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FIXED POINTS OF OPERATORS SATISFYING VARIOUS CONTRACTIVE CONDITIONS IN COMPLETE PARTIAL METRIC SPACES

A.P. BAISNAB*, PRATULANANDA DAS** AND SUMANA PAL***

*Department of Mathematics, Lady Brabourne College
Kolkata 700017, West Bengal, India
E-mail: baisnababhoypada@yahoo.com

**Department of Mathematics, Jadavpur University
Kolkata 700032, West Bengal, India
E-mail: pratulananda@yahoo.co.in

***Department of Mathematics, Aliah University
IIA/27, New Town, Kolkata 700160, West Bengal, India
E-mail: sumana.pal@gmail.com

Abstract. In this paper, we give a generalized definition of diameter of a set in a partial metric space and as a consequence, a Cantor's Intersection like Theorem for partial metric spaces follows. We apply this theorem to study some fixed point results for generalized contractive type mappings over a complete partial metric space and also give some results on continuity of fixed points and simultaneous fixed point.
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1. Introduction

In 1994, Matthews [13] introduced the concept of partial metric spaces as a generalization of metric spaces, motivated by the notion of non-zero self-distance from his experience in computer science. He also studied Banach contraction mapping theorem in the setting of partial metric spaces. Since then, many authors have studied fixed point theory in the realm of these spaces using various contractive type mappings. The study was later continued by several authors like Oltra and Valero [15], Altun et al. [3], [2], [1], Gangopadhyay [5]. As of now, many researchers are pursuing a variety of results in fixed point theory in partial metric spaces [11], [12], [14], [4].

First we recall some basic definitions on a partial metric space.

A partial metric on X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

(i) $x = y$ if and only if $p(x, x) = p(y, y) = p(x, y)$,
(ii) $p(x, x) \leq p(x, y)$,

(iii) $p(x,y) = p(y,x)$,
(iv) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

A partial metric space is a pair (X,p) such that X is a non-empty set and p is a partial metric on X. In a partial metric space (X,p) each point does not necessarily possess zero distance from itself. Of course a metric space is a partial metric space while the converse is false. An example of a partial metric space is the pair $(\mathbb{R}^+,p)$ where $p(x,y) = \max\{x,y\}$, for all $x,y \in \mathbb{R}^+$. It is not a metric space.

If $x \in X$ and $\epsilon > 0$, then the set $p\text{-}B_\epsilon(x) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ is called a p-open ball in (X,p). By routine check up one finds $\{B_\epsilon(x)\}, x \in X$ and $\epsilon > 0$, is a base to generate a topology τ_p called the partial metric topology on X, and this topology τ_p is T_0 in nature.

We have the following definitions in a partial metric space (X,p).

Definition 1.1. A sequence $\{x_n\}$ in a partial metric space (X,p) is said to be p-Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and is finite).

Definition 1.2. A sequence $\{x_n\}$ in a partial metric space (X,p) is said to be p-convergent at $x_0 \in X$ if $\lim_{n\to\infty} p(x_n,x_0) = p(x_0,x_0)$.

Definition 1.3. A partial metric space (X,p) is said to be complete if every p-Cauchy sequence in (X,p) p-converges to a point of X, i.e., if $\{x_n\}$ is p-Cauchy in (X,p), there is a point $x_0 \in X$ such that

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x_0) = p(x_0,x_0).$$

Definition 1.4. A function $h : (X,p) \to \mathbb{R}$ is said to be p-lower semi-continuous (p-l.s.c.) at $u \in X$ if given $\epsilon > 0$, there is a $\delta > 0$ such that

$$h(x) > h(u) - \epsilon \text{ for } x \in p\text{-}B_\delta(u),$$

or equivalently, if $\{x_n\}$ is a sequence in (X,p) p-converging to u, then

$$\underline{\lim}_{n\to\infty} f(x_n) \geq f(u).$$

Also h is said to be a p-l.s.c. function on X if it is so at every point of X.

Similarly, we define a p-upper semi-continuous function on a partial metric space (X,p).

2. Main results

Here we introduce a notion of p-boundedness and p-diameter of a set in a partial metric space (X,p) and establish a Cantor's Intersection like theorem in a complete partial metric space. Then employing Cantor's Intersection like theorem, we have proved a fixed point theorem for a mixed type mapping instead of applying Picard's iteration scheme.

Definition 2.1. (a) Let (X,p) be a partial metric space and $G \subset X$. If

$$0 \leq \sup\{p(x,y) - p(x,x) - p(y,y) : p(x,y) - p(x,x) - p(y,y) \geq 0, x,y \in G\} < \infty,$$

then G is called p-bounded and the diameter of a p-bounded set G, denoted by p-Diam(G) is defined by

$$p\text{-Diam}(G) = \sup\{z : z = p(x,y) - p(x,x) - p(y,y) \geq 0; x, y \in G\}.$$

(b) A subset in (X,p) is called p-closed if it contains all its limit points with respect to p.

Lemma 2.2. *Let G be a p-bounded set in a partial metric space (X,p). Then its closure $\bar{G}$ is p-bounded and p-Diam$(\bar{G}) = p$-Diam(G).*

Proof. Since $G \subset \bar{G}$, we have

$$\begin{aligned} 0 &\leq \sup_{x,y\in G}\{p(x,y) - p(x,x) - p(y,y)\} \\ &\leq \sup_{x,y\in \bar{G}}\{p(x,y) - p(x,x) - p(y,y)\}, \end{aligned}$$

so that

$$p\text{-Diam}(G) \leq p\text{-Diam}(\bar{G}). \tag{2.1}$$

On the other hand, let $u, v \in \bar{G}$ such that $p(u,v) - p(u,u) - p(v,v) \geq 0$. Then given $\epsilon > 0$, there exists $x, y \in G$ to satisfy

$$p(u,x) < \frac{\epsilon}{2} + p(u,u)$$

and

$$p(v,y) < \frac{\epsilon}{2} + p(v,v).$$

Therefore

$$\begin{aligned} p(u,v) &\leq p(u,x) + p(x,y) + p(y,v) - p(x,x) - p(y,y) \\ &< \epsilon + p(x,y) - p(x,x) - p(y,y) + p(u,u) + p(v,v) \end{aligned}$$

which implies

$$\begin{aligned} p(u,v) - p(u,u) - p(v,v) &< \epsilon + p(x,y) - p(x,x) - p(y,y) \\ &\leq \epsilon + \sup_{x,y\in G}\{p(x,y) - p(x,x) - p(y,y) \geq 0\} \\ &\leq \epsilon + p\text{-Diam}(G). \end{aligned}$$

Thus

$$p\text{-Diam}(\bar{G}) \leq p\text{-Diam}(G). \tag{2.2}$$

Combining (2.1) and (2.2),

$$p\text{-Diam}(\bar{G}) = p\text{-Diam}(G). \qquad \square$$

Theorem 2.3. *(Cantor's Intersection like Theorem) Let (X,p) be a complete partial metric space. If $\{G_n\}$ is a monotonically decreasing sequence of non-empty p-closed sets in (X,p) with p-Diam$(G_n) \to 0$ as $n \to \infty$, then $G = \bigcap_{n=1}^{\infty} G_n$ is a singleton.*

Proof. The sequence $\{G_n\}$ is a monotonically decreasing sequence of p-closed sets, i.e.,

$$G_1 \supset G_2 \supset \ldots \supset G_n \supset G_{n+1} \supset \ldots.$$

Let $x_n \in G_n, n = 1, 2, \ldots$. Then if $n < m$, we have $x_n, x_m \in G_n$. So

$$\begin{aligned} 0 &\leq \sup_{x_m \in G_n, m>n} \{p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \geq 0\} \\ &\leq p\text{-Diam}(G_n) \to 0 \text{ as } m, n \to \infty. \end{aligned}$$

Therefore, without loss of generality, we have

$$\begin{aligned} 0 &\leq p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \\ &\to 0 \text{ as } m, n \to \infty. \end{aligned}$$

So we get

$$\begin{aligned} -p(x_n, x_n) - p(x_m, x_m) &\leq p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \\ &\to 0 \text{ as } m, n \to \infty, \end{aligned}$$

and thus $p(x_n, x_m) \to 0$ as $m, n \to \infty$.
Since (X, p) is complete, there exists $x \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x) = 0.$$

As each set G_n is p-closed, $x \in G_n$, for all n which implies that $x \in \bigcap_{n=1}^{\infty} G_n$ so that

$$G = \bigcap_{n=1}^{\infty} G_n \neq \phi.$$

Again, let $x, y \in G, x \neq y$, then $x, y \in G_n$, for all n. We have

$$\begin{aligned} 0 &\leq p(x, y) - p(x, x) - p(y, y) \\ &\leq p\text{-Diam}(G_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $p(x, y) = p(x, x) + p(y, y)$. Also $x, x \in G_n$, for all n which gives

$$p(x, x) - p(x, x) - p(x, x) = 0, \text{ or } p(x, x) = 0.$$

Similarly, $p(y, y) = 0$.
Thus $p(x, y) = 0 = p(x, x) = p(y, y)$ giving $x = y$. Hence G is a singleton. □

As an application of the above theorem, we find a result for which we begin with the next lemma.

Lemma 2.4. *Let (X, p) be a partial metric space and T be a self-mapping on X satisfying the following condition*

$$p(Tx, Ty) \leq \alpha\, p(x, Tx) + \beta\, p(y, Ty) + \gamma\, p(x, y)$$

where

$$\alpha + \beta + \gamma < 1 \ \text{ and } \ \alpha, \beta, \gamma \geq 0, \ \text{ for all } x, y \in X,$$

and let $\{\alpha_n\}$ be a sequence of reals with $0 < \alpha_n < 1$, for all n and $\lim \alpha_n = 0$. For each $n \in \mathbb{N}$, if the set $G_n = \{x \in X : p(x, Tx) \leq \alpha_n\}$ is non-empty, then $\{G_n\}$ is a decreasing sequence of sets with p-Diam$(G_n) \to 0$.

Proof. Clearly $\{G_n\}$ is monotone decreasing. Let x, y be elements in G_n so that $p(x, Tx) \leq \alpha_n$ and $p(y, Ty) \leq \alpha_n$. Now

$$\begin{aligned} p(x,y) &\leq p(x,Tx) + p(Tx,Ty) + p(Ty,y) - p(Tx,Tx) - p(Ty,Ty) \\ &\leq p(x,Tx) + p(Tx,Ty) + p(Ty,y) \\ &\leq 2\alpha_n + \alpha\, p(x,Tx) + \beta\, p(y,Ty) + \gamma\, p(x,y). \end{aligned}$$

So

$$p(x,y) \leq \frac{2+\alpha+\beta}{1-\gamma}\ \alpha_n \to 0 \text{ as } n \to \infty.$$

Since $0 \leq p(x,y) - p(x,x) - p(y,y) \leq p(x,y)$, we get

$$\sup_{x,y \in G_n} \{p(x,y) - p(x,x) - p(y,y)\} = 0.$$

Therefore p-Diam$(G_n) \to 0$ as $n \to \infty$. □

Lemma 2.5. *Let (X,p) be a partial metric space. If the function $f : X \to \mathbb{R}^+$ defined by $f(x) = p(x, Tx)$ is a p-l.s.c. function, then the sets G_n as constructed in Lemma 2.4 are p-closed.*

Proof. It is a consequence of p-l.s.c. property of f. □

Lemma 2.6. *Let (X,p) be a partial metric space and T be a self-mapping on X satisfying the conditions of Lemma 2.4. Then $T(G_n) \subset G_n$, for all n, where the sets G_n appear there.*

Proof. Let $x \in G_n$. Then $p(x,Tx) \leq \alpha_n$. Now

$$\begin{aligned} p(Tx,T^2x) &= p(Tx,T(Tx)) \\ &\leq \alpha p(x,Tx) + \beta p(Tx,T^2x) + \gamma p(x,Tx) \\ \text{or, } (1-\beta)p(Tx,T^2x) &\leq (\alpha+\gamma)p(x,Tx) \\ \text{or, } p(Tx,T^2x) &\leq \frac{\alpha+\gamma}{1-\beta}\alpha_n < \alpha_n \text{ since } \alpha+\beta+\gamma < 1. \end{aligned}$$

Therefore, $Tx \in G_n$, i.e., $T(G_n) \subset G_n$. □

Theorem 2.7. *Let (X,p) be a complete partial metric space and $T : X \to X$ be a self mapping on X which satisfies the following conditions:*

(i) $p(Tx,Ty) \leq \alpha\, p(x,Tx) + \beta\, p(y,Ty) + \gamma\, p(x,y)$ *where*

$$\alpha+\beta+\gamma < 1 \ \text{ and } \ \alpha,\beta,\gamma \geq 0, \ \text{ for all } x, y \in X,$$

and

(ii) $p(x,Tx)$ *is a p-l.s.c. function on X, for all $x \in X$.*

Then T has a fixed point in X.

Proof. Let $x_0 \in X$. Also let $x_{n+1} = Tx_n$, for all $n \geq 1$. So we get

$$\begin{aligned} p(x_1, x_2) &= p(Tx_o, Tx_1) \\ &\leq \alpha p(x_0, x_1) + \beta p(x_1, Tx_1) + \gamma p(x_0, x_1) \\ &= \alpha p(x_0, x_1) + \beta p(x_1, x_2) + \gamma p(x_0, x_1), \\ \text{or, } p(x_1, x_2) &\leq \frac{\alpha + \gamma}{1 - \beta} p(x_0, x_1). \end{aligned}$$

Similarly

$$p(x_2, x_3) \leq \left(\frac{\alpha + \gamma}{1 - \beta}\right)^2 p(x_0, x_1)$$

and so on. Proceeding in this way, we obtain

$$p(x_n, x_{n+1}) \leq \left(\frac{\alpha + \gamma}{1 - \beta}\right)^n p(x_0, x_1) \to 0 \text{ as } n \to \infty,$$

since $\alpha + \beta + \gamma < 1$.

Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < \alpha_n < 1$, for all n and $\lim \alpha_n = 0$. Let us construct the set

$$G_n = \{x \in X : p(x, Tx) \leq \alpha_n\}.$$

Then $G_n \neq \phi$ and by Lemma 2.4, $\{G_n\}$ is monotone decreasing with p-Diam(G_n) $\to 0$. By condition (ii) and Lemma 2.5, it follows that the sets G_n are p-closed. Now we apply Theorem 2.3 to obtain $G = \bigcap_{n=1}^{\infty} G_n$ to be a singleton set $\{u\}$, say. Using Lemma 2.6, we obtain $Tu = u$. Therefore, u is a fixed point of T. □

Remark 2.8. Theorem 2.7 has been proved under completeness assumption of partial metric space (X, p). This version of Theorem 2.7 is partially included in Theorem 2.6 proved by Haghi et al in [6], in proof of which the authors (i) assumed (X, p) as 0-complete, and (ii) had changed partial metric p into a full metric; and the rest of their proof of concerned theorem owes to Picard's iteration scheme for deriving a fixed point of mapping in question. Ours is an attempt to explore an alternative route by employing Cantor's Intersection like theorem in a complete partial metric space to produce desired fixed point of the mapping.

When operator $T : X \to X$ in Theorem 2.7 is purely of contractive type, i.e., when $\alpha = \beta = 0$, the hypothesis of completeness of the partial metric space (X, p) is not redundant as supported by Example 2.9 below.

Example 2.9. Consider the set $\mathbb{N}$ of natural numbers. We take $p : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ given by

$$p(m, n) = \begin{cases} \frac{1}{n} & \text{if } m = n, \\ \frac{1}{n} + \frac{1}{m} & \text{if } m \neq n. \end{cases}$$

Then $(\mathbb{N}, p)$ is a partial metric space which is not complete. For, if we consider the sequence $\{n\}$, then $p(n, m) = \frac{1}{n} + \frac{1}{m} \to 0$ as $n, m \to \infty$ showing that $\{n\}$ is a Cauchy sequence in $(\mathbb{N}, p)$. But for a fixed number n_0,

$$\begin{aligned} p(n_0, n_0) &= \frac{1}{n_0} > 0, \\ \lim_{n\to\infty} p(n, n_0) &= \lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n_0}\right) = \frac{1}{n_0}, \\ \text{but } \lim_{n,m\to\infty} p(n, m) &= 0. \end{aligned}$$

Therefore $\lim_{n,m\to\infty} p(n,m) \neq \lim_{n\to\infty} p(n, n_0) = p(n_0, n_0)$, which shows that $\{n\}$ does not p-converge to any point of $\mathbb{N}$. So $(\mathbb{N}, p)$ is not p-complete. Now let us consider a self-mapping T on $\mathbb{N}$ defined by $Tn = 2n$, for all $n \in \mathbb{N}$.

For any $m, n \in \mathbb{N}, m \neq n$, we have

$$\begin{aligned} p(Tm, Tn) &= \frac{1}{2m} + \frac{1}{2n} \\ &= \frac{1}{2}\, p(m,n) < \frac{3}{4}\, p(m,n) \end{aligned}$$

so that T satisfies the condition (i) of Theorem 2.7 with $\alpha = 0 = \beta, \gamma = \frac{3}{4}$. Also condition (ii) of Theorem 2.7 is satisfied. But T does not have a fixed point in $(\mathbb{N}, p)$.

When operator $T : X \to X$ is purely Kannan type (see [7], [8], [9], [10]), i.e., when $\gamma = 0$ with $\alpha = \beta$ and $2\beta < 1$, then assumption (ii) of Theorem 2.7 cannot be dispensed with. The next Example 2.10 shall bear it out.

Example 2.10. We consider the set $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and define $p : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{R}^+$ by

$$p(m,n) = \begin{cases} \frac{1}{n} & \text{if } m \neq 0, n \neq 0, m = n, \\ \frac{1}{n} + \frac{1}{m} & \text{if } m \neq 0, n \neq 0, m \neq n; \end{cases}$$

$$p(0,0) = 0, \text{ and } p(0,n) = p(n,0) = \frac{1}{n}.$$

It can be verified that $(\mathbb{N}^*, p)$ is a complete partial metric space. We define a self-mapping $T : \mathbb{N}^* \to \mathbb{N}^*$ by

$$Tn = 2n, \text{for all } n \in \mathbb{N}$$

and

$$T(0) = 1.$$

We now show that

$$p(Tx, Ty) \leq 2\beta\{p(x, Tx) + p(y, Ty)\}, \text{ for all } x, y \in \mathbb{N}^*, \text{ where } 2\beta < 1.$$

We consider the following cases:

Case (i). Let $x = 0, y = n$, for some $n \in \mathbb{N}$. Then

$$p(Tx, Ty) = p(T0, Tn) = p(1, 2n) = 1 + \frac{1}{2n}$$

and

$$p(x, Tx) + p(y, Ty) = p(0, 1) + p(n, 2n) = 1 + \frac{1}{n} + \frac{1}{2n}.$$

So,

$$p(Tx,Ty) \leq 2\beta[p(x,Tx)+p(y,Ty)],$$

if

$$2\beta \geq \frac{1+\frac{1}{2n}}{1+\frac{1}{n}+\frac{1}{2n}} > \frac{\frac{1}{2n}}{1+\frac{1}{n}+\frac{1}{2n}} = \frac{1}{2n+3},\ n \geq 1.$$

Case (ii). Let $x=n, y=m$ for some $n,m \in \mathbb{N}, n \neq m$. Then

$$p(Tx,Ty) = p(Tn,Tm) = p(2n,2m) = \frac{1}{2n}+\frac{1}{2m}$$

and

$$p(x,Tx)+p(y,Ty) = p(n,2n)+p(m,2m) = \frac{3}{2}\left(\frac{1}{n}+\frac{1}{m}\right).$$

Therefore,

$$p(Tx,Ty) \leq 2\beta[p(x,Tx)+p(y,Ty)]$$

if

$$2\beta \geq \frac{\frac{1}{2}(\frac{1}{n}+\frac{1}{m})}{\frac{3}{2}(\frac{1}{n}+\frac{1}{m})} = \frac{1}{3}.$$

Case (iii). Let $x=y=n, n \in \mathbb{N}$. Then

$$p(Tx,Ty) = p(Tn,Tn) = p(2n,2n) = \frac{1}{2n}$$

and

$$p(x,Tx)+p(y,Ty) = p(n,Tn)+p(n,Tn) = 2\left(\frac{3}{2n}\right) = \frac{3}{n}.$$

Therefore,

$$p(Tx,Ty) \leq 2\beta[p(x,Tx)+p(y,Ty)]$$

if

$$2\beta \geq \frac{\frac{1}{2n}}{\frac{3}{n}} = \frac{1}{6}.$$

Thus when $\frac{1}{6} \leq 2\beta < 1$ we have

$$p(Tx,Ty) \leq 2\beta[p(x,Tx)+p(y,Ty)]$$

for all $x,y \in \mathbb{N}^*$, and so condition (i) of Theorem 2.7 is satisfied.
Next we check that the function $f : X \to \mathbb{R}^*$ given by $f(x) = p(x,Tx)$ is not p-l.s.c. at 0.
Let us suppose that f is p-l.s.c. at 0. Then given $\epsilon > 0$, there exists $\delta > 0$ such that

$$p\text{-}B_\delta(0) \subset \{n \in \mathbb{N}^* : p(n,Tn) > p(0,T0)-\epsilon\}.$$

Let $\epsilon = \frac{1}{2}$ and δ may be any positive number. If $n \in p\text{-}B_\delta(0)$, then

$$p(0,n) < p(0,0)+\delta, \text{ or, } \frac{1}{n} < \delta.$$

If we choose δ sufficiently close to 0, then the condition

$$p(n,Tn) > p(0,T(0))-\epsilon = 1-\epsilon$$

is violated.
Therefore f is not p-l.s.c. at 0 and T does not have a fixed point in $\mathbb{N}^*$.

Now we deal with simultaneous fixed points, continuity of fixed points of mappings in a partial metric space where mappings display some kind of contractive nature.

Definition 2.11. Let (X, p) be a partial metric space. A sequence $\{T_n\}$ of self-mappings on X is said to be strongly uniformly convergent to a self-mapping T_0 on X if given $\epsilon > 0$, there exists a natural number N such that for every $x \in X$,

$$p(T_n x, T_0 x) < \epsilon, \text{ for } n \geq N.$$

Theorem 2.12. *Let (X, p) be a complete partial metric space and $\{T_n\}$ be a sequence of self-mappings on X which is strongly uniformly convergent to $T_0 : (X, p) \to (X, p)$ satisfying the following condition:*

$$p(T_0 x, T_0 y) \leq \alpha[p(x, T_0 x) + p(y, T_0 y)] + \beta p(x, y), \text{ for all } x, y \in X,$$

where $\alpha > 0$ and $\beta > 0$ with $\alpha + \beta < 1$. If u_n is a fixed point of T_n in X and u_0 is a fixed point of T_0 in X, then $p\text{-}\lim_{n\to\infty} u_n = u_0$.

Proof. Let $\epsilon > 0$ be given. Then one can find N such that

$$p(T_n x, T_0 x) < \frac{1-\beta}{1+\alpha}\,\epsilon, \text{ for } n > N, \text{ for all } x \in X.$$

Now

$$\begin{aligned} p(u_0, u_n) &= p(T_0 u_0, T_n u_n) \\ &\leq p(T_0 u_0, T_0 u_n) + p(T_0 u_n, T_n u_n) - p(T_0 u_n, T_0 u_n) \end{aligned}$$

and

$$p(T_0 u_0, T_0 u_n) \leq \alpha[p(u_0, T_0 u_0) + p(u_n, T_0 u_n)] + \beta p(u_0, u_n).$$

Therefore

$$\begin{aligned} p(u_0, u_n) &\leq \alpha[p(u_0, u_0) + p(T_n u_n, T_0 u_n)] + \beta p(u_0, u_n) + p(T_0 u_n, T_n u_n), \\ \text{or, } p(u_0, u_n) &\leq \frac{\alpha}{1-\beta}\, p(u_0, u_0) + \frac{1+\alpha}{1-\beta}\, p(T_n u_n, T_0 u_n) \\ &= \frac{\alpha}{1-\beta}\, p(u_0, u_0) + \epsilon, n \geq N. \end{aligned}$$

Therefore $p\text{-}\lim_n u_n < u_0$. □

The condition of strong uniform convergence of the sequence of self-mappings on the partial metric space (X, p) cannot be dispensed with in the above theorem as can be found from the following example.

Example 2.13. We take $X = \{0, \frac{1}{n}\}_{n=1,2,\cdots}$ with usual metric d of reals. We define

$$T_n(0) = T_n(1) = 1$$

and for $m \geq 2$,

$$T_n\left(\frac{1}{m}\right) = \begin{cases} \frac{1}{m}, & \text{if } n = m, \\ 1, & \text{if } n \neq m, \end{cases}$$

where $\{T_n\}$ converges to T_0 pointwise in X, with $T_0(x) = 1$ for $x \in X$. Also T_0 satisfies

$$d(T_0x, T_0y) \leq \alpha[d(x, T_0x) + d(y, T_0y)] + \beta d(x, y), x, y \in X$$
$$\text{and } \alpha > 0, \beta > 0, \alpha + \beta < 1.$$

Here fixed point of T_n is $\frac{1}{n}, n = 1, 2, ...$ and $\frac{1}{n} \to 0$ which is not a fixed point of T_0. Further routine verification shows that $\{T_n\}$ does not strongly uniformly converge to T_0 over X.

Theorem 2.14. *Let (X, p) be a complete partial metric space and T_1, T_2 be two self-mappings on X satisfying*

$$p(T_1x, T_2y) \leq \alpha\ p(x, y), \text{ for all } x, y \in X, 0 \leq \alpha < 1.$$

Then T_1 and T_2 have a common fixed point in X.

Proof. Let $x_0 \in X$. We take $x_{2n+1} = T_1(x_{2n}), x_{2n+2} = T_2(x_{2n+1}), n = 0, 1, 2, \cdots$. Now

$$\begin{aligned} p(x_2, x_1) &= p(T_2(x_1), T_1(x_0)) \\ &\leq \alpha\ p(x_0, x_1), \\ p(x_3, x_2) &= p(T_1(x_2), T_2(x_1)) \\ &\leq \alpha\ p(x_2, x_1) \leq \alpha^2 p(x_0, x_1). \end{aligned}$$

In general

$$p(x_{n+1}, x_n) \leq \alpha^n p(x_0, x_1).$$

So

$$\begin{aligned} p(x_{n+m}, x_n) &\leq p(x_{n+m}, x_{n+m-1}) + p(x_{n+m-1}, x_{n+m-2}) + \cdots + p(x_{n+1}, x_n) \\ &\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{n+m-1}) p(x_0, x_1) \\ &< \frac{\alpha^n}{1-\alpha}\ p(x_0, x_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies that $\{x_n\}$ is Cauchy in X and since X is complete, we can find $u \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0 = \lim_{n\to\infty} p(x_n, u) = p(u, u).$$

Now

$$p(T_1u, T_2(x_{2n-1})) \leq \alpha p(u, x_{2n-1}) \to 0 \text{ as } n \to \infty \quad \text{or,}$$
$$p(T_1u, x_{2n}) \to 0 \text{ as } n \to \infty.$$

Therefore

$$\begin{aligned} p(u, T_1u) &\leq p(u, x_{2n}) + p(x_{2n}, T_1u) - p(x_{2n}, x_{2n}) \\ &\leq p(u, x_{2n}) + p(x_{2n}, T_1u) \to 0 \text{ as } n \to \infty. \end{aligned}$$

So

$$p(u, T_1u) = 0 = p(u, u) = p(T_1u, T_1u),$$

which gives $u = T_1(u)$. Again

$$p(T_2u, T_1x_{2n}) \leq \alpha\ p(x_{2n}, u) \to 0 \text{ as } n \to \infty,$$

or,

$$p(T_2u, x_{2n+1}) \to 0 \text{ as } n \to \infty.$$

Now $p(u, T_2u) \leq p(u, x_{2n+1}) + p(x_{2n+1}, T_2u) \to 0$ as $n \to \infty$ giving $u = T_2(u)$. Hence u is a common fixed point of T_1 and T_2. □

Theorem 2.15. *Let (X, p) be a complete partial metric space and $T : X \to X$ be a mapping satisfying*

$$p(Tx, Ty) \leq h \max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\}, 0 < h < \frac{1}{2}.$$

Then T has a unique fixed point u in X.

Proof. We take any $x_0 \in X$ and consider the iterated sequence $x_n = T^n(x_0)$, $n = 1, 2, \cdots$. Then

$$\begin{aligned} & p(Tx_{n-1}, Tx_n) \\ = \ & p(x_n, x_{n+1}) \\ \leq \ & h \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), p(x_{n-1}, x_{n+1}), p(x_n, x_n)\} \\ = \ & h \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), p(x_{n-1}, x_{n+1})\}. \end{aligned}$$

Now $p(x_n, x_{n+1})$ is not the maximum value of $\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), p(x_{n-1}, x_{n+1})\}$ because it leads to $p(x_n, x_{n+1}) \leq hp(x_n, x_{n+1})$ which is impossible. Now following two cases arise:

Case I. If the maximum value is $p(x_{n-1}, x_n)$, then $p(x_n, x_{n+1}) \leq hp(x_{n-1}, x_n)$.

Case II. If the maximum value is $p(x_{n-1}, x_{n+1})$, then

$$\begin{aligned} p(x_n, x_{n+1}) & \leq hp(x_{n-1}, x_{n+1}) \\ & \leq h[p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n)] \\ & \leq h[p(x_{n-1}, x_n) + p(x_n, x_{n+1})] \\ \text{or, } p(x_n, x_{n+1}) & \leq \frac{h}{1-h} p(x_{n-1}, x_n). \end{aligned}$$

From these two cases, we find that

$$\begin{aligned} p(x_n, x_{n+1}) & \leq \max\left(h, \frac{h}{1-h}\right) p(x_{n-1}, x_n) \\ & = \frac{h}{1-h}\, p(x_{n-1}, x_n) \\ & \leq \left(\frac{h}{1-h}\right)^2 p(x_{n-2}, x_{n-1}) \\ & \cdots \\ & \leq \left(\frac{h}{1-h}\right)^n p(x_0, x_1) \\ & = \lambda^n p(x_0, x_1), \text{ where } \lambda = \frac{h}{1-h} < 1. \end{aligned}$$

Let $m > n$, then

$$\begin{aligned} & p(x_n, x_m) \\ \leq \; & p(x_n, x_{n+1}) + \cdots + p(x_{m-1}, x_m) \\ & -\{p(x_{n+1}, x_{n+1}) + \cdots + p(x_{m-1}, x_{m-1})\} \\ \leq \; & \lambda^n \left(1 + \lambda + \cdots + \lambda^{m-n-1}\right) p(x_0, x_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $\{x_n\}$ is Cauchy in (X, p). Hence we can find $u \in X$ to satisfy

$$\lim_{m,n\to\infty} p(x_m, x_n) = 0 = \lim_{n\to\infty} p(x_n, u) = p(u, u).$$

Now

$$\begin{aligned} p(x_{n+1}, Tu) &= p(Tx_n, Tu) \\ &\leq h \max\{p(x_n, u), p(x_n, x_{n+1}), p(u, Tu), p(x_n, Tu), p(u, Tx_n)\}. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} p(x_{n+1}, Tu) &\leq h \max\{p(u, Tu), \lim_{n\to\infty} p(x_n, Tu)\} \\ &= h\, p(u, Tu) \\ &\leq h[p(u, x_{n+1}) + p(x_{n+1}, Tu)], \text{ for all } n \\ &\to h \lim_{n\to\infty} p(u, x_{n+1}) + h \lim_{n\to\infty} p(x_{n+1}, Tu) \end{aligned}$$

giving

$$\lim_{n\to\infty} p(x_{n+1}, Tu) = 0.$$

Thus

$$\begin{aligned} p(u, Tu) &\leq p(u, x_{n+1}) + p(x_{n+1}, Tu) - p(x_{n+1}, x_{n+1}) \\ &\leq p(u, x_{n+1}) + p(x_{n+1}, Tu) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $p(u, Tu) = 0$ and so $u = Tu$.
Suppose $u = Tu$ and $v = Tv$ for some $u, v \in X$. Then

$$\begin{aligned} p(u, v) &= p(Tu, Tv) \\ &\leq h \max\{p(u, v), p(u, Tu), p(v, Tv), p(u, Tv), p(v, Tu)\}, 0 < h < \frac{1}{2} \\ &= h\, p(u, v). \end{aligned}$$

Thus $p(u, v) = 0$ and this gives $u = v$. □

Theorem 2.16. *Let (X, p) be a complete partial metric space and $\{T_n\}$ be a sequence of self-mappings on X satisfying*

$$p(T_n x, T_n y) \leq h \max\{p(x, y), p(x, T_n x), p(y, T_n y), p(x, T_n y), p(y, T_n x)\}, 0 < h < \frac{1}{2}.$$

Let $T_0 : X \to X$ such that $\{T_n\}$ converges to T_0 pointwise. If $p(x, y)$ is sectionally continuous in each variable x and y, then T_0 has a fixed point $u_0 \in X$ such that $u_0 = \lim_{n\to\infty} u_n$, where u_n is a fixed point of $T_n (n = 1, 2, \cdots)$.

Proof. Given

$$p(T_nx, T_ny) \leq h\max\{p(x, y), p(x, T_nx), p(y, T_ny), p(x, T_ny), p(y, T_nx)\},$$

p being sectionally continuous, passing on to limit as $n \to \infty$ gives

$$p(T_0x, T_0y) \leq h\max\{p(x, y), p(x, T_0x), p(y, T_0y), p(x, T_0y), p(y, T_0x)\}.$$

By the previous theorem, T_0 has a fixed point $u_0 \in X$, i.e., $T_0(u_0) = u_0$. Taking $u_n = T_n(u_n), n = 1, 2, \cdots$, we have

$$\begin{aligned} p(u_0, u_n) &= p(T_0u_0, T_nu_n) \\ &\leq p(T_0u_0, T_nu_0) + p(T_nu_0, T_nu_n) - p(T_nu_0, T_nu_0). \end{aligned}$$

Now,

$$p(T_nu_0, T_nu_n) \leq h\max\{p(u_0, u_n), p(u_0, T_nu_0), p(u_n, T_nu_n), p(u_0, T_nu_n), p(u_n, T_nu_0)\}.$$

Therefore

$$\begin{aligned} & p(u_0, u_n) \\ \leq\ & p(T_0u_0, T_nu_0) \\ & +h\max[p(u_0, u_n), p(u_0, T_nu_0), p(u_n, T_nu_n), p(u_0, u_n), p(u_n, T_nu_0)] \\ \leq\ & p(T_0u_0, T_nu_0) \\ & +h\max\{p(u_0, u_n), p(T_0u_0, T_nu_0), p(u_n, u_n), \\ & p(u_0, u_n), p(u_n, u_0) + p(u_0, T_nu_0)\} \\ \leq\ & p(T_0u_0, T_nu_0) + h[p(u_n, u_0) + p(T_0u_0, T_nu_0)]. \end{aligned}$$

And so

$$p(u_0, u_n) \leq \frac{1+h}{1-h}\, p(T_0u_0, T_nu_0).$$

Hence

$$\lim_{n\to\infty} p(u_0, u_n) \leq \frac{1+h}{1-h} \lim_{n\to\infty} p(T_0u_0, T_nu_0) = 0.$$

Therefore

$$\lim_{n\to\infty} p(u_0, u_n) = 0$$

and so $u_0 = \lim_{n\to\infty} u_n$. □

Theorem 2.17. *(Caccioppoli like theorem)Let (X, p) be a complete partial metric space and $T_i : (X, p) \to (X, p)$ satisfy*

$$p(T_ix, T_jy) \leq c_{ij}\max\{p(x, y), p(x, T_ix), p(y, T_jy), \frac{1}{2}[p(x, T_jy) + p(y, T_ix)]\},$$

$x \neq y$ in X, where $0 \leq c_{ij} \leq \alpha, 0 < \alpha < 1$ for all i, j, such that $\sum_{i=1}^{\infty} c_{i,i+1}$ is $(C, 1)$ summable. Then $\{T_i\}$ has a unique common fixed point in X.

Proof. We take $x_0 \in X$ and construct a sequence $\{x_n\}$ by $x_n = T_n(x_{n-1})$, $n = 1, 2, \cdots$. Now

$$\begin{aligned} p(x_n, x_{n+1}) &= p(T_n x_{n-1}, T_{n+1} x_n) \\ &\leq c_{n,n+1} \max\{p(x_{n-1}, x_n), p(x_{n-1}, T_n x_{n-1}), p(x_n, T_{n+1} x_n), \\ &\quad \frac{1}{2}[p(x_{n-1}, T_{n+1} x_n) + p(x_n, T_n x_{n-1})]\} \\ &= c_{n,n+1} \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \\ &\quad \frac{1}{2}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)]\}. \end{aligned}$$

Now $p(x_{n-1}, x_{n+1}) \leq p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n)$.
Therefore the above inequality reads as

$$\begin{aligned} p(x_n, x_{n+1}) &\leq c_{n,n+1} \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \\ &\quad \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})]\} \\ &\leq c_{n,n+1} \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}. \end{aligned}$$

Now max value $\neq p(x_n, x_{n+1})$ because $p(x_n, x_{n+1}) \leq c_{n,n+1} p(x_n, x_{n+1})$ is untenable. Therefore, we have

$$\begin{aligned} p(x_n, x_{n+1}) &\leq c_{n,n+1}\ p(x_{n-1}, x_n) \\ &\leq c_{n,n+1}\ c_{n-1,n}\ p(x_{n-2}, x_{n-1}) \\ &\leq c_{n,n+1}\ c_{n-1,n} \cdots c_{1,2}\ p(x_0, x_1) \\ &= \prod_{i=1}^{n} c_{i,i+1} p(x_0, x_1). \end{aligned}$$

Now, for $m > n$,

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + \cdots + p(x_{m-1}, x_m) \\ &= p(x_n, x_{n+1}) + \cdots + p(x_{n+(m-n-1)}, x_{n+(m-n)}) \\ &\leq \sum_{k=n}^{n+(m-n-1)} \left(\prod_{i=1}^{k} c_{i,i+1} \right) p(x_0, x_1) \\ &\leq \sum_{k=n}^{n+(m-n-1)} \left(\frac{\sum_{i=1}^{k} c_{i,i+1}}{k} \right)^k p(x_0, x_1). \end{aligned}$$

Putting $s_k = \sum_{i=1}^{k} c_{i,i+1}$ and $S_k = \frac{\sum_{\nu=1}^{k} s_\nu}{k}$, by $(C, 1)$ summability of $\sum_{i=1}^{\infty} c_{i,i+1}$, we have $\sum_{k=1}^{\infty} S_k < \infty$. Therefore $\lim_{k \to \infty} S_k = 0$. As $0 < \alpha < 1$, then $S_k < \alpha$ for large values of k, and thus

$$\frac{S_k}{k} \leq S_k < \alpha, \text{ for large values of } k.$$

Therefore
$$\lim_{k\to\infty}\left(\frac{S_k}{k}\right)^k \le \lim_{k\to\infty}\alpha^k = 0.$$
Treating case of $n > m$ in a similar way, one concludes that $\{x_n\}$ is a Cauchy sequence in (X, p) which is complete. Let $u \in X$ with
$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, u) = p(u, u),$$
i.e.,
$$0 = \lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, u) = p(u, u).$$
Given m, we have
$$\begin{aligned}
& p(T_m u, u) \\
\le\ & p(T_m u, T_n x_{n-1}) + p(T_n x_{n-1}, u) \\
=\ & p(T_m u, T_n x_{n-1}) + p(x_n, u) \\
\le\ & c_{m,n} \max\{p(u, x_{n-1}), p(u, T_m u), p(x_{n-1}, T_n x_{n-1}), \\
& \frac{1}{2}[p(u, T_n x_{n-1}) + p(x_{n-1}, T_m u)]\} + p(x_n, u) \\
<\ & \alpha \max\{p(u, x_{n-1}), p(u, T_m u), p(x_{n-1}, x_n), \frac{1}{2}[p(u, x_n) + p(x_{n-1}, T_m u)]\} \\
& + p(x_n, u).
\end{aligned}$$
Now passing on to limit as $n \to \infty$, we have
$$p(T_m u, u) \le \alpha \max\{p(u, u), p(u, T_m u), p(u, u), \frac{1}{2}[p(u, u) + p(u, T_m u)]\} + p(u, u)$$
$$\text{or, } p(T_m u, u) \le \alpha \max\{p(u, T_m u), p(u, u)\} + p(u, u).$$
As $p(u, u) = 0$, we have
$$p(T_m u, u) \le \alpha\ p(u, T_m u).$$
That means $p(T_m u, u) = 0$ giving $T_m u = u$.
For uniqueness of u as a common fixed point of $\{T_n\}$, suppose $T_n u = u$ and $T_n v = v$ for all n, and $u, v \in X$, we have
$$\begin{aligned}
p(u, v) &= p(T_n u, T_n v) \\
&\le c_{n,n} \max\{p(u, v), p(u, T_n u), p(v, T_n v), \frac{1}{2}[p(u, T_n v) + p(v, T_n u)]\} \\
&= c_{n,n} \max\{p(u, v), p(u, u), p(v, v), \frac{1}{2}[p(u, v) + p(v, u)]\}
\end{aligned}$$
As $p(u, u) \le p(u, v)$ and $p(v, v) \le p(u, v)$, we get
$$p(u, v) \le c_{n,n}\ p(u, v)$$
That means $p(u, v) = 0$ and hence $u = v$. □

Theorem 2.17 gives following fixed point theorem due to Ray in [16] as a Corollary.

Theorem 2.18. *(Ray's Theorem) Let X be a complete metric space, $0 < c_{ij} < 1$ and $\sum_{i=1}^{\infty} c_{i,i+1}$ is $(C, 1)$ summable, and $T_i : X \to X$ satisfies*

$$d(T_i x, T_j y) \leq c_{ij} d(x, y) \text{ for } x, y \in X; x \neq y; i, j = 1, 2, \cdots .$$

Then $\{T_i\}$ has a unique common fixed point in X.

Remark 2.19. One may be inclined to infer from Ray's Theorem above that at a non-isolated point $x \in X$, all T_i's agree, and in that case a single contraction mapping acting on the closed subspace of all non-isolated points of X does the job. This inclination is false as we see that a contraction mapping may send a non-isolated point to an isolated point of a metric space.

Example 2.20. We take $S_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}, n = 1, 2, \cdots$ and

$$X = \{1, 2, S_{n+1}, 1 + S_n, n \in \mathbb{N}\}.$$

Then X is a complete metric space with usual metric d of reals. Now we define $T : X \to X$ where

$$\begin{aligned} T(1) &= T(S_{n+1}) = 1 + S_1, \\ T(1 + S_n) &= 1 + S_{n+1} \\ \text{and } T(2) &= 2. \end{aligned}$$

By routine check up we see that T is a contraction mapping satisfying

$$d(Tx, Ty) \leq \frac{1}{2} d(x, y),$$

for all $x, y \in X$. Here T sends non-isolated point $x = 1$ to $T(1) = 1 + \frac{1}{2}$ that is an isolated point in X.

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