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EXTRAGRADIENT AND LINESEARCH ALGORITHMS FOR SOLVING EQUILIBRIUM PROBLEMS, VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS IN BANACH SPACES

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Abstract. Using generalized metric projection, new extragradient and linesearch algorithms are presented for finding a common element of the solution set of an equilibrium problem and the solution set of variational inequality problem which is also an element of the set of fixed points of a weakly relatively nonexpansive mapping in Banach spaces. To prove strong convergence of the iterates in the extragradient method, a ϕ-Lipschitz-type condition is introduced and is assumed that the equilibrium bifunction satisfies in this condition. To avoid using this condition, the linesearch method is applied instead of the extragradient method. Using FMINCON optimization toolbox in MATLAB, some numerical examples are given to illustrate the usability of obtained results.
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1. Introduction

Assume that C is a nonempty closed convex subset of a real Banach space E with dual space E^*. In this paper, we consider the Variational Inequality (VI) as follow: " find $u \in C$ such that $\langle x^*, y-u\rangle \geq 0,\ \forall\, y \in C\ \&\ \forall x^* \in Au$ ", where $A : C \to 2^{E^*}$ is a given mapping and $\langle .,.\rangle$ denotes the generalized duality pairing. The solution set of (VI) is denoted by $SOL(C, A)$.

An operator $A : C \to 2^{E^*}$ is called monotone if $\langle x^* - y^*, x - y\rangle \geq 0$, for all $x, y \in C$, all $x^* \in Ax$ and all $y^* \in Ay$. If there exists a constant $\alpha > 0$ such that

$$\langle x^* - y^*, x - y\rangle \geqslant \alpha\|x^* - y^*\|^2, \quad \forall\, x, y \in C,\ \forall\, x^* \in Ax\ \&\ \forall\, y^* \in Ay,$$

then, A is called α-inverse-strongly monotone. A monotone operator A is called to be maximal if its graph $G(A) = \{(x, x^*) : x \in D(A)\ \&\ x^* \in Ax\}$ is not contained in the graph of any other monotone operator, where $D(A)$ is the domain of A.

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Also, we consider the equilibrium problem (EP) [5], which consists in finding a point $x^* \in C$ such that for any $y \in C$, $f(x^*, y) \geq 0$, where $f : C \times C \to \mathbb{R}$ is an equilibrium bifunction, i.e., $f(x,x) = 0$ for all $x \in C$. The solution set of (EP) is denoted by $E(f)$. Many well-known problems have been covered by (EP) [21] such as the optimization problem, the variational inequality problem, the generalized Nash equilibrium problem in game theory, the fixed point problem and others; (see [2, 17, 22, 28]). Also numerous problems in physic and economic reduce to find a solution of an equilibrium problem. For such wide applications, many methods have been proposed to solve the equilibrium problems see for instance [5, 6, 9, 22, 33, 34]. In 1980, Cohen [11] introduced a useful tool for solving optimization problem which is known as auxiliary problem principle and then extended it to variational inequality [12]. In auxiliary problem principle a sequence $\{x_k\}$ is generated as follows: $x_{k+1} \in C$ is a unique solution of the following strongly convex problem

$$\min_{y \in C} \Big\{ c_k f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 \Big\}, \tag{1.1}$$

where $c_k > 0$ and C is a nonempty closed convex subset of a real Hilbert space. Recently, Mastroeni [20] extended the auxiliary problem principle to equilibrium problems under the assumptions that the equilibrium function f is strongly monotone on $C \times C$ and that f satisfies the following Lipschitz-type condition:

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|y - x\|^2 - c_2\|z - y\|^2, \quad \forall\, x, y, z \in C, \tag{1.2}$$

where $c_1, c_2 > 0$. To avoid the monotonicity of f, motivated by Antipin [4], Tran et al. [32] have used an extrapolation step in each iteration after solving (1.1) and supposed that f is pseudomonotone on $C \times C$ which is weaker than monotonicity assumption. They assumed y_k was the unique solution of (1.1) and the unique solution of the strongly convex problem $\min_{y \in C} \Big\{ c_k f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 \Big\}$ is denoted by x_{k+1}. In special case , when (EP) is (VI), this method reduces to the classical extragradient method which has been introduced by Korpelevich [19]. The extragradient method is well known because of its efficiency in numerical tests. In recent years, many authors introduced extragradient algorithms for solving (EP) in Hilbert spaces such that the convergence of the proposed algorithms requires f which satisfies in a certain Lipschitz-type condition [23, 32, 34]. Lipschitz-type condition depends on two positive parameters c_1 and c_2 which in some cases, they are unknown or difficult to approximate. In otder to avoid this requirement, the authors used the linesearch technique in Hilbert space to obtain convergent algorithms for solving equilibrium problem [13, 23, 32, 34].

In this paper, we consider the auxiliary equilibrium problem $(AUEP)$ as follow: "find $x^* \in C$ such that $\rho f(x^*, y) + L(x^*, y) \geq 0$, for all $y \in C$", where $\rho > 0$ is a regularization parameter, C is a nonempty subset of a real smooth Banach space and $L : C \times C \to \mathbb{R}$ is a nonnegative differentiable convex bifunction on C with respect to y, for any fixed $x \in C$, such that $L(x,x) = 0$ and $\nabla_2 L(x,x) = 0$, for all $x \in C$, where $\nabla_2 L(x,x)$ is the gradient of $L(x,.)$ at x.

Recently, many authors studied the problem of finding a common element of the set of fixed points of a nonlinear mapping, the solution set of an equilibrium problem and the solution set of variational inequalities in the framework of Hilbert spaces and Banach spaces, see for instance [7, 8, 14, 15, 24, 26, 31, 34, 35]. For solving variational inequalities in Banach spaces [14, 15, 35], the authors often supposed the following strong condition:

$$\|Ax\| \leq \|Ax - Au\|, \qquad \forall\, x \in C, \quad u \in SOL(C, A), \tag{1.3}$$

where $A : C \to E^*$ is an α-inverse-strongly monotone operator and α is a positive real number depends on 2-uniformly convexity constant of Banach space E.

In this paper, motivated D. Q. Tran et al. [32] and P. T. Vuong et al. [34], we introduce new extragradient and linesearch algorithms for finding a common element of the set of fixed points of a weakly relatively nonexpansive mapping, the solution set of an equilibrium problem and the solution set of a variational inequality problem in real Banach spaces, by using generalized metric projection and without considering condition (1.3). Using these algorithms, we prove strong convergence theorems under suitable conditions.

2. Preliminaries

Suppose that E^* is the dual of a real Banach space E. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Assume that $S(E)$ is the unite sphere centered at the origin of E. A Banach space E is strictly convex if $\|\frac{x+y}{2}\| < 1$, whenever $x, y \in S(E)$ and $x \neq y$. Modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}\|(x+y)\| : \ \|x\|, \|y\| \leq 1, \ \|x - y\| \geq \epsilon\},$$

for all $\epsilon \in [0, 2]$.

A Banach space E is said to be uniformly convex if $\delta_E(0) = 0$ and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. Let $p \geq 2$ be a fixed real number. The Banach space E is called p-uniformly convex [30] if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in [0, 2]$. The Banach space E is called smooth if the limit $\lim_{t\to 0} \dfrac{\|x + ty\| - \|x\|}{t}$, exists for all $x, y \in S(E)$. The smoothness modules of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 \ : \ x \in S(E), \ \|y\| \leq t\}.$$

A Banach space E is said to be uniformly smooth if $\dfrac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Every uniformly smooth Banach space E is smooth. If E is uniformly convex, then E is reflexive and strictly convex [1, 29].

The mapping J_p from real Banach space E to 2^{E^*} defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \ \|x^*\| = \|x\|^{p-1}\}, \quad \forall\, x \in E,$$

is called the generalized duality mapping, where $p > 1$ is a real number. If $p = 2$, then $J_2 = J$ is called the normalized duality mapping. If E is uniformly convex and uniformly smooth, then J is uniformly norm-to-norm continuous on bounded sets of

E and $J^{-1} = J^*$ (the normalized duality mapping on E^*) is also uniformly norm-to-norm continuous on bounded sets of E^*. Many other properties of J have been given in [1, 29].

Let E be a smooth and real Banach space, we define the function $\phi : E \times E \to \mathbb{R}$ by $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, for all $x, y \in E$.

Observe that, in a real Hilbert space H, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. It is clear from the definition of ϕ that for all $x, y, z \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \tag{2.1}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \tag{2.2}$$

if E additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \iff x = y. \tag{2.3}$$

Also, we define the function $V : E \times E^* \to \mathbb{R}$ by $V(x, x^*) = \|x\|^2 - 2 < x, x^* > + \|x^*\|^2$, for all $x \in E$ and all $x^* \in E^*$. It is easy to see that, $V(x, x^*) = \phi(x, J^{-1}x^*)$.
It is well known that, if E is a reflexive strictly convex, smooth and real Banach space with E^* as its dual, then for any $x \in E$ and any $x^*, y^* \in E^*$ [27], the following inequality holds:

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*). \tag{2.4}$$

Let C be a closed convex subset of a real Banach space E and $T : C \to C$ be a mapping. A point p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x^k\}$ which converges strongly to p such that $\lim_{k\to\infty} \|Tx^k - x^k\| = 0$. Let $\tilde{F}(T)$ be the set of asymptotic fixed points of T. A mapping $T : C \to C$ is called weakly relatively nonexpansive if $\tilde{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and all $p \in F(T)$ and T is called to be quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and all $p \in F(T)$. The class of quasi-ϕ-nonexpansive mappings is broader than the class of weakly relatively nonexpansive mappings which requires $\tilde{F}(T) = F(T)$. It is well known that, if C is a nonempty convex closed subset of a strictly convex, smooth and real Banach space E and $T : C \to C$ is a quasi-ϕ-nonexpansive mapping, then $F(T)$ is a convex closed subset of C [25].

For a convex subset C of a real Banach space E, we denote by $N_C(\nu)$ the normal cone for C at a point $\nu \in C$, which is defined by

$$N_C(\nu) := \{x^* \in E^* : \langle \nu - y, x^*\rangle \geq 0, \ \forall\, y \in C\}.$$

Suppose that E is a real Banach space and let $f : E \to (-\infty, +\infty]$ be a proper function. For $x_0 \in D(f)$, we define the subdifferential of f at x_0 as the subset of E^* given by

$$\partial f(x_0) = \{x^* \in E^* : \ f(x) \geq f(x_0) + \langle x^*, x - x_0\rangle, \ \forall x \in E\}.$$

If $\partial f(x_0) \neq \emptyset$, then we say f is subdifferentiable at x_0.

3. An extragradient algorithm

In this section, we present an extragradient algorithm for finding a common solution of (EP) and (VI) which is also an element of the fixed points set of a weakly relatively nonexpansive mapping.

For the proof of the following lemmas, we refer the readers to [16].

Lemma 3.1. *Let C be a nonempty convex subset of a smooth and real Banach space E and assume that $f : C \times C \to \mathbb{R}$ is an equilibrium bifunction, which is also convex with respect to the second variable. Then $x^* \in E(f)$ if and only if $x^* = \arg\min_{y\in C} f(x^*, y)$.*

Equivalence between (EP) and $(AUEP)$ is expressed in the following lemma.

Lemma 3.2. *Assume that C is nonempty, closed and convex subset of a reflexive, smooth and real Banach space E and $f : C \times C \to \mathbb{R}$ is an equilibrium bifunction and let $x^* \in C$. Suppose that $f(x^*, .) : C \to \mathbb{R}$ is convex and subdifferentiable on C. Let $L : C \times C \to \mathbb{R}_+$ be a differentiable convex function on C with respect to the second variable such that $L(x^*, x^*) = 0$ and $\nabla_2 L(x^*, x^*) = 0$. Then $x^* \in E(f)$ if and only if x^* is a solution to $(AUEP)$.*

Throughout this paper, we suppose that C is nonempty, convex and closed subset of 2-uniformly convex, uniformly smooth real Banach space E, S is a weakly relatively nonexpansive self-mapping of C and $A : C \to E^*$ is an α-inverse-strongly monotone operator. Also, we assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$,

(A2) f is pseudomonotone on C, i.e., $f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0$ for all $x, y \in C$,

(A3) f is jointly weakly continuous on $C \times C$, i.e., if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in C converge weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$,

(A4) $f(x, .)$ is convex, lower semicontinuous and subdifferentiable on C for every $x \in C$,

(A5) f satisfies ϕ-Lipschitz-type condition: there exist two constants $c_1 > 0$ and $c_2 > 0$ such that $f(x, y) + f(y, z) \geq f(x, z) - c_1\phi(y, x) - c_2\phi(z, y)$, for all $x, y, z \in C$.

It is easy to see that if f satisfies the properties $(A1) - (A4)$, then the set $E(f)$ is closed and convex. Furthermore, if E is a Hilbert space, then ϕ-Lipschitz-type condition reduces to Lipschitz-type condition (1.2).

Example 3.3. Let $f : C \times C \to \mathbb{R}$ be a bifunction defined by

$$f(x, y) = 4\|y\|^2 + 2\langle y, Jx\rangle - 6\|x\|^2.$$

We have

$$\begin{aligned} f(x, y) + f(y, z) &= f(x, z) - (\|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2) - (\|y\|^2 - 2\langle z, Jy\rangle + \|z\|^2) \\ &\qquad + (\|x\|^2 - 2\langle z, Jx\rangle + \|z\|^2) \\ &\geq f(x, z) - \phi(y, x) - \phi(z, y), \end{aligned}$$

i.e., f satisfies the ϕ-Lipschitz-type condition with $c_1, c_2 = 1$.

Algorithm 1

Step 0: Suppose that $\{\alpha_n\} \subseteq [a, e]$ for some $0 < a \leq e < 1$, $\{\beta_n\} \subseteq [d, b]$ for some $0 < d \leq b < 1$ and $\{\tau_n\}, \{\lambda_n\} \subseteq (0, 1]$ where $\lim_{n\to\infty} \tau_n = 0$ and

$$0 < \lambda_{min} \leq \lambda_n \leq \lambda_{max} < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}.$$

Step 1: Let $x_0 \in C$. Set n=0.

Step 2: Compute y_n and z_n, such that

$$y_n = \arg\min_{y\in C}\{\lambda_n f(x_n, y) + \frac{1}{2}\phi(y, x_n)\}, \tag{3.1}$$

$$z_n = \arg\min_{y\in C}\{\lambda_n f(y_n, y) + \frac{1}{2}\phi(y, x_n)\}. \tag{3.2}$$

Step 3: Put $p_n = \Pi_C J^{-1}(Jx_n - \tau_n Ax_n)$ and

$$t_n = \Pi_C J^{-1}(\alpha_n Jp_n + (1-\alpha_n)(\beta_n Jz_n + (1-\beta_n)JSz_n)).$$

Step 4: Compute $x_{n+1} = \Pi_{C_n\cap D_n} x_0$, where $\Pi_{C_n\cap D_n}$ is the generalized metric projection from E onto $C_n \cap D_n$ in which
$C_n = \{z \in C : \phi(z, t_n) \leq \phi(z, x_n) + \frac{4}{c^2}\alpha_n\tau_n^2\|Ax_n\|^2\}$ and
$D_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}$,
and $\frac{1}{c}$ $(0 < c \leq 1)$ is the 2-uniformly convexity constant of E.

Step 5: Put $n := n + 1$ and go to Step 2.

Lemma 3.4. *For every $x^* \in E(f)$ and each $n \in \mathbb{N} \cup \{0\}$, we obtain*

(i) $\langle Jx_n - Jy_n, y - y_n\rangle \leq \lambda_n f(x_n, y) - \lambda_n f(x_n, y_n), \quad \forall y \in C,$

(ii) $\phi(x^*, z_n) \leq \phi(x^*, x_n) - (1 - 2\lambda_n c_1)\phi(y_n, x_n) - (1 - 2\lambda_n c_2)\phi(z_n, y_n).$

Proof. Using a similar argument such as the proof of Lemma 3.6 in [16], we can deduce the desired results. □

Remark 3.5. In a real Hilbert space E, Lemma 3.4 is reduced to Lemma 3.1 in [3].

Lemma 3.6. *In Algorithm 1, the optimal solutions y_n and z_n are uniquely determined.*

Proof. Let $y_n, \acute{y}_n \in \arg\min_{y\in C}\{\lambda_n f(x_n, y) + \frac{1}{2}\phi(y, x_n)\}$, so using Lemma 3.4 (i), we have

$$\langle Jx_n - Jy_n, y - y_n\rangle \leq \lambda_n f(x_n, y) - \lambda_n f(x_n, y_n), \quad \forall y \in C, \tag{3.3}$$

$$\&$$

$$\langle Jx_n - J\acute{y}_n, y - \acute{y}_n\rangle \leq \lambda_n f(x_n, y) - \lambda_n f(x_n, \acute{y}_n), \quad \forall y \in C. \tag{3.4}$$

Putting $y = \acute{y}_n$ in inequality (3.3) and $y = y_n$ in inequality (3.4) and adding them, we get $\langle J\acute{y}_n - Jy_n, \acute{y}_n - y_n\rangle \leq 0$. Since J is monotone and one to one, we obtain $y_n = \acute{y}_n$. In a similar way, we can conclude that z_n is uniquely determined. □

Lemma 3.7. *For every $x^* \in E(f) \cap SOL(C, A) \cap F(S)$ and each $n \in \mathbb{N} \cup \{0\}$, we get*

$$\phi(x^*, t_n) \leq \phi(x^*, x_n) + \frac{4}{c^2}\alpha_n\tau_n^2\|Ax_n\|^2.$$

Proof. Utilizing Proposition 5 of [18], the definition of function V and inequality (2.4), we get

$$\begin{aligned}\phi(x^*,p_n) &\leq \phi(x^*,J^{-1}(Jx_n-\tau_nAx_n)) \\ &= V(x^*,Jx_n-\tau_nAx_n) \\ &\leq V(x^*,(Jx_n-\tau_nAx_n)+\tau_nAx_n)-2\langle J^{-1}(Jx_n-\tau_nAx_n)-x^*,\tau_nAx_n\rangle \\ &= \phi(x^*,x_n)-2\tau_n\langle x_n-x^*,Ax_n\rangle-2\langle J^{-1}(Jx_n-\tau Ax_n)-x_n,\tau_nAx_n\rangle.\end{aligned} \tag{3.5}$$

Since A is an α-inverse strongly monotone operator and $x^*\in SOL(C,A)$, we have

$$\begin{aligned}-2\tau_n\langle x_n-x^*,Ax_n\rangle &= -2\tau_n\langle x_n-x^*,Ax_n-Ax^*\rangle-2\tau_n\langle x_n-x^*,Ax^*\rangle \\ &\leq -2\alpha\tau_n\|Ax_n-Ax^*\|^2,\end{aligned} \tag{3.6}$$

using Lemma 2.1 of [35], we obtain

$$\begin{aligned}2\langle J^{-1}(Jx_n-\tau_nAx_n)-x_n,-\tau_nAx_n\rangle &\leq 2\|J(J^{-1}(Jx_n-\tau_nAx_n))-J(J^{-1}Jx_n)\|\|\tau_nAx_n\| \\ &\leq \frac{4}{c^2}\tau_n^2\|Ax_n\|^2.\end{aligned} \tag{3.7}$$

Utilizing Proposition 5 of [18], Lemma 3.4, the convexity of $\|.\|^2$, the definition of functions ϕ and S and inequalities (3.5) and (3.6), we have

$$\phi(x^*,t_n) \leq \phi(x^*,J^{-1}(\alpha_nJp_n+(1-\alpha_n)(\beta_nJz_n+(1-\beta_n)JSz_n)))$$

$$\leq \|x^*\|^2+\alpha_n\|p_n\|^2+(1-\alpha_n)(\beta_n\|z_n\|^2+(1-\beta_n)\|Sz_n\|^2)$$

$$-2\alpha_n\langle x^*,Jp_n\rangle-2(1-\alpha_n)\beta_n\langle x^*,Jz_n\rangle-2(1-\alpha_n)(1-\beta_n)\langle x^*,JSz_n\rangle$$

$$\leq \phi(x^*,x_n)+\frac{4}{c^2}\alpha_n\tau_n^2\|Ax_n\|^2.$$

□

Theorem 3.8. *If $\Omega:=E(f)\cap SOL(C,A)\cap F(S)\neq\emptyset$, then the sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, $\{z_n\}_{n=0}^\infty$, $\{p_n\}_{n=0}^\infty$ and $\{t_n\}_{n=0}^\infty$ generated by Algorithm 1 converge strongly to the same solution $u^*\in\Omega$, where $u^*=\Pi_\Omega x_0$ and Π_Ω is generalized metric projection from E onto Ω.*

Proof. At First, using induction we show that $\Omega\subseteq C_n\cap D_n$ for all $n\geq 0$. Let $x^*\in\Omega$, utilizing Lemma 3.7, we get $\Omega\subseteq C_n$ for all $n\geq 0$. Now, we show that $\Omega\subseteq D_n$ for all $n\geq 0$. It is clear that $\Omega\subseteq D_0$. Suppose that $\Omega\subseteq D_n$, i.e., $\langle x_n-x^*,Jx_0-Jx_n\rangle\geq 0$, for all $x^*\in\Omega$. Since $x_{n+1}=\Pi_{C_n\cap D_n}x_0$, using Proposition 4 of [18], we get $\langle x_{n+1}-z,Jx_0-Jx_{n+1}\rangle\geq 0$, for all $z\in C_n\cap D_n$. This implies that $x^*\in D_{n+1}$. Therefore $\Omega\subseteq D_{n+1}$.

Let $x^*\in\Omega\subseteq D_{n+1}$. Since $x_{n+1}\in D_n$, using successively equality (2.2), it is easy to see that $\{\phi(x_n,x_0)\}$ is increasing and bounded from above by $\phi(x^*,x_0)$, so $\lim_{n\to\infty}\phi(x_n,x_0)$ exists. This yields that $\{\phi(x_n,x_0)\}$ is bounded. From inequality (2.1), we know that $\{x_n\}$ is also bounded. It is clear that $\lim_{n\to\infty}\phi(x_{n+1},x_n)=0$, so, Proposition 2 of [18] implies that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and therefore $\{x_n\}$ converges strongly to $\bar{x} \in C$. Since A is α-inverse-strongly monotone, we have $Ax_n \to A\bar{x}$. From inequality (3.7), we obtain

$$\begin{aligned}\phi(x_n, p_n) &\leq \phi(x_n, J^{-1}(Jx_n - \tau_n Ax_n))\\ &\leq V(x_n, Jx_n - \tau_n Ax_n + \tau_n Ax_n) - 2\langle J^{-1}(Jx_n - \tau_n Ax_n) - x_n, \tau_n Ax_n\rangle\\ &\leq \frac{4}{c^2}\tau_n^2\|Ax_n\|^2\\ &\to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

because of $\lim_{n\to\infty} \tau_n = 0$. So, Proposition 2 of [18] implies that $\lim_{n\to\infty} \|x_n - p_n\| = 0$. Consequently, $\{p_n\}$ converges strongly to $\bar{x} \in C$. Since $x_{n+1} \in C_n$, we have $\lim_{n\to\infty} \phi(x_{n+1}, t_n) = 0$ and utilizing Proposition 2 of [18], we deduce that

$$\lim_{n\to\infty} \|x_{n+1} - t_n\| = 0.$$

Thus $\lim_{n\to\infty} \|x_n - t_n\| = 0$ which implies that $\{t_n\}$ converges strongly to $\bar{x} \in C$. Using norm-to-norm continuity of J on bounded sets, we conclude that $\lim_{n\to\infty} \|Jx_n - Jt_n\| = 0$ and therefore

$$\lim_{n\to\infty} \phi(x^*, x_n) = \lim_{n\to\infty} \phi(x^*, t_n). \tag{3.8}$$

Applying Lemma 3.4 (ii), we obtain $\phi(x^*, z_n) \leq \phi(x^*, x_n)$. From inequality (2.1) and the definition of S, we derive that $\{z_n\}$ and $\{Sz_n\}$ are bounded. Let

$$r_1 = \sup_{n\geq 0}\{\|p_n\|, \|z_n\|\} \text{ and } r_2 = \sup_{n\geq 0}\{\|z_n\|, \|Sz_n\|\}.$$

So, by Lemma 1.4 of [10], there exists a continuous, strictly increasing and convex function $g_1 : [0, 2r_1] \to \mathbb{R}$ with $g_1(0) = 0$ such that for $x^* \in \Omega$, we get

$$\begin{aligned}\phi(x^*, t_n) &\leq \phi(x^*, J^{-1}(\alpha_n Jp_n + (1 - \alpha_n)(\beta_n Jz_n + (1 - \beta_n)JSz_n)))\\ &\leq \phi(x^*, x_n) + \frac{4}{c^2}\alpha_n\tau_n^2\|Ax_n\| - \alpha_n(1 - \alpha_n)\beta_n g_1(\|Jz_n - Jp_n\|),\end{aligned} \tag{3.9}$$

and using the same argument, there exists a continuous, strictly increasing and convex function $g_2 : [0, 2r_2] \to \mathbb{R}$ with $g_2(0) = 0$ such that for $x^* \in \Omega$, we have

$$\phi(x^*, t_n) \leq \phi(x^*, x_n) + \frac{4}{c^2}\alpha_n\tau_n^2\|Ax_n\| - (1 - \alpha_n)^2\beta_n(1 - \beta_n)g_2(\|Jz_n - JSz_n\|),$$

which imply

$$\alpha_n(1 - \alpha_n)\beta_n g_1(\|Jz_n - Jp_n\|) \leq \phi(x^*, x_n) - \phi(x^*, t_n) + \frac{4}{c^2}\alpha_n\tau_n^2\|Ax_n\|, \tag{3.10}$$

$$\&$$

$$(1-\alpha_n)^2\beta_n(1-\beta_n)g_2(\|Jz_n-JSz_n\|) \leq \phi(x^*, x_n)-\phi(x^*, t_n)+\frac{4}{c^2}\alpha_n\tau_n^2\|Ax_n\|. \tag{3.11}$$

By letting $n \to \infty$ in inequalities (3.10) and (3.11), using (3.7) and (3.8), we obtain

$$\lim_{n\to\infty} g_1(\|Jz_n - Ju_n\|) = 0 \quad \& \quad \lim_{n\to\infty} g_2(\|Jz_n - JSz_n\|) = 0.$$

Utilizing the properties of g_1 and g_2, we have

$$\lim_{n\to\infty} \|Jz_n - Jp_n\| = 0 \quad \& \quad \lim_{n\to\infty} \|Jz_n - JSz_n\| = 0. \tag{3.12}$$

So, we obtain

$$\lim_{n\to\infty}\|z_n - p_n\| = 0 \quad \& \quad \lim_{n\to\infty}\|z_n - Sz_n\| = 0, \tag{3.13}$$

$$\|z_n - x_n\| \le \|z_n - p_n\| + \|p_n - x_n\| \to 0 \ \& \ \lim_{n\to\infty}\|Jz_n - Jx_n\| = 0, \tag{3.14}$$

since J^{-1} and J are uniformly norm-to-norm continuous on bounded sets.
By the same reason as in the proof of (3.8), we can conclude from (3.14) that

$$\lim_{n\to\infty}\phi(x^*, x_n) = \lim_{n\to\infty}\phi(x^*, z_n), \tag{3.15}$$

for all $x^* \in \Omega$. Using Lemma 3.4 (ii), we have

$$(1 - 2\lambda_n c_1)\phi(y_n, x_n) + (1 - 2\lambda_n c_2)\phi(z_n, y_n) \le \phi(x^*, x_n) - \phi(x^*, z_n), \tag{3.16}$$

for all $x^* \in \Omega$. Taking the limits as $n \to \infty$ in inequality (3.16) and using equality (3.15), we get

$$\lim_{n\to\infty}\phi(y_n, x_n) = 0 \quad \& \quad \lim_{n\to\infty}\phi(z_n, y_n) = 0. \tag{3.17}$$

Since $\{x_n\}$ and $\{z_n\}$ are bounded, it follows from Proposition 2 of [18] that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0 \quad \& \quad \lim_{n\to\infty}\|z_n - y_n\| = 0,$$

which imply $\{y_n\}$ and $\{z_n\}$ converges strongly to $\bar{x} \in C$.
Now, we prove that $\bar{x} \in E(f)$. It follows from the definition of y_n that for any $y \in C$,

$$\lambda_n f(x_n, y_n) + \frac{1}{2}\phi(y_n, x_n) \le \lambda_n f(x_n, y) + \frac{1}{2}\phi(y, x_n). \tag{3.18}$$

By letting $n \to \infty$ in inequality (3.18), it follows from equality (3.17), conditions $(A1)$ and $(A3)$ and uniformly norm-to-norm continuity of J on bounded sets that $0 \le f(\bar{x}, y) + \phi(y, \bar{x})$, because of $\lambda_{min} \le \lambda_n \le 1$. Letting $\frac{1}{2}\phi(y, \bar{x}) = L(\bar{x}, y)$, Lemma 3.2 implies that $\bar{x} \in E(f)$.
Now, we prove that $\bar{x} \in SOL(C, A)$. Let $B \subset E \times E^*$ be defined as follows:

$$B\nu = \begin{cases} A\nu + N_C(\nu), & \nu \in C, \\ \emptyset, & \nu \notin C. \end{cases}$$

It follows from Theorem 2.1 of [15] that B is a maximal monotone operator and $B^{-1}(0) = SOL(C, A)$. Let $(\nu, w) \in G(B)$. Since $w \in B\nu = A\nu + N_C(\nu)$, we get $w - A\nu \in N_C(\nu)$. Because $p_n \in C$, utilizing the definition of $N_C(\nu)$ and Proposition 4 of [18], we get

$$\tau_n\langle \nu - p_n, w - A\nu\rangle \ge 0 \quad \& \quad \langle \nu - p_n, Jp_n - (Jx_n - \tau_n Ax_n)\rangle \ge 0. \tag{3.19}$$

Since $\tau_n \le 1$, using the definition of A and inequality (3.19), we have

$$\begin{aligned} \langle \nu - p_n, w\rangle &\ge \tau_n\langle \nu - p_n, A\nu\rangle - \langle \nu - p_n, Jp_n - Jx_n + \tau_n Ax_n\rangle \\ &= \tau_n\langle \nu - p_n, A\nu - Ap_n\rangle + \tau_n\langle \nu - p_n, Ap_n - Ax_n\rangle - \langle \nu - p_n, Jp_n - Jx_n\rangle \\ &\ge -(\tau_n\|Ax_n - Ap_n\| + \|Jx_n - Jp_n\|)\|\nu - p_n\| \\ &\to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

So, $\langle \nu - \bar{x}, w\rangle \ge 0$ and consequently $\bar{x} \in B^{-1}(0) = SOL(C, A)$.

Now, since $z_n \to \bar{x}$, from (3.13), we get $\bar{x} \in \tilde{F}(S)$. Thus, using the definition of S, we have $\bar{x} \in F(S)$. Set $u^* = \Pi_\Omega x_0$. Since $u^* \in C_n \cap D_n$, $x_{n+1} = \Pi_{C_n \cap D_n} x_0$ and ϕ is continuous with respect to the first argument, from proposition 4 of [18], we obtain

$$\phi(u^*, x_0) \geq \lim_{n\to\infty} \phi(x_{n+1}, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) = \phi(\bar{x}, x_0),$$

also, using Proposition 4 of [18], we have $\phi(u^*, x_0) \leq \phi(\bar{x}, x_0)$, because of $u^* = \Pi_\Omega x_0$ and $\bar{x} \in \Omega$. Therefore $\bar{x} = u^*$ and consequently the sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, $\{z_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$ and $\{t_n\}_{n=0}^\infty$ converge strongly to $\Pi_\Omega x_0$. □

4. A linesearch algorithm

As we see in the previous section, ϕ-Lipschitz-type condition $(A5)$ depends on two positive parameters c_1 and c_2. It is worth noting that, in some cases, these parameters are unknown or difficult to approximate. To overcome this drawback, using linesearch method, we modify Extragradient Algorithm. We prove the strong convergence of this new algorithm without assuming the ϕ-Lipschitz-type condition.

Here, we assume that bifunction $f : \Delta \times \Delta \to \mathbb{R}$ satisfies the conditions $(A1)$, $(A2)$ and $(A4)$ in which Δ is an open convex set containing C, and also it satisfies the following condition:
$(A3^*)$ f is jointly weakly continuous on $\Delta \times \Delta$, i.e., if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in Δ converge weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$.

Algorithm 2

Step 0: Let $\xi \in (0,1)$, $\gamma \in (0,1)$ and suppose that $\{\alpha_n\} \subseteq [a,e]$ for some $0 < a \leq e < 1$, $\{\beta_n\} \subseteq [d,b]$ where $0 < d \leq b < 1$, $0 < \tau_n \leq 1$ in which $\lim_{n\to\infty} \tau_n = 0$, $\{\lambda_n\} \subseteq [\lambda, 1]$ where $0 < \lambda \leq 1$ and $0 < \nu < \frac{c^2}{2}$ where $\frac{1}{c}$ $(0 < c \leq 1)$ is the 2-uniformly convexity constant of E.

Step 1: Let $x_0 \in C$. Set $n = 0$.

Step 2: Obtain the unique optimal solution y_n by solving the following convex optimization problem

$$\min_{y\in C}\{\lambda_n f(x_n, y) + \frac{1}{2}\phi(x_n, y)\}. \tag{4.1}$$

Step 3: If $y_n = x_n$, then set $z_n = x_n$. Otherwise

Step 3.1: Choose the smallest nonnegative integer m such that

$$f(z_{n,m}, x_n) - f(z_{n,m}, y_n) \geq \frac{\xi}{2\lambda_n}\phi(y_n, x_n) \text{ where } z_{n,m} = (1-\gamma^m)x_n + \gamma^m y_n. \tag{4.2}$$

Step 3.2: Set $\rho_n = \gamma^m$, $z_n = z_{n,m}$ and go to Step 4.

Step 4: Choose $g_n \in \partial_2 f(z_n, x_n)$ and compute $w_n = \Pi_C J^{-1}(Jx_n - \sigma_n g_n)$.

If $y_n \neq x_n$, then $\sigma_n = \dfrac{\nu f(z_n, x_n)}{\|g_n\|^2}$ and $\sigma_n = 0$ otherwise.

Step 5: Compute $p_n = \Pi_C J^{-1}(Jx_n - \tau_n A x_n)$ and

$$t_n = \Pi_C J^{-1}(\alpha_n J p_n + (1-\alpha_n)(\beta_n J w_n + (1-\beta_n) JSw_n)).$$

Step 6: Compute $x_{n+1} = \Pi_{C_n \cap D_n} x_0$, where
$D_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}$ and
$C_n = \{z \in C : \phi(z, t_n) \leq \phi(z, x_n) + \frac{4}{c^2} \alpha_n \tau_n^2 \|Ax_n\|^2\}$.
Step 7: Put n:=n+1, and go to Step 2.

In the following, we show that linesearch corresponding to x_n and y_n (Step 3.1) is well defined.

Lemma 4.1. *Let $y_n = x_n$ for some $n \in \mathbb{N} \cup \{0\}$. Then (i) there exists a nonnegative integer m such that the inequality in (4.2) is satisfied, (ii) $f(z_n, x_n) > 0$ and (iii) $0 \notin \partial_2 f(z_n, x_n)$.*

Proof. Let $n \geq 0$. Assume towards a contradiction that for each nonnegative integer m,

$$f(z_{n,m}, x_n) - f(z_{n,m}, y_n) < \frac{\xi}{2\lambda_n} \phi(y_n, x_n), \tag{4.3}$$

where $z_{n,m} = (1 - \gamma^m) x_n + \gamma^m y_n$. It is easy to see that $z_{n,m} \to x_n$ as $m \to \infty$. Using condition ($A3^*$), we obtain $f(z_{n,m}, x_n) \to f(x_n, x_n)$ and $f(z_{n,m}, y_n) \to f(x_n, y_n)$ as $m \to \infty$. Since $f(x_n, x_n) = 0$, letting $m \to \infty$ in inequality (4.3), we get

$$0 \leq f(x_n, y_n) + \frac{\xi}{2\lambda_n} \phi(y_n, x_n). \tag{4.4}$$

Because of y_n is a solution of (4.1), we deduce

$$\lambda_n f(x_n, y) + \frac{1}{2} \phi(y, x_n) \geq \lambda_n f(x_n, y_n) + \frac{1}{2} \phi(y_n, x_n),$$

for all $y \in C$. If $y = x_n$, then

$$\lambda_n f(x_n, y_n) + \frac{1}{2} \phi(y_n, x_n) \leq 0. \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$f(x_n, y_n) + \frac{1}{2\lambda_n} \phi(y_n, x_n) \leq f(x_n, y_n) + \frac{\xi}{2\lambda_n} \phi(y_n, x_n).$$

Therefore, $\frac{1-\xi}{2} \phi(y_n, x_n) \leq 0$, since $\lambda_n \leq 1$. It follows from (2.3) that $\phi(y_n, x_n) > 0$, because of $y_n \neq x_n$. Thus, $1 - \xi \leq 0$ which contradicts the assumption $\xi \in (0, 1)$. So, (i) is proved.

Now, we prove (ii). Since f is convex, we obtain

$$\rho_n f(z_n, y_n) + (1 - \rho_n) f(z_n, x_n) \geq f(z_n, z_n) = 0. \tag{4.6}$$

Consequently from (4.6), since $y_n \neq x_n$, we get

$$f(z_n, x_n) \geq \rho_n [f(z_n, x_n) - f(z_n, y_n)] \geq \frac{\xi \rho_n}{\lambda_n} \phi(y_n, x_n) > 0.$$

So, $f(z_n, x_n) > 0$. The proof of (iii) can be found in [32, Lemma 4.5]. □

Remark 4.2. If E is a real Hilbert space, then Lemma 4.1 is reduced to Proposition 4.1 in [34] when $\xi \in (0, 1)$.

Lemma 4.3. *Suppose that $f : \Delta \times \Delta \to \mathbb{R}$ is a bifunction satisfying conditions ($A3^*$) and ($A4$). Let $\{x_n\}$ and $\{z_n\}$ be two sequences in Δ such that $x_n \rightharpoonup \bar{x}$ and $z_n \rightharpoonup \bar{z}$, where $\bar{x}, \bar{z} \in \Delta$. Then, for any $\varepsilon > 0$, there exist $\eta > 0$ and $n_\varepsilon \in \mathbb{N}$ such that $\partial_2 f(z_n, x_n) \subseteq \partial_2 f(\bar{z}, \bar{x}) + \frac{\varepsilon}{\eta} B$, for all $n \geq n_\varepsilon$, in which B is the closed unit ball in E.*

Proof. Using a similar argument as in the proof of Proposition 4.3 of [34] for a Banach space E, we can get the result. $\square$

Proposition 4.4. *For each $x^* \in E(f) \cap F(S)$ and each $n \in \mathbb{N} \cup \{0\}$, we get*

(i) $\phi(x^*, w_n) \leq \phi(x^*, x_n) - (\frac{2}{\nu} - \frac{4}{c^2})\sigma_n^2 \|g_n\|^2$,

(ii) $\phi(x^*, t_n) \leq \phi(x^*, x_n) + \frac{4}{c^2}\alpha_n \tau_n^2 \|Ax_n\|^2 - (1-\alpha_n)(\frac{2}{\nu} - \frac{4}{c^2})\sigma_n^2 \|g_n\|^2$.

Proof. Using Proposition 5 of [18], the definition of V and inequality (2.4), we have

$$\begin{aligned} \phi(x^*, w_n) &= \phi(x^*, \Pi_C J^{-1}(Jx_n - \sigma_n g_n)) \\ &\leq V(x^*, Jx_n - \sigma_n g_n + \sigma_n g_n) - 2\langle J^{-1}(Jx_n - \sigma_n g_n) - x^*, \sigma_n g_n \rangle \\ &= \phi(x^*, x_n) - 2\sigma_n \langle x_n - x^*, g_n \rangle + 2\langle J^{-1}(Jx_n - \sigma_n g_n) - x_n, -\sigma_n g_n \rangle. \end{aligned} \tag{4.7}$$

Since $g_n \in \partial_2 f(z_n, x_n)$, we get

$$\langle x_n - x^*, g_n \rangle \geq f(z_n, x_n) - f(z_n, x^*) \geq \frac{\sigma_n \|g_n\|^2}{\nu}.$$

Therefore,

$$-\frac{2}{\nu}\sigma_n^2 \|g_n\|^2 \geq -2\sigma_n \langle x_n - x^*, g_n \rangle. \tag{4.8}$$

On the other hand, utilizing Lemma 2.1 of [35], we get

$$2\langle J^{-1}(Jx_n - \sigma_n g_n) - x_n, -\sigma_n g_n \rangle \leq \frac{4}{c^2}\sigma_n^2 \|g_n\|^2. \tag{4.9}$$

Thus, combining inequalities (4.8) and (4.9), we can derive (i). Applying a similar argument as in the proof of Lemma 3.7 and using (i), we can prove (ii). $\square$

Theorem 4.5. *Assume that $\Omega := E(f) \cap SOL(C, A) \cap F(S) \neq \emptyset$, then the sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, $\{z_n\}_{n=0}^\infty$, $\{w_n\}_{n=0}^\infty$, $\{p_n\}_{n=0}^\infty$ and $\{t_n\}_{n=0}^\infty$ generated by Algorithm 2 converge strongly to the same solution $u^* \in \Omega$, where $u^* = \Pi_\Omega x_0$.*

Proof. Let $x^* \in \Omega$. Similar to the proof of Theorem 3.8, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \quad \& \quad \lim_{n\to\infty} \|x_n - t_n\| = 0 \quad \& \quad \lim_{n\to\infty} \|x_n - p_n\| = 0,$$

which imply that $\{x_n\}$ and consequently $\{t_n\}$ and $\{p_n\}$ converge strongly to $\bar{x} \in C$ and $\lim_{n\to\infty} (\phi(x^*, x_n) - \phi(x^*, t_n)) = 0$. Since $(1-\alpha_n)(\frac{2}{\nu} - \frac{4}{c^2}) > 0$, it follows from Lemma 4.4 (ii) that $\lim_{n\to\infty} \sigma_n \|g_n\| = 0$.

Now, suppose that $A(y) = \lambda_n f(x_n, y) + \frac{1}{2}\phi(y, x_n)$, for all $y \in C$. Similar to the proof of Theorem 3.2 in [16], we can prove that $\{y_n\}$ and $\{z_n\}$ converge strongly to $\bar{x} \in E(f)$ and also we can show that $\bar{x} \in SOL(C, A) \cap F(S)$ in which $\bar{x} = \Pi_\Omega x_0$. Using the analogous argument such as Theorem 3.8, we can establish $\{w_n\}$ converge strongly to $\bar{x}$. $\square$

5. Numerical examples

In this section, we illustrate theorems 3.8 and 4.5 with numerical examples. Also, we investigate the behavior of the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{p_n\}$, $\{t_n\}$ and $\{w_n\}$ generated by algorithms 1 and 2. We have solved the optimization subproblems in algorithms 1 and 2 with the solver FMINCON from optimization toolbox in MATLAB software.

5.1. **Numerical example in $\mathbb{R}^k$.**

Now, we give an example in $\mathbb{R}^k$. In the one dimensional case, we present some figures and tables to clarify our results.

Example 5.1. Let $k \in \mathbb{N}$, $E = \mathbb{R}^k$ and $C = \underbrace{[-2,2] \times [-2,2] \times ... \times [-2,2]}_{k\ times}$. For all $x = (x_1, x_2, ..., x_k)$ and $y = (y_1, y_2, .., y_k) \in \mathbb{R}^k$, we define $\|x\| = (\sum_{i=1}^k |x_i|^2)^{\frac{1}{2}}$, $\langle x, y\rangle = \sum_{i=1}^k x_i y_i$ and $f(x,y) := 7\|y\|^2 + \langle x, y\rangle - 8\|x\|^2 = \sum_{i=1}^k (7y_i^2 + x_i.y_i - 8x_i^2)$. It is readily seen that f satisfies the conditions $(A1) - (A3)$ and also it satisfies in the conditions $(A4)$ and $(A5)$ as follows:

(A4) Since $\partial_2 f(x,y) = 14y + x = (14y_1 + x_1, 14y_2 + x_2, ..., 14y_k + x_k)$, thus $f(x,.)$ is subdifferentiable on C for each $x \in C$.

(A5) Since $\phi(y,x) = \|y - x\|^2 = (\sum_{i=1}^k |y_i - x_i|^2)^{\frac{1}{2}}$, we get

$$f(x,y) + f(y,z) \geq f(x,z) - \frac{1}{2}\|y - x\|^2 - \frac{1}{2}\|y - z\|^2,$$

i.e., f satisfies the ϕ-Lipschitz-type condition with $c_1, c_2 = \frac{1}{2}$.

Now, we define $S : C \to C$ by

$$Sx = \frac{x}{5} = \frac{1}{5}(x_1, x_2, ..., x_k)$$

for all $x \in C$, so $F(S) = \{0, 0, ..., 0\}$.

Table 1. Numerical results for extragradient and linesearch algorithms in $\mathbb{R}$.

	Numerical results for Algorithm 1					*Numerical results for Algorithm 2*					
n	x_n	y_n	z_n	p_n	t_n	x_n	y_n	z_n	p_n	t_n	w_n
0	2.000	0.5000	0.5750	1.9216	1.5313	2.000	0.5000	1.7000	1.9216	1.7436	1.9303
1	1.7853	0.4463	0.5133	1.7166	1.2940	1.9078	0.4769	1.6216	1.8344	1.6155	1.8413
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
19	0.0620	0.0155	0.0178	0.0602	0.0369	0.2626	0.0657	0.2232	0.2551	0.1965	0.2535
20	0.0496	0.0124	0.0143	0.0482	0.0294	0.2305	0.0576	0.1959	0.2240	0.1722	0.2225
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
30	0.0051	0.0013	0.0015	0.0049	0.0030	0.0602	0.0150	0.0511	0.0587	0.0444	0.0581
31	0.0040	0.0010	0.0012	0.0039	0.0023	0.0131	0.0524	0.0446	0.0512	0.0387	0.0506
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
45	...	...	...	...	...	0.0086	0.0025	0.0073	0.0084	0.0063	0.0083
46	...	...	...	...	...	0.0074	0.0021	0.0063	0.0073	0.0054	0.0072

It is easy to see that S is weakly relatively nonexpansive mapping. Moreover, if for each $y \in C$, $f(x,y) \geq 0$, then

$$E(f) = \{x = (x_1, x_2, ..., x_k) \in C : \ \|x\|^2 = \sum_{i=1}^{k} |x_i|^2 = 0\} = \{(0,0,...,0)\}.$$

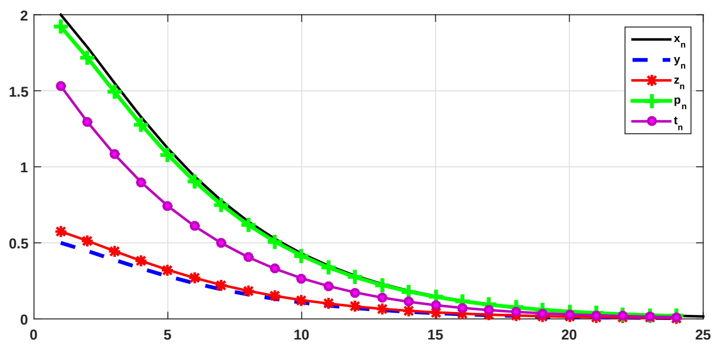
Also, we define $A : C \to \mathbb{R}^k$ by $A = 2I$ and we suppose that $\alpha = \frac{1}{2}$ and $c = 1$, so

$$\{(0,0,...,0)\} \subseteq SOL(C,A)$$

$$= \left\{ x = (x_1, x_2, ..., x_k) \in C : \ 2\langle u, y-u \rangle = 2\sum_{i=1}^{k} u_i y_i - u_i^2 \geq 0 \right\}.$$

Consequently, $\Omega = E(f) \cap SOL(C,A) \cap F(S) = \{(0,0,....,0)\}$.
Now, we assume that $\alpha_n = \frac{1}{2} + \frac{1}{3+n}$, $\beta_n = \frac{1}{3} + \frac{1}{4+n}$, $\tau_n = \frac{1}{n+50}$ and $\lambda_n = \frac{1}{6}$ for all $n \geq 0$.

(a) Extragradient Algorithm



(b) Linesearch Algorithm

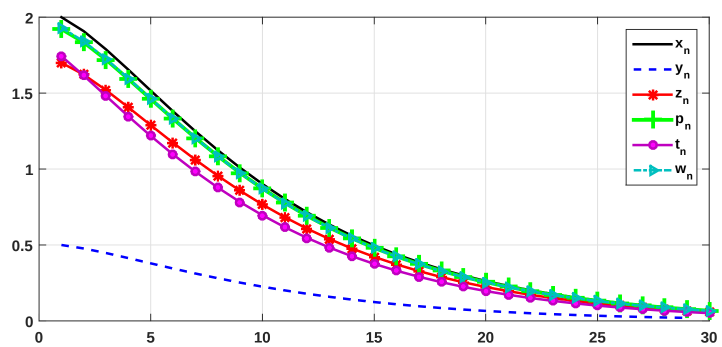


FIGURE 1. The convergence behavior of the generated sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{p_n\}$, $\{t_n\}$ and $\{w_n\}$ by extragradient and linesearch algorithms.

Case I. Extragradient Algorithm: From (3.1) and (3.2), we get $y_{n_i} = \frac{1}{4}x_{n_i}$ and $z_{n_i} = \frac{23}{20}y_{n_i} = \frac{23}{80}x_{n_i}$, for $i = 1, 2, ..., k$, therefore

$$\|p_n - ((1-2\tau_n)x_n)\|^2 = \min_{z\in C} \|z - ((1-2\tau_n)x_n)\|^2,$$

$$t_{n_i} = \alpha_n p_{n_i} + (1-\alpha_n)[\beta_n z_{n_i} + \frac{1}{5}(1-\beta_n)z_{n_i}],$$

and

$$\|x_{n+1} - x_0\|^2 = \min_{z\in C_n\cap D_n} \|z - x_0\|^2,$$

where

$$\begin{cases} D_n = \{z \in C : \sum_{i=1}^{k}(x_{n_i} - z_i)(x_{0_i} - x_{n_i}) \geq 0\} \\ \& \\ C_n = \{z \in C : \|t_n - z\|^2 \leq \|x_n - z\|^2 + \frac{16}{c^2}\alpha_n\tau_n^2\|x_n\|^2\}. \end{cases}$$

Since $\Omega = \{(0, 0, ..., 0)\}$, we get $\Pi_\Omega(x_0) = (0, 0, ..., 0)$.

Case II. Linesearch Algorithm: In this case x_n, t_n and p_n are the same as Extragradient Algorithm. Assume that $\xi = \frac{1}{2}$, $\gamma = 0.2$ and $\nu = \frac{1}{4}$. So, $y_{n_i} = \frac{1}{4}x_{n_i}$, and m is the smallest nonnegative integer such that

$$11\|x_n\|^2 + \langle x_n, 2z_n + 6y_n\rangle \geq 17\|y_n\|^2 + 2\langle z_n, y_n\rangle,$$

where $z_n = z_{n,m} = (1-(0.2)^m)x_n + (0.2)^m y_n$. Also, $g_n = 14x_n + z_n$ and

$$\|w_n - (J^{-1}(Jx_n - \sigma_n g_n)\| = \min_{z\in C} \|z - (J^{-1}(Jx_n - \sigma_n g_n)\|.$$

Since $y_n \neq x_n$, then $\sigma_n = \dfrac{7\|x_n\|^2 + \langle z_n, x_n\rangle - 8\|z_n\|^2}{4\|g_n\|^2}$.

Numerical results for the algorithms 1 and 2 show that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{p_n\}$, $\{t_n\}$ and $\{w_n\}$ converge strongly to 0. See Table 5.1 and Figure 1 with $k = 1$, starting point $x_0 = 2$ and stopping criterion $\|x_{n+1} - x_n\| < 10^{-3}$. Also, since the CPU time to get a solution in Extragradient method is $8.254\, s$ and in Linesearch method is $11.311\, s$, we see that the speed of convergence of the generated sequences by Extragradient Algorithm is faster than Linesearch Algorithm.

Let $O(n) = \left|\dfrac{ln(\frac{\|x_{n+1}\|}{\|x_n\|})}{ln(\frac{n+1}{n})}\right|$ for all $n \geq 0$, where $O(n)$ is the order of convergence of $\{x_n\}$ for all $n \geq 0$. Now, we consider Example 5.1 for Extragradient Algorithm and obtain $O(n)$ for the starting points as follows:

1. $(-1, 1) \in \mathbb{R}^2$
2. $(-1.8, 1, 1.9) \in \mathbb{R}^3$
3. $(-1, 0.5, 1.75, -1.5) \in \mathbb{R}^4$
4. $(-1, -1.5, 1.25, 0.6, -0.75) \in \mathbb{R}^5$

Table 2 shows that by increasing n, convergence order of $\{x_n\}$ increases. Similarly, we can show that the order of convergence of the sequences $\{y_n\}$, $\{z_n\}$, $\{p_n\}$ and $\{t_n\}$ increases. This means our Extragradient Algorithm has good efficiency.

It is worth emphasizing that the condition $\lim\limits_{n\to\infty} \tau_n = 0$ is necessary in our Extragradient Algorithm. Because, if we put $\tau_n = \frac{1}{3} + \frac{1}{n+50}$ for all $n \geq 0$, then $\lim\limits_{n\to\infty} \tau_n = \frac{1}{3} \neq 0$, and $O(n) = 0$ for all $n \geq 0$. i.e., the generated sequences will not converge to 0.

TABLE 2. Order of convergence of $\{x_n\}$ generated by Algorithm 1

n	4	10	14	20	25
$O_i(n)$	0.7596	2.1581	3.1284	4.5999	5.8317

It should be noted that we can derive similar results for our Linesearch Algorithm.

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