Fixed Point Theory, 20(2019), No. 2, 507-510
DOI: 10.24193/fpt-ro.2019.2.32
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON FIXED POINTS OF SOME FUNCTIONS

ZBIGNIEW GRANDE

Institute of Mathematics, Kazimierz Wielki University
Plac Weyssenhoffa 11, 85-072 Bydgoszcz, Poland
E-mail: grande@ukw.edu.pl

Abstract. Let $f : [0, 1] \to [0, 1]$ be a Darboux function of Baire class one. Then f has fixed point $x \in [0, 1]$, **ie.** there is a point $x \in [0, 1]$ such that $f(x) = x$. So approximate continuity of f implies that f has a fixed point. In this article I investigate when f has a fixed point x satisfying some other conditions (for example f is bilaterally quasicontinuous at x or even continuous at x).
Key Words and Phrases: Darboux property, Baire class 1, density topologies, quasi-continuity.
2010 Mathematics Subject Classification: 26A05, 26A15, 28A05.

1. Introduction

It is well known ([2] or [1]) that all Baire 1 Darboux functions $f : [0, 1] \to [0, 1]$ have fixed points $x \in [0, 1]$, (i.e. $f(x) = x$). Really, if $f(0) = 0$ or $f(1) = 1$ then f has fixed points. In the other case let $g(x) = f(x) - x$ for $x \in [0, 1]$ and observe that $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$. Since g has the Darboux property (as the difference of Baire 1 Darboux function f and a continuous function x) ([2]), there is a point $x \in (0, 1)$ with $g(x) = 0$. Evidently, $0 = g(x) = f(x) - x$ and $f(x) = x$.

On the other side some examples of Darboux functions $f : [0, 1] \to [0, 1]$ without the fixed point property are well known. For example in my article ([3]) is shown a construction of Darboux function $f : [0, 1] \to [0, 1]$, which has not fixed point and which is bilaterally quasicontinuous and satisfies some special condition (s_1) based on the density topology and implying the continuity almost everywhere of f.

2. Main results

I. Quasicontinuities

For the formulation of the main results of this note recall that a function $g : [0, 1] \to \mathbb{R}$ is bilaterally quasicontinuous at a point $x \in (0, 1)$ if for each real $\eta > 0$ there are open intervals $I_1 \subset (x - \eta, x) \cap [0, 1]$ and $I_2 \subset (x, x + \eta) \cap [0, 1]$ such that $g(I_1 \cup I_2) \subset (g(x) - \eta, g(x) + \eta)$ (see [5, 6]).

Analogously we define the quasicontinuity from the right at 0 and the quasicontinuity from the left at 1. A function $g : [0, 1] \to [0, 1]$ is bilaterally quasicontinuous if it is bilaterally quasicontinuous at each point $x \in (0,1)$, quasicontinuous from the right at 0 and quasicontinuous from the left at 1.

Since every bilaterally quasicontinuous function of Baire one class has the Darboux property ([2]) we obtain the following

Remark 2.1. Every bilaterally quasicontinuous Darboux of Baire one class function $f : [0,1] \to [0,1]$ has fixed point.

II. Density topologies

Let $\mathbb{R}$ be the set of all reals. Denote by μ the Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$. For a set $A \subset \mathbb{R}$ and a point$x$ we define the upper (lower) outer density $D_u(A,x)$ $(D_l(A,x))$ of the set A at the point x as

$$\limsup_{h \to 0^+} \frac{\mu_e(A \cap [x-h, x+h])}{2h}$$

$$(\liminf_{h \to 0^+} \frac{\mu_e(A \cap [x-h, x+h])}{2h}, \text{ respectively}).$$

A point x is said to be an outer density point (a density point) of a set A if $D_l(A,x) = 1$ (if there is a Lebesgue measurable set $B \subset A$ such that $D_l(B,x) = 1$).

Taking the extremal limits for the expressions $\frac{A \cap [x-h,x]}{h}$ and $\frac{A \cap [x,x+h}{h}$ we obtain respectively the left or the right upper (lower) densities of A at x.The family T_d of all sets A for which the implication $x \in A$ implies x is a density point of A is true is a topology called the density topology ([2] or [8]). The sets $A \in T_d$ are Lebesgue measurable ([2]). Let T_n be the natural euclidean topology in $\mathbb{R}$.

The following family $T_{ae} = \{A \in T_d; \mu(A \setminus int(A)) = 0\}$, where $int(A)$ denotes the natural Euclidean interior of A, was examined by O'Malley [7]. If T is a topology in $\mathbb{R}$ then $C(T)$ denotes the class of functions $f : \mathbb{R} \to \mathbb{R}$ continuous as applications from $(\mathbb{R}, T)$ to $(\mathbb{R}, T_n)$. It is well known that

$$C(T_n) \subset C(T_{ae}) \subset C(T_d)$$

and $f \in C(T_{ae}$ if and only if $f \in C(T_d)$ and f is almost everywhere continuous ([7]).

Moreover if $f \in C(T_d)$ then f is of the first Baire class and f has the Darboux property ([2]). So all functions $f : [0,1] \to [0,1]$ belonging to $C(T_d)$ have fixed points.

All functions $f \in C(T_{ae})$ are bilaterally quasicontinuous, but there are functions $g \in C(T_d)$ which are not quasicontinuous ([2]).

III. Existence of fixed points satisfying some special conditions

Theorem 2.2. *There is a function $f : [0,1] \to [0,1]$ belonging to $C(T_d)$ and such that if x is a fixed point of f then f is not bilaterally quasicontinuous at x.*

Proof. Let Q be the set of all rationals and let $A \supset Q$ be a G_δ-set of Lebesgue measure 0 containing Q. Moreover let $B = A \setminus \{0\}$. Then B is a G_δ-set of measure 0 and by Theorem 6.5 of [2] there is an upper semicontinuous function $f \in C(T_d)$ such that $0 < f() \leq 1$ for all $x \in \mathbb{R} \setminus B$ and $f(x) = 0$ for all $x \in B$.

Since $f(0) > 0$, the point 0 is not any fixed point of f. Since $f(1) = 0$, the point 1 is not any fixed point of f. Let $u \in (0,1)$ be a fixed point of f. Then $u = f(u) > 0$. If f is bilaterally quasicontinuous at u then there is an open interval $I \subset (0,1)$ with $f(I) \subset (\frac{f(u)}{2}, \infty)$. But Q is dense, so there is a point $t \in I \cap (Q \setminus \{0\})$. Since $f(t) = 0$, we obtain a contradiction, which finishes the proof. □

As evident conclusions we obtain

Corollary 2.3. *There is a function $f \in C(T_d)$ such that if x is a fixed point of f then f is not bilaterally quasicontinuous at x.*

and

Corollary 2.4. *There is a function $f \in C(T_d)$ such that if x is a fixed point of f then f is not T_{ae}-continuous at x.*

3. Final problems

Problem 3.1. *Let $f : [0,1] \to [0,1]$ be a Darboux function of Baire class one. Does it exist a fixed point x of f at which f is T_d-continuous?*

Problem 3.2. *Let $f \in T_{ae}$ be a function from $[0,1]$ to $[0,1]$. Does it exist a fixed point x of f at which f is T_n-continuous?*

References

[1] J.B. Brown, *Almost continuous Darboux functions and Reed's pointwise convergence criteria*, Fund Math., **86**(1974), 1–17.

[2] A.M. Bruckner, *Differentiation of real functions*, Lecture Notes in Math. 659, Springer-Verlag, Berlin, 1978.

[3] Z. Grande, *An example of a Darboux function having no fixed points*, Real Anal. Exch., **28**(2002–2003), No. 2, 375–379.

[4] Z. Grande, *On some special notions of approximate quasicontinuity*, Real Anal. Exch., **24** (1998–1999), No. 1, 171–184.

[5] S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math., **19**(1932), 184–197.

[6] T. Neubrunn, *Quasi-continuity*, Real Anal. Exch., **14**(1988/89), No. 2, 259–306.

[7] R.J. O'Malley, *Approximately differentiable functions. The r topology*, Pacific J. Math., **72** (1977), 207–222.

[8] F.D. Tall, *The density topology*, Pacific J. Math., **62**(1976), 275–284.

Received: December 4, 2017; Accepted: January 15, 2018.