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A DOMAIN-THEORETIC BISHOP-PHELPS THEOREM

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Abstract. In this paper, the notion of c-support points of a set in a semitopological cone is introduced. It is shown that any nonempty convex Scott closed bounded set has a c-support point in a cancellative bd-cone under certain condition. We also introduce the notion of wd-cone and then we prove a Bishop-Phelps type theorem for wd-cones, especially for normed cones, under appropriate conditions. Finally, using of the Bishop-Phelps technique, we obtain a result about the fixed points of a mapping on s-cones.
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1. Introduction

Domain theory which is based on logic and computer science, started as an outgrowth of theories of order. Progress in this domain rapidly required a lot of material on (non-Hausdorff) topologies. After about 40 years of domain theory, one is forced to recognize that topology and domain theory have been beneficial to each other [5, 7].

One of Klaus Keimel's many mathematical interests is the interaction between order theory and functional analysis. In recent years this has led to the beginnings of a domain-theoretic functional analysis, which may be considered to be a topic within positive analysis in the sense of Jimmie Lawson [11]. In the latter, notions of positivity and order play a key role, as do lower semicontinuity and so T_0 spaces. Some basic functional analytic tools were developed by Roth and Tix and later by Plotkin and Keimel for these structures. Roth has written several papers in this area including his papers [13, 14] on Hahn-Banach type theorems for locally convex cones. Tix in her 1999 Ph.D. thesis gave a domain-theoretic version of these theorems in the framework of d-cones (see [17, 18]). Plotkin subsequently gave another separation theorem, which was incorporated, together with other improvements, into a revised version

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of Tix's thesis [19, 12]. Finally, Keimel [9] improved the Hahn-Banach theorems to semitopological cones.

The theory of locally convex cones, with applications to Korovkin type approximation theory for positive operators and to vector-measure theory, was developed in the books by Keimel and Roth [10] and Roth [15], respectively.

The Bishop-Phelps theorem [3] is a fundamental theorem in functional analysis which has many applications in the geometry of Banach spaces, fixed point theory and optimization (for instance see [8, 4]). The classical Bishop-Phelps theorem states that "the set of support functionals for a closed bounded convex subset B of a real Banach space X is norm dense in X^* and the set of support points of B is dense in the boundary of B" [3]. The present paper contributes a domain-theoretic analogue of the classical Bishop-Phelps theorem for semitopological cone.

The work on Hahn-Banach-type theorems has found application in theoretical computer science, *viz*. the study of powerdomains. It was a pleasant surprise that the separation theorems found application in this development and we anticipate that so too will the domain-theoretic Bishop-Phelps theorem given here. As an application of the Bishop-Phelps theorem, we show that a mapping on a *wd*-cone has a fixed point under some conditions.

2. Preliminaries

For convenience of the reader we give a survey of the relevant materials from [1], [2], [7] and [9], without proofs, thus making our exposition self-contained.

Let B be a nonempty subset of a real Banach space X and f be a nonzero continuous linear functional on X. If f attains either its maximum or its minimum over B at the point $x \in B$, we say that f supports B at x and that x is a *support point* of B.

For subsets A of a partially ordered set P we use the following notations:

$$\downarrow A =: \{x \in P | x \leq a \text{ for some } a \in A\},$$
$$\uparrow A =: \{x \in P | x \geq a \text{ for some } a \in A\}.$$

It is called that A is a lower or upper set, if $\downarrow A = A$ or $\uparrow A = A$, respectively.

We denote by $\mathbb{R}_+$ the subset of all nonnegative reals. Further, $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$. Addition, multiplication and the order are extended to $+\infty$ in the usual way. In particular, $+\infty$ becomes the greatest element and we put $0 \cdot (+\infty) = 0$.

According to [9], a *cone* is a set C, together with two operations $+ : C \times C \to \mathrm{C}$ and $\cdot : \mathbb{R}_+ \times C \to C$ and a neutral element $0 \in C$, satisfying the following laws for all $v, w, u \in C$ and $\lambda, \mu \in \mathbb{R}_+$:

$$\begin{array}{ll} 0 + v = v, & 1v = v, \\ v + (w + u) = (v + w) + u, & (\lambda\mu)v = \lambda(\mu v), \\ v + w = w + v, & (\lambda + \mu)v = \lambda v + \mu v, \\ & \lambda(v + w) = \lambda v + \lambda w. \end{array}$$

An *ordered cone* C is a cone endowed with a partial order $\leq$ such that the addition and multiplication by fixed scalars $r \in \mathbb{R}_+$ are order preserving, that is, for all $x, y, z \in C$ and all $r \in \mathbb{R}_+$:

$$x \leq y \Rightarrow x + z \leq y + z \quad \text{and} \quad rx \leq ry.$$

Let us recall that a *linear function* from a cone $(C, +, \cdot)$ to a cone $(C', +, \cdot)$ is a function $f : C \to C'$ such that $f(v + w) = f(v) + f(w)$ and $f(\lambda v) = \lambda f(v)$, for all $v, w \in C$ and $\lambda \in \mathbb{R}_+$.

A subset D of a cone C is said to be *convex* if for all $u, v \in D$ and $\lambda \in [0, 1], \lambda u + (1 - \lambda)v \in D$. The convex hull of a set D is defined to be the smallest convex set containing D.

For example, $(\mathbb{R}_+)^n$ is a cone, with respect to the coordinate-wise operations. On $\mathbb{R}_+$, the cone order is just the usual order $\leq$ of the reals. On $(\mathbb{R}_+)^n$, it is the coordinate-wise order.

Recall that a partially ordered set $(A, \leq)$ is called directed if for every $a, b \in A$ there exists $c \in A$ with $a, b \leq c$. A partially ordered set $(D, \leq)$ is called a directed complete partial order (*dcpo*) if every directed subset A of D, has a least upper bound in D. The least upper bound of a directed subset A is denoted by $\sqcup^{\uparrow} A$, and it is also called the directed supremum.

In any partially ordered set P, the *way-below* relation $x \ll y$ is defined by: $x \ll y$ iff, for any directed subset $D \subset P$ for which supremum of D exists, the relation $y \leq \sqcup^{\uparrow} D$ implies the existence of a $d \in D$ with $x \leq d$. An element $y \in P$ is called finite if, $y \ll y$.

The partially ordered set P is called continuous if, for every element y in P, the set $\Downarrow y =: \{x \in P; x \ll y\}$ is directed and $y = \sqcup^{\uparrow} \Downarrow y$. Note that $x \ll y$ implies $x \leq y$ [7, Prop. 5.1.4].

Any T_0-space X comes with an intrinsic order, the *specialization order* which is defined by $x \leq y$ if the element x is contained in the closure of the singleton $\{y\}$ or, equivalently, if every open set containing x also contains y.

Given any ordering $\leq$, there are at least two topologies with $\leq$ as specialization ordering, the coarsest possible one (the upper topology) and the finest possible one (the Alexandroff topology) (see [7, Sec. 4.2.2] for more details). Additionally, there are some other interesting topologies in between. An important example of a topology that sits in between is the Scott topology.

Let D be a partially ordered set. A subset A is called Scott closed if it is a lower set and is closed under supremum of directed subsets, as far as these suprema exist. Complements of Scott closed sets are called Scott open. The collection of all Scott open sets is a topology, called the *Scott topology* on D [7, Prop. 4.2.18]. We write D_σ for the set D with the Scott topology.

The basic notion is that of a *Scott continuous* function: A function f from a partially ordered set P to a partially ordered set Q is called Scott continuous if it is order preserving and if, for every directed subset D of P which has a least upper bound in P, the image $f(D)$ has a least upper bound in Q and $f(\sqcup^{\uparrow} D) = \sqcup^{\uparrow} f(D)$.

Let P, Q be two partially ordered sets. A map $f : P_\sigma \to Q_\sigma$ is continuous iff $f : P \to Q$ is Scott continuous [7, Prop. 4.3.5].

In a continuous partially ordered set C, the set $\Uparrow x$ is Scott open for all x. More generally, for every subset E of C, the subset $\Uparrow E$ is open in σ_C [7, Prop. 5.1.16], so $\Uparrow E \subset int(\uparrow E)$. If the subset $E \subset C$ is finite, then $\Uparrow E = int(\uparrow E)$ [7, Prop. 5.1.35].

On the extended reals $\overline{\mathbb{R}}$ and on its subsets $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ we use the *upper topology*, the only open sets for which are the open intervals $\{s : s > r\}$. This upper topology is T_0, but far from being Hausdorff.

2.1. **Semitopological Cones.** According to [9], a *semitopological cone* is a cone with a T_0-topology such that the addition and scalar multiplication are separately continuous, that is:

$$\begin{array}{ll} a \mapsto ra : C \to C & \text{is continuous for every fixed } r > 0, \\ r \mapsto ra : \mathbb{R}_+ \to C & \text{is continuous for every fixed } a \in C, \\ b \mapsto a + b : C \to C & \text{is continuous for every fixed } a \in C. \end{array}$$

An *s-cone* is a cone with a partial order such that addition and scalar multiplication:

$$(a, b) \mapsto a + b : C \times C \to C, \quad (r, a) \mapsto ra : \mathbb{R}_+ \times C \to C$$

are Scott continuous. A s-cone is called a [b]d*-cone* if its order is [bounded] directed complete, i.e., if each [upper bounded] directed subset has a least upper bound.

Note that every s-cone is a semitopological cone with respect to its Scott topology [9, Prop. 6.3].

A cone C with a topology is called *locally convex*, if each point has a neighborhood basis of open convex neighborhoods.

Let C be a semitopological cone. The cone C^* of all linear continuous functionals $f : C \to \overline{\mathbb{R}}_+$ are called dual of C.

We shall use the following *separation theorem* [9, Theorem 9.1]: in a semitopological cone C consider a nonempty convex subset A and an open convex subset U. If A and U are disjoint, then there exists a continuous linear functional $f : C \to \overline{\mathbb{R}}_+$ such that $f(a) \leq 1 < f(u)$ for all $a \in A$ and all $u \in U$.

Finally, we shall use the following *strict separation theorem* [9, Theorem 10.5]: let C be a locally convex semitopological cone. Suppose that K is a compact convex set and that A is a nonempty closed convex set disjoint from K. Then there is a continuous linear functional f and an r such that $f(b) \geq r > 1 \geq f(a)$ for all b in K and all a in A.

2.2. **Normed Cones.** A cancellative cone (more precisely cancellative asymmetric cone) is a cone C, satisfying the following laws for all $v, w, u \in C$:

$$\begin{array}{ll} v + u = w + u \Rightarrow v = w, & (\textit{cancellation}) \\ v + w = 0 \Rightarrow v = w = 0. & (\textit{strictness}) \end{array}$$

Let C be a cancellative cone, we define a partial order on C by $x \preccurlyeq y \Leftrightarrow y \in x + C$, called the *cone order* on C.

According to [16], a norm on a cancellative cone C is a function $\| \cdot \|: C \to \mathbb{R}_+$ satisfying the following conditions for all $v, w \in C$ and $\lambda \in \mathbb{R}_+$:

$$\begin{aligned} &\| v + w \| \leq \| v \| + \| w \|, \\ &\| \lambda v \| = \lambda \| v \|, \\ &\| v \| = 0 \Rightarrow v = 0, \\ &v \preccurlyeq w \Rightarrow \| v \| \leq \| w \| . \end{aligned}$$

A *normed cone* $C = \langle C, \| \cdot \| \rangle$ is a cancellative cone equipped with a norm. The *unit ideal* of a normed cone C is the set

$$U_C = \{u \in C; \| u \| \leq 1\}.$$

A normed cone C is called *complete* if its unit ideal is a dcpo. For example the normed cones $\mathbb{R}_+, \mathbb{R}_+^n$, l_∞^+ (the set of all bounded sequences in $\mathbb{R}_+$) together with the supremum norm $\|(x_i)_i\| = \sup_i x_i$ and l_1^+ (the set of all sequences in $\mathbb{R}_+$ of bounded sum) together with the sum norm $\|(x_i)_i\| = \sum_i x_i$ are all complete and continuous [16, Exam. 2.7]. We will say simply continuous normed cone for continuous complete normed cone.

3. Main results

The purpose of this section is to establish the Bishop-Phelps type theorem for semitopological cones. Indeed we want to study the Bishop-Phelps theorem in non-Hausdorff setting.

Remark 3.1. (a_1) Let B be a nonempty Scott closed set in a semitopological cone C. Since $0 \in B$, so for any linear functionals $f : C \to \overline{\mathbb{R}}_+$ we have $f(0) = \inf f(B)$.

(a_2) If B is a nonempty compact set in a semitopological cone C and $f : C \to \overline{\mathbb{R}}_+$ is a continuous map, then there is an element $z \in B$ such that $f(z) = \inf f(B)$ [6, Lemma 3.8]. Since in a semitopological cone, a compact set is not necessarily closed, so the proof of this statement is different from the method of the classical analysis and the result is not true for supremum (for details see [6]).

Note that a subset B in a partially ordered set C, is called *bounded* if there exists an element $d \in C$ such that for any $b \in B$, $b \leq d$.

Let C be an s-cone. We will say that C has the *additive property*, if the following axioms are satisfied:
(i) $x' \ll x$ and $y' \ll y$ implies $x' + y' \ll x + y$.
(ii) $x \ll \lambda x$ for any scalar $\lambda > 1$ and $x > 0$.

Example 3.2. (b_1) $\mathbb{R}_+^n$ is a cancellative continuous *bd*-cone with the ordering:

$$(x_1, ..., x_n) \leq (y_1, ..., y_n) \Longleftrightarrow \forall n \;\; x_n \leq y_n.$$

In $\mathbb{R}_+^n$, we have $(x_1, ..., x_n) \ll (y_1, ..., y_n)$ iff for all i, $x_i = 0$ or $x_i < y_i$. It is easy to see that $\mathbb{R}_+^n$ has the additive property.

(b_2) The cones ℓ_1^+ and ℓ_∞^+ are cancellative continuous *bd*-cones under usual point-wise ordering which also have the additive property.

It is known that the interior of a convex set in a topological linear space is a convex set, but this is not true in semitopological cones in general [9].

Remark 3.3. (c_1) In a continuous s-cone C, which has the additive property, the interior of every convex upper set is convex [9, Lemmas 4.10 & 6.14].

(c_2) In a continuous normed cone C, for any convex set B, the open set $\Uparrow B$, is convex [16, Lemma 2.16].

(c_3) In a continuous s-cone C, which has the additive property, the interior of every upper set is nonempty. To see this, let A be an upper set. For $x \in A$ consider $\Uparrow x$, which is a nonempty open set in A, so $int(A)$ is nonempty.

Proposition 3.4. *Let C be a continuous s-cone which has the additive property, and $B \subset C$ be an upper convex set. If $x \in B$ such that $\lambda x \notin B$, whenever $\lambda < 1$, then there exists a continuous linear functional $f : C \to \overline{\mathbb{R}}_+$ such that $f(x) = \inf f(B)$.*

Proof. Let $x \in B$. By the assumption, $\lambda x \notin B$ for every $0 < \lambda < 1$, so $x \notin int(B)$. The interior of B is a nonempty convex open set, therefore by the separation theorem, there exists a continuous linear functional $f : C \to \overline{\mathbb{R}}_+$ such that $f(x) \leq f(b)$ for all $b \in int(B)$. By continuity of C, for each $y \in B$, $\Uparrow y =: \{a : y \ll a\}$ is an open set in B and so it is included in $int(B)$, so $f(x) \leq f(a)$ for all $a \in \Uparrow y$. By the additive property $f(x) \leq f(y)$ for each $y \in B$, hence $f(x) = \inf f(B)$. □

In the sequel, we consider suprema instead of infima. Let B be a convex closed set in a semitopological cone C. A point $x \in B$ is called a *c-support point* for B, if there exists a linear continuous functional $f : C \to \overline{\mathbb{R}}_+$ such that $f(x) = \sup f(B)$ and $f(x) < \infty$; such a functional f is said a *c-support functional*.

Remark 3.5. Let B be a convex closed set in a semitopological cone C. Then we have the following facts:

- (d_1) If the set B has a maximum, then any linear continuous functional $f : C \to \overline{\mathbb{R}}_+$ is a c-support functional for B.
- (d_2) If the set B has nonempty interior, then B is an unbounded set, so any linear continuous functional $f : C \to \overline{\mathbb{R}}_+$ on B is unbounded.
- (d_3) If C is a d-cone and B is a directed Scott closed set, then $\sqcup^{\uparrow} B \in B$, and so every linear Scott continuous functional is a c-support functional for B.

Now we restrict our attention to the case that B is a nonempty convex closed set with empty interior. To establish the Bishop-Phelps theorem for semitopological cones, we need a discussion of certain cones:

Let C be a cancellative semitopological cone and $f : C \to \overline{\mathbb{R}}_+$ be a continuous linear functional. For $0 < \delta < 1$ and $d \in C$, we define

$$K(f, \delta, d) = \{x \in C : f(x) < \infty \text{ and } \delta x \leq f(x)d\}.$$

Note that $K(f, \delta, d)$ is a convex subcone of C. Since C is a cancellative cone, the order $x \sqsubseteq y \Leftrightarrow y \in\ x + K$, defines a partial order on C, which is called the *subcone order* on C. If $x \sqsubseteq y$, then we sometimes write $x - y$ for the unique element z such that $x + z = y$.

Lemma 3.6. *Let C be a cancellative semitopological cone. Then for every $x, y \in C$ we have*

$$x \sqsubseteq y \ (y \in x + K) \Rightarrow x \leq y \ (\textit{with the specialization order}).$$

Proof. Let $x \sqsubseteq y$. For some $z \in K$, $y = x + z$. By the definition of semitopological cone, we know that the function $S : b \mapsto x + b : C \to C$ is continuous. So $S(\overline{\{z\}}) \subset \overline{S(z)}$ and then $x \in \overline{\{y\}}$ and so $x \leq y$. □

Now we investigate the first part of the Bishop-Phelps type theorem for *bd*-cones.

Let X be a partially ordered set, with ordering $\leq$. The specialization ordering of the Scott topology is the original ordering $\leq$ [7, Prop. 4.2.18].

Remark 3.7. Note that in a continuous cancellative semitopological cone C, we have $x \notin \Uparrow x$, in fact there is no finite element in C. Because, if $x \in \Uparrow x$, since C is a continuous cone, so $B := \Uparrow x$ is an open set. With considering the directed set $A := \{\lambda x \ : \ \lambda \in \mathbb{R}_+, \lambda < 1\}$ which its supremum is in B. Since C is cancellative, it follows that $A \cap B = \varnothing$, and this contrary to opening of $\Uparrow x$.

Theorem 3.8. *Let B $(\neq \{0\})$ be a nonempty convex dcpo in a continuous cancellative bd-cone C, where C has the additive property. Then there exists some $m \in B$ such that $B \cap (m + C) = \{m\}$. Such an m is also a c-support point for B.*

Proof. It is easy to see that the partially ordered set $(B, \leq)$ has a maximal element, in fact by Zorn's lemma, it suffices to prove that every chain in $(B, \leq)$ has an upper bound in B.

By Lemma 3.6 $(B, \preccurlyeq)$ (with the cone order) has a maximal element, say m. It follows that $B \cap \uparrow m = \{m\}$ or $B \cap \ (m + \ C) = \ \{m\}$. Since C is continuous and has the additive property, so $\Uparrow m$ is a nonempty convex open set, hense $m \notin \Uparrow m$ and $B \cap (\Uparrow m) = \varnothing$. By the separation theorem there exists a Scott continuous linear functional $f : C \to \overline{\mathbb{R}}_+$ satisfying $f(b) \leq f(y)$ for all $b \in B$ and $y \in \Uparrow m$. So $f(b) \leq f(\lambda m)$ for all $b \in B$ and $\lambda > 1$. Therefore, m is a c-support point. □

3.1. ***wd*-Cones.** In this section we introduce and study the notion of *wd*-cones.

Definition 3.9. Let C be a semitopological cone. The net $\{x_\alpha\}$ is Cauchy if there exists $0 < d \in C$, satisfying the condition that, for any $\epsilon > 0$ there exists α_0, such that for $\alpha, \beta \geq \alpha_0$, $x_\alpha \leq x_\beta + \epsilon d$ and $x_\beta \leq x_\alpha + \epsilon d$.

In the sequel, by an order on a semitopological cone we will always mean the specialization order $\leq$, if not specified otherwise.

Definition 3.10. A *s*-cone C is called a *wd*-cone, if each increasing Cauchy net, has a least upper bound.

Clearly every *bd*-cone is a *wd*-cone. The following example shows that the converse does not hold in general.

Example 3.11. Let $C^+[0,1]$ denote the cone of all continuous functions $f : [0,1] \to \mathbb{R}_+$, which is also an ordered cone under the usual pointwise ordering. Note that

$C^+[0,1]$ is not a bd-cone. To see this, consider the sequence of piecewise linear function in $C^+[0,1]$ defined by

$$f_n(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2} - \frac{1}{n}, \\ -n(x-\frac{1}{2}) & \text{if } \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2}, \\ x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Thus $0 \le f_n \le \mathbf{1}$ in $C^+[0,1]$ and is an increasing sequence, where $\mathbf{1}$ is the constant function one, but $\{f_n\}$ does not have a supremum in $C^+[0,1]$. It is easy to see that the functions $(f,g) \mapsto f+g : C^+[0,1] \times C^+[0,1] \to C^+[0,1]$ and $(r,f) \mapsto rf : \mathbb{R}_+ \times C^+[0,1] \to C^+[0,1]$ are Scott continuous. So $C^+[0,1]$ is an s-cone. Furthermore, $C^+[0,1]$ is a continuous s-cone. Now we show that $C^+[0,1]$ is a wd-cone. Indeed, if $\{f_\alpha\}$ is an increasing Cauchy net in $C^+[0,1]$, then it is a norm Cauchy net. Since $C[0,1]$ is a Banach space, the net $\{f_\alpha\}$ is norm-convergent to some f. This means that $f_\alpha(x) \to f(x)$, Thus $f(x) = \sup_\alpha f_\alpha(x)$ for each $x \in [0,1]$. It follows that $\sup_\alpha f_\alpha = f$ and $f \in C^+[0,1]$.

In the sequel, the mean of a wd-cone will always a cancellative continuous wd-cone, if not specified otherwise.

3.2. **A Bishop-Phelps type Theorem.** In this section, we prove the Bishop-Phelps type theorem for wd-cones.

Lemma 3.12. *Let f be a continuous linear functional on a wd-cone C, $0 < \delta < 1$ and $d \in C$. If B is a nonempty convex bounded closed subset of C, then for each $b \in B$ there exists a maximal element $m \in B$ satisfying $B \cap (m + K(f,\delta,d)) = \{m\}$ and $b \sqsubseteq m$.*

Proof. It is sufficient to show that the set $B_b = \{y \in B : b \sqsubseteq y\}$ with the order $\sqsubseteq$ has a maximal element.

By Zorn's Lemma, it suffices to prove that every chain in $(B_b, \sqsubseteq)$ has an upper bound in B_b. Let Z be a chain in B_b. If we let $x_\alpha = \alpha$ for each $\alpha \in Z$, we can identify Z with the increasing net $\{x_\alpha\}$.

Without loss of generality, let x_α and x_β be two elements of the net. Without lose of the generality, we can suppose that $x_\alpha \sqsubseteq x_\beta$. So there exists $k \in K$ such that $x_\beta = x_\alpha + k$ and $\delta k \le f(k)d$. Therefore, $\delta x_\beta \le \delta x_\alpha + (f(x_\beta) - f(x_\alpha))d$. By the boundedness of B and continuity of f, it follows that $f(x_\alpha)$ is a bounded net, and so $f(x_\alpha)$ is convergent and Cauchy. It is easy to see that the net $\{x_\alpha\}$ is a directed Cauchy and so has a supremum, say x (with specialization order), that means $\sup_\le x_\alpha = x$. Since B is a Scott closed set, so $x \in B$. Now fix $\beta \in Z$, so we have $x_\beta \sqsubseteq x_\alpha$ for $\beta \sqsubseteq \alpha$. Thus $\delta x_\alpha \le (f(x_\alpha) - f(x_\beta))d + \delta x_\beta$, which follows that $\delta x \le (f(x) - f(x_\beta))d + \delta x_\beta$. Hence $x_\beta \sqsubseteq x$ and x is an upper bound of the net $(\{x_\alpha\}, \sqsubseteq)$. It follows that $x \in B_b$, hence $(B_b, \sqsubseteq)$ has a maximal element; say m. Therefore $B \cap (m + K) = \{m\}$ and $b \sqsubseteq m$. □

Lemma 3.13. *Let C be a cancellative continuous s-cone and let K be a subcone of C. If B is a nonempty subset of C and $0 \ne m \in B$, such that $B \cap (m + K) = \{m\}$, then $B \cap \uparrow (m + K \backslash \{0\}) = \varnothing$. In particular,*

$$B \cap int(\uparrow (m + K\backslash\{0\})) = \varnothing, \; B \cap \Uparrow (m + K\backslash\{0\}) = \varnothing.$$

Proof. Let $x \in B \cap \uparrow (m+K\backslash\{0\})$, then there exists a $k \in K\backslash\{0\}$ suth that $x = m+k$. By the assumption, $m + k = m$, so $k = 0$. This leads to a contradiction. □

Applying the separation theorem and Lemmas 3.12 and 3.13, we obtain the following Bishop-Phelps type theorem for *wd*-cones, the main result of this paper.

Theorem 3.14. *Let B be a nonempty convex bounded closed set in a locally convex wd-cone C, such that C has the additive property. Then we have:*

(e_1) Fix $\epsilon > 0$ and $d \in C$". For each $x_0 \in B$, such that $\lambda x_0 \notin B$ whenever $\lambda > 1$, there exist a continuous linear functional $f : C \to \overline{\mathbb{R}}_+$ and an $m \in B$ such that $f(m) = \sup f(B)$ and $x_0 \leq m \leq x_0 + \epsilon d$.

(e_2) For each continuous linear functional $f : C \to \overline{\mathbb{R}}_+$, there exists a c-support functional h for B such that $0 \leq h \leq f$ on a subcone of C.

Proof. (e_1) Let $x_0 \in B$ satisfies the conditions of the theorem, then $(1 + \epsilon)x_0 \notin B$. Now we take $E = \uparrow (1 + \epsilon)x_0$. E is a compact set, so by the strict separation theorem, cited in Section 2, there exists a continuous linear functional g such that $g(b) < (1+\epsilon)g(x_0)$ for all $b \in B$. Since B is bounded, so the function g can be chosen such that $g(B) \leq 1$.

Now, let $0 < \delta < 1$, by Lemma 3.12, there exists an $m \in B$ satisfying $B \cap (m + K(g, \delta, d)) = \{m\}$, $x_0 \sqsubseteq m$ and $x_0 \leq m$. Hence, $\delta(m - x_0) \leq g(m - x_0)d$. Therefore, $\delta(m - x_0) \leq \epsilon g(x_0).d$ and so $\delta(m - x_0) \leq \epsilon d$. By Lemma 3.13, $B \cap int(\uparrow (m + K\backslash\{0\})) = \varnothing$. Applying the separation theorem, there exists a continuous linear functional $f : C \to \overline{\mathbb{R}}_+$ such that $f(b) \leq 1 < f(w)$ for all $b \in B$ and all $w \in int(\uparrow (m + K\backslash\{0\}))$. By the continuity of *wd*-cone C, we have $f(b) \leq f(w)$ for all $b \in B$ and all $w \in \uparrow (m+K\backslash\{0\})$. By additive property(ii), $f(b) \leq f(m+k)$ for all $b \in B$ and all $k \in K\backslash\{0\}$, and so $f(b) \leq f(m)$ for all $b \in B$. Hence $\sup f(B) = f(m)$ and $f(m) < \infty$.

(e_2) Let f be a continuous linear functional and let $0 < \delta < 1$. We consider the subcone $K = K(f, \delta, d)$. By Lemmas 3.12 and 3.13, there exists an $m \in B$ such that $B \cap int(\uparrow (m + K(f, \delta, d)\backslash\{0\})) = \varnothing$. So by the separation theorem there exists a continuous linear functional $h : C \to \overline{\mathbb{R}}_+$ satisfying $h(b) \leq h(m + c)$ for all $b \in B$ and $c \in K(f, \delta, d)$. This implies that h attains its maximum. It follows that $h(\delta c) \leq f(c)h(d)$. The number δ and the element d can be taken to have $h(d) = \delta$. Thus $0 \leq h \leq f$ on a subcone of C. □

It is well known that each norm on a linear space X induces a metric on X. Note that, using the same method as in classical case, each norm on a cone C necessarily does not induce a metric on C. For example, the usual norm $\|x\| := x$ on $\mathbb{R}_+$, does not induce a metric on $\mathbb{R}_+$. The cone order of each normed cone produce a topology (named the Scott topology) on a normed cone. To know more about the relationship between this topology and concept of norm refer to [16].

Corollary 3.15. *Let B be a nonempty convex bounded Scott closed set in a continuous normed cone such that for any scalar $\lambda > 1$ and $x > 0$, $x \ll \lambda x$. Then the results of Theorem 3.14 are still true.*

Proof. In a continuous normed cone, addition and scalar multiplication are Scott continuous [16, Lemma 2.12], so any continuous normed cone is a cancellative s-cone with the Scott topology. It is easy to see that every continuous normed cone is a bd-cone. Appealing to [16, Lemma 2.16], C has the additive property, thus assumptions of Theorem 3.14 hold and so we can conclude the desired statement. □

Let us illustrate the above theorem with some examples:

Example 3.16. (f_1) Let $C = \overline{\mathbb{R}}^2_+$ and $B = \{(x, y) \in \overline{\mathbb{R}}^2_+; x + y \leq 1\}$. Then B is a convex Scott closed set which has no any maximum. It can be easily checked that the c-support points of B is the set $\{(x, y) \in \overline{\mathbb{R}}^2_+; x + y = 1\}$.

(f_2) For $d = (d_1, d_2, ...) \in \ell_1^+$, the set

$$B_d := \{x = (x_1, x_2, ...) \in \ell_1^+ \quad : \quad x \leq d\}$$

is a bounded Scott closed set in ℓ_1^+ that has a maximum, so any linear Scott continuous functional $f : \ell_1^+ \to \overline{\mathbb{R}}_+$ takes its supremum on B_d at the point d. Observe that the set of c-support points of B_d is

$$\{x \in \ell_1^+ \; : \; \exists f \in (\ell_1^+)^* \text{ s.t. } f(x) = f(d)\}.$$

One can check that $\ell_\infty^+ \subset (\ell_1^+)^*$. Let z belong to the following set,

$$D := \{x = (x_1, x_2, ...) \in B_d \; : \; \text{for some } i, \; x_i = d_i\}.$$

If we take $a = (a_1, a_2, ...) \in \ell_\infty^+$, such that $a_i = 0$ whenever $z_i \neq d_i$, then a is a c-support functional and z is a c-support point for B_d. Hence, D is the set of c-support points of B_d.

3.3. **A fixed point result in s-cones.** What follows is an application of the Bishop-Phelps technique in some fixed point results. Let X be any space and f a map of X, or of a subset of X, into X. A point $x \in X$ is called a fixed point for f if $x = f(x)$. The set of all fixed points of f is denoted by $Fix(f)$.

Let C be a d-cone. In [1, Theorem 2.1.19.] the authors proved that every continuous function f on C has a least fixed point. The property that "in d-cones, every directed subset has a supremum" is applied in the proof of the theorem. Since in wd-cones this property does not remain true in general, we establish a fixed point result in wd-cones by using the Bishop-Phelps technique.

Theorem 3.17. *Let C be a wd-cone and $f : C \to \overline{\mathbb{R}}_+$ be a continuous linear functional and $0 < \delta < 1$ and $d \in C$. Suppose that B is a bounded closed set in C and $T : C \to C$ is a mapping such that $f(B) \subset B$. Then the following assertion holds.*

If for each $x \in C$, $x \sqsubseteq Tx$, then there exists $m \in B$ such that $T(m) = m$.

Proof. By applying Lemma 3.12, B has a maximal element m. Now, using the assumption, $m \sqsubseteq T(m)$. The maximality of m implies $T(m) = m$. □

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