathbb{N}$. Since $(x_{n_i})_{i\in\mathbb{N}}$ converges weakly to $x$ then, arguing as before, we may extract a subsequence $(x_{n_{i_{j_k}}})_{k\in\mathbb{N}}$ of $(x_{n_i})_{i\in\mathbb{N}}$, such that $Tx_{n_{i_{j_k}}} \rightharpoonup Tx$, which is a contradiction since $Tx_{n_{i_{j_k}}} \notin V^w$, for all $k \in \mathbb{N}$. As a result, $T$ is weakly sequentially continuous. Now use Theorem 3.2. □

If we take $A = 1_X$ in the above theorem, we obtain the following known result in [22].

**Corollary 3.6.** *Let $S$ be a nonempty, bounded, closed, and convex subset of a Banach algebra $X$, and let $C : X \to X$ and $B : S \to X$ be weakly sequentially continuous operators satisfying the following conditions:*

*(i) $C$ is a contraction,*

*(i) $B(S)$ is relatively weakly compact, and*

*(i) $(x = By + Cx,\ y \in S) \Longrightarrow x \in S$.*

*Then, $B + C$ has, at least, a fixed point in $S$.*

## 4. Example

Let us consider the nonlinear functional integral equation:

$$x(t) = k(t, x(t)) + Tx(t) \left[ \left( q(t) + \int_0^{\sigma(t)} p(t, s, x(\xi(s)), x(\eta(s)))ds \right) \cdot u \right], \tag{4.1}$$

for all $t \in J = [0,1]$, where $X$ is a real $WC$-Banach algebra, $u \neq 0$ is a fixed vector of $X$, and the functions $k$, $q$, $\sigma$, $\xi$, $\eta$, $p$, and $T$ are given, while $x = x(t)$ is an unknown function. As usual, we will denote by $E = \mathcal{C}(J, X)$ the Banach space of all $X$-valued continuous functions defined on $J$, endowed with its standard norm $\|x\| = \sup_{t\in J} \|x(t)\|$. Notice that $E$ is a $WC$-Banach algebra (cf. Example 3.1). We will prove the existence of solutions for the functional integral equation (4.1) under some suitable conditions. Suppose that the functions involved in (4.1) verify the following conditions:

($\mathcal{A}_1$) $\sigma, \xi, \eta : J \to J$ are continuous.
($\mathcal{A}_2$) $q : J \to \mathbb{R}$ is a continuous function.
($\mathcal{A}_3$) The operator $T : \mathcal{C}(J, X) \to \mathcal{C}(J, X)$ satisfies:
  ($a$) there exists a continuous function $\gamma : J \to \mathbb{R}_+$ with bound $\Gamma = \|\gamma\|_\infty$ such that for all $x, y \in X$ and $t \in [0,1]$,

$$\|Tx(t) - Ty(t)\| \leq \gamma(t)\|x(t) - y(t)\|,$$

  ($b$) $T$ is regular on $\mathcal{C}(J, X)$, and verifies ($\mathcal{H}1$),
  ($c$) $T$ is weakly sequentially continuous on $\mathcal{C}(J, X)$.
($\mathcal{A}_4$) The function $k : J \times X \to X$ satisfies:
  ($d$) for all $t \in [0,1]$, $k(t, \cdot) : X \to X$ is weakly sequentially continuous on $X$,
  ($e$) for all $x \in X$, $k(\cdot, x) : J \to X$ is continuous,
  ($f$) there is a continuous function $\delta : J \to \mathbb{R}_+$ with bound $\Delta = \|\delta\|_\infty$ such that for all $x, y \in X$ and $t \in [0,1]$, $\|k(t, x) - k(t, y)\| \leq \delta(t)\|x - y\|$.
($\mathcal{A}_5$) the function $p : J^2 \times X^2 \to \mathbb{R}$ is continuous such that for arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \to p(t, s, x, y)$ is continuous uniformly for $(s, x, y) \in J \times X \times X$.
($\mathcal{A}_6$) There exists $M > 0$ such that:
  ($g$) for all $r > 0$, $|p(t, s, x, y)| \leq M$, for each $t, s \in J$; $x, y \in X$, where $\|x\| \leq r$ and $\|y\| \leq r$,
  ($h$) $(M + \|q\|_\infty)\|u\|\Gamma + \Delta < 1$.

**Theorem 4.1.** *Let $X$ be a real $WC$-Banach algebra. Under the assumptions* ($\mathcal{A}_1$) $-$ ($\mathcal{A}_6$)*, the functional integral equation* (4.1) *has, at least, one solution $x = x(t)$ which belongs to the space $E = \mathcal{C}(J, X)$.*

*Proof.* Let us define the subset $S$ of $\mathcal{C}(J, X)$ by:

$$S := \{x \in \mathcal{C}(J, X) : \|x\|_\infty \leq r_0\},$$

where

$$r_0 = \frac{\|k(t, 0)\| + \|A0\|(M + \|q\|_\infty)\|u\|}{1 - ((M + \|q\|_\infty)\|u\|\Gamma + \Delta)}.$$

Obviously, $S$ is a nonempty, closed, convex, and bounded subset of $\mathcal{C}(J, X)$. To make lecture of the functional integral equation (4.1) easier, let us consider two operators

$A$, $C$ defined on $\mathcal{C}(J,X)$ and $B$ defined on $S$ as follow:

$$\begin{aligned}(Ax)(t) &= (Tx)(t),\\ (Bx)(t) &= \left[\left(q(t)+\int_0^{\sigma(t)} p(t,s,x(\xi(s)),x(\eta(s)))ds\right)\cdot u\right],\ and\\ (Cx)(t) &= k(t,x(t)).\end{aligned}$$

This means that equation (4.1) is equivalent to the operator equation $x = Ax\cdot Bx+Cx$. Now, we will prove that the operator equation $x = Ax \cdot Bx + Cx$ satisfies all the conditions of Theorem 3.5.

$(i)$ We show that $A$ and $C$ are Lipschitzian functions. Firstly, we verify that the mapping $C$ is well defined. Let $x \in E$ and let $(t_n)_{n\in\mathbb{N}}$ be sequence in $J$ converging to a point $t$ in $J$. Thus,

$$\begin{aligned}\|Cx(t_n)-Cx(t)\| &= \|k(t_n,x(t_n))-k(t,x(t))\|\\ &\leq \|k(t_n,x(t_n))-k(t_n,x(t))\|+\|k(t_n,x(t))-k(t,x(t))\|\\ &\leq \Delta\|x(t_n)-x(t)\|+\|k(t_n,x(t))-k(t,x(t))\|.\end{aligned}$$

Since $k(\cdot,x)$ is continuous (from $\mathcal{A}_4(e)$), it follows that $\|Cx(t_n)-Cx(t)\| \to 0$. It follows that $Cx \in E$. Now we prove that $A$ and $C$ are Lipschitzian functions. To do it, let us fix arbitrary $x, y \in E$. If we take an arbitrary $t \in J$, then we get

$$\|Cx(t)-Cy(t)\| \leq \delta(t)\|x(t)-y(t)\|.$$

From the last inequality and taking the supremum over $t$, we obtain

$$\|Cx-Cy\| \leq \Delta\|x-y\|.$$

This proves that $C$ is Lipschitzian with a Lipschitz constant $\Delta$. By using a similar reasoning, we may prove that $A$ is Lipschitzian with constant $\Gamma$ (from $\mathcal{A}_3(a)$).

$(ii)$ From assumption $(\mathcal{A}_3)$, $A$ is weakly sequentially continuous, regular, and satisfies $(\mathcal{H}1)$ on $E$. Next we prove that the operators $B$ and $C$ are weakly sequentially continuous. Firstly, we verify that $C$ is weakly sequentially continuous. To see this, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $E$ such that $x_n \rightharpoonup x$ for some $x \in E$. Then $(x_n)_{n\in\mathbb{N}}$ is bounded in $E$. Applying Theorem 3.1, we get $x_n(t) \rightharpoonup x(t)$. From the weak sequential continuity of $k(t,\cdot)$, it follows that $C$ is weakly sequentially continuous. Now we show that $B$ is weakly sequentially continuous on $S$. Firstly, we verify that if $x \in S$, then $Bx \in E$. For this, let $(t_n)_{n\in\mathbb{N}}$ be any sequence in $J$ converging to a point $t$ in $J$. Then,

$$\begin{aligned}\|Bx(t_n)-Bx(t)\| \leq |q(t_n)-q(t)|\|u\| &+ \left[\int_{\sigma(t_n)}^{\sigma(t)} |p(t,s,x(\xi(s)),x(\eta(s)))ds|\right]\|u\|\\ +\left[\int_0^{\sigma(t_n)} |p(t_n,s,x(\xi(s)),x(\eta(s)))-p(t,s,x(\xi(s)),x(\eta(s)))ds|\right]\|u\|\\ \leq \|q(t_n)-q(t)\|\|u\|+M\|\sigma(t_n)-\sigma(t)\|\|u\|\end{aligned}$$

$$+\left[\int_0^1 |p(t_n,s,x(\xi(s)),x(\eta(s)))-p(t,s,x(\xi(s)),x(\eta(s)))ds|\right]\|u\|$$

Since $t_n \to t$ and taking into account the hypothesis $(\mathcal{A}_5)$, it follows that for all $s \in J$,

$$p(t_n,s,x(\xi(s)),x(\eta(s))) \to p(t,s,x(\xi(s)),x(\eta(s))).$$

Moreover, the use of assumption $(\mathcal{A}_6)$ leads to

$$|p(t_n,s,x(\xi(s)),x(\eta(s)))-p(t,s,x(\xi(s)),x(\eta(s)))| \leq 2M$$

for all $t$, $s \in J$. Now, we can apply the dominated convergence theorem and since assumption $(\mathcal{A}_2)$ holds, we get

$$Bx(t_n) \to Bx(t).$$

It follows that

$$Bx \in E.$$

Next, let $(x_n)_{n\in\mathbb{N}}$ be any sequence in $S$ weakly convergent to a point $x \in S$. By using Theorem 3.1 combined with assumptions $(\mathcal{A}_5)$ and $(\mathcal{A}_6)$ and the dominated convergence theorem, we obtain:

$$\int_0^{\sigma(t)} p(t,s,x_n(\xi(s)),x_n(\eta(s)))ds \to \int_0^{\sigma(t)} p(t,s,x(\xi(s)),x(\eta(s)))ds$$

as $n \to \infty$, which implies that

$$\lim_{n\to\infty}\left(q(t)+\int_0^{\sigma(t)} p(t,s,x_n(\xi(s)),x_n(\eta(s)))ds\right)\cdot u$$
$$=\left(q(t)+\int_0^{\sigma(t)} p(t,s,x(\xi(s)),x(\eta(s)))ds\right)\cdot u.$$

Hence,

$$Bx_n(t) \to Bx(t) \text{ in } X$$

and so,

$$Bx_n(t) \rightharpoonup Bx(t) \text{ in } X.$$

It is clear that the sequence $(Bx_n)_{n\in\mathbb{N}}$ is bounded by $(\|q\|+M)\|u\|$. Then, by using Theorem 3.1, we get

$$Bx_n \rightharpoonup Bx.$$

Thus, we conclude that $B$ is weakly sequentially continuous on $S$.

$(iii)$ Now we prove that $B(S)$ is relatively weakly compact. For all $t \in J$, we have

$$B(S)(t) := \{Bx(t) : \|x\| \leq r_0\}.$$

We claim that $B(S)(t)$ is sequentially relatively weakly compact in $X$. To show it, let $(x_n)_{n\in\mathbb{N}}$ be any sequence in $S$. We have $Bx_n(t) = r_n(t)\cdot u$, where

$$r_n(t) = q(t)+\int_0^{\sigma(t)} p(t,s,x_n(\xi(s)),x_n(\eta(s)))ds.$$

It is clear that $|r_n(t)| \leq (\|q\| + M)$ and $(r_n(t))_{n\in\mathbb{N}}$ is a real sequence, so, by using Bolzano-Weirstrass theorem, there is renamed subsequence such that

$$r_n(t) \to r(t),$$

which implies

$$r_n(t) \cdot u \to r(t) \cdot u,$$

and consequently,

$$(Bx_n)(t) \to r(t) \cdot u.$$

Hence, we conclude that $B(S)(t)$ is sequentially relatively compact in $X$. Then, $B(S)(t)$ is sequentially relatively weakly compact in $X$. Now, we have to prove that $B(S)$ is weakly equicontinuous on $J$. For this purpose, let $\varepsilon > 0$; $x \in S$, $x^* \in X^*$; $t, t' \in J$ such that $t \leq t'$ and $t' - t \leq \varepsilon$. Then,

$$\begin{aligned}
\|x^*(Bx)(t) - Bx(t')\| &\leq |q(t) - q(t')||x^*(u)| \\
&+ \left| \int_0^{\sigma(t)} \chi(t,s) - \int_0^{\sigma(t')} \chi(t',s)ds \right| |x^*(u)| \\
&\leq |q(t) - q(t')||x^*(u)| \\
+ \left[ \int_0^{\sigma(t)} |\chi(t,s) - \chi(t',s)|ds \right] |x^*(u)| &+ \left| \int_{\sigma(t)}^{\sigma(t')} \chi(t',s)ds \right| |x^*(u)| \\
&\leq [w(q,\varepsilon) + w(p,\varepsilon) + Mw(\sigma,\varepsilon)] |x^*(u)|,
\end{aligned}$$

where

$$\begin{aligned}
\chi(t,s) &:= p(t,s,x(\xi(s)),x(\eta(s))), \\
w(q,\varepsilon) &:= \sup\{|q(t) - q(t')| : t,t' \in J; |t'-t| \leq \varepsilon\}, \\
w(p,\varepsilon) &:= \sup\{|p(t',s,x,y) - p(t,s,x,y)|, t,t',s \in J; |t'-t| \leq \varepsilon; x,y \in S\}, \ \textit{and} \\
w(\sigma,\varepsilon) &:= \sup\{|\sigma(t) - \sigma(t')| : t,t' \in J; |t'-t| \leq \varepsilon\}.
\end{aligned}$$

Taking into account the hypothesis $(\mathcal{A}_5)$, and in view of the uniform continuity of the functions $q$ and $\sigma$, it follows that $w(p,\varepsilon) \to 0$, $w(q,\varepsilon) \to 0$, and $w(\sigma,\varepsilon) \to 0$ when $\varepsilon \to 0$. Hence, the application of Arzelà-Ascoli's theorem (see [29]) implies that $B(S)$ is sequentially relatively weakly compact. Now, the use of Eberlein-Šmulian's theorem [26, Theorem 2.8.6] allows $B(S)$ is relatively weakly compact.

*(iv)* Finally, we shall show that the hypothesis *(iv)* of Theorem 3.5 is satisfied. In fact, we fix arbitrarily $x \in E$ and $y \in S$ such that

$$x = Ax \cdot By + Cx,$$

or equivalently for all $t \in J$,

$$x(t) = Ax(t) \cdot By(t) + Cx(t).$$

Then by using $(\mathcal{A}_3) - (\mathcal{A}_5)$, we have

$$\begin{aligned} \|x(t)\| &\leq \|Cx(t)\| + \|Ax(t) \cdot By(t)\| \\ &\leq \Delta\|x(t)| + \|k(t,0)\| + (\Gamma\|x(t)\| + \|A0\|)(M + \|q\|)\|u\| \\ &\leq ((M + \|q\|_\infty)\|u\|\Gamma + \Delta)\, \|x(t)\| + \|k(t,0)\| + \|A0\|(M + \|q\|)\|u\|. \end{aligned}$$

Then, we have

$$\|x(t)\| \leq \frac{\|k(t,0)\| + \|A0\|(M + \|q\|_\infty)\|u\|}{1 - ((M + \|q\|_\infty)\|u\|\Gamma + \Delta)} = r_0.$$

Taking the supremum over $t$, we obtain

$$\|x\|_\infty \leq r_0.$$

As a result, $x$ is in $S$. Now, by applying Theorem 3.5, we deduce that the functional integral equation (4.1) has, at least, one solution in $E$. □

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*Received: March 1st, 2017; Accepted: January 11, 2018.*