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AUXILIARY PRINCIPLE TECHNIQUE FOR SOLVING REGULARIZED NONCONVEX MIXED EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we introduce a regularized nonconvex mixed equilibrium problem and suggest iterative algorithms for solving such a problem by using the auxiliary principle technique. The convergence analysis of the proposed iterative algorithms is discussed either under pseudomonotonicity or partially mixed relaxed and strong monotonicity of type (I) property of the bifunctions involved in the formulation. We also point out some fatal errors in [M.A. Noor et al.: On nonconvex bifunction variational inequalities. Optim. Lett. **6**, 1477-1488 (2012)] and [M.A. Noor et al.: Some iterative methods for solving nonconvex bifunction equilibrium variational inequalities. J. Appl. Math. Volume 2012, Article ID 280451]. Finally, we present the correct version of the results presented in these references.

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1. Introduction

The equilibrium problem (in short, EP) provides an unified framework to study a large variety of problems such as optimization problems, variational inequality problems, saddle point problems, complementarity problems, Nash equilibria problems, fixed point problems, etc, see, for example, [2, 10, 11, 17, 20] and the references therein. The mixed equilibrium problem is one of the most useful generalizations of EP which contains hemivariational inequality problems [22], nonsmooth variational inequality problems [5] and several other problems studied in [24, 25] and the references therein. In the recent years much attention has been devoted to the study of different classes of variational inequalities in the setting of uniformly prox-regular sets

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[13, 14], such sets are nonconvex but include the convex sets as special cases, see, for example, [1, 3, 4, 7, 8, 16, 19, 26, 28].

Recently, Noor et al. [24] considered so-called nonconvex bifunction variational inequalities. They suggested and analyzed some iterative methods for solving them by using the auxiliary principle technique [18]. They also studied the convergence analysis of the proposed iterative algorithms under certain conditions.

Very recently, Noor et al. [25] introduced and studied so-called nonconvex bifunction equilibrium variational inequalities. By using the auxiliary principle technique [18], they suggested some iterative methods for solving such class of problems. They claimed that the convergence of the sequences generated by suggested algorithms requires only monotonicity property of the bifunctions involved in the considered problem.

The main aim of this work is to introduce of a class of regularized nonconvex mixed equilibrium problems and to propose and analyze some iterative schemes for finding the approximate solutions of this class of problems. We also point out some errors in the results given in [24, 25] and present the correct version of these results.

The outline of this paper is as follows. Next section presents some known definitions, notations and results. In Section 3, we give the formulation of a regularized nonconvex mixed equilibrium problem (in short, RNMEP). By using the auxiliary principle technique, some iterative algorithms for solving RNMEP are suggested and analyzed. The convergence analysis of the proposed iterative algorithms is studied either under pseudomonotonicity or partially mixed relaxed and strong monotonicity of type (I) property of the bifunctions involved in RNMEP. The results presented in this section represent an improvement and generalization of the results in [23, 24] and the references therein. The final section is devoted to investigate and analyze the results given in [24, 25]. Some errors in the main results of [24, 25] are detected. We have pointed out that contrary to the claim in [25], the monotonicity property of the bifunctions involved in the considered problem is not enough for proving the convergence of the sequences generated by the proposed algorithms. By using the results presented in Section 3, we derive the correct version of the main results in [24, 25].

2. Preliminaries and basic facts

Throughout the paper, unless otherwise specified, let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let K be a nonempty closed subset of $\mathcal{H}$. We denote by $d_K(.)$ or $d(.,K)$ the usual distance function from a point to the set K, that is, $d_K(u) = \inf\limits_{v \in K} \|u - v\|$.

Definition 2.1. Let $u \in \mathcal{H}$. A point $v \in K$ is called the *projection* of u onto K if $d_K(u) = \|u - v\|$. The set of all such points is denoted by $P_K(u)$, that is,

$$P_K(u) := \{v \in K : d_K(u) = \|u - v\|\}.$$

Definition 2.2. The *proximal normal cone* of K at a point $u \in K$ is given by

$$N_K^P(u) := \{\xi \in \mathcal{H} : \exists \alpha > 0 \text{ such that } u \in P_K(u + \alpha\xi)\}.$$

Clarke et al. [15] introduced a nonconvex set, called *proximally smooth set*. Subsequently, it has been investigated by Poliquin et al. [27] but under the name of uniformly prox-regular set. Such kind of sets are used in many nonconvex applications in optimization, economic models, dynamical systems, differential inclusions, etc.

Definition 2.3. [15] For a given $r \in (0, +\infty]$, a subset K_r of $\mathcal{H}$ is said to be *normalized uniformly prox-regular* (or *uniformly r-prox-regular*) if every nonzero proximal normal to K_r can be realized by an r-ball.

This means that for all $\bar{x} \in K_r$ and all $0 \neq \xi \in N^P_{K_r}(\bar{x})$,

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2, \quad \text{for all } x \in K_r.$$

The class of normalized uniformly prox-regular sets includes the class of convex sets, p-convex sets [12], $C^{1,1}$ submanifolds (possibly with boundary) of $\mathcal{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets [15].

Lemma 2.4. [15] *A closed set $K \subseteq \mathcal{H}$ is convex if and only if it is uniformly r-prox-regular for every $r > 0$.*

If $r = +\infty$, then in view of Definition 2.3 and Lemma 2.4, the uniform r-prox-regularity of K_r is equivalent to the convexity of K_r. That is, for $r = +\infty$, we set $K_r = K$.

The union of two disjoint intervals $[a, b]$ and $[c, d]$ is uniformly r-prox-regular with $r = \frac{c-b}{2}$ [14, 27].

3. Main results

From now onward, unless otherwise specified, we assume that K_r is a closed and uniformly r-prox-regular set in $\mathcal{H}$ and $\varphi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is an univariate proper extended real-valued continuous function. Suppose further that $F, B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ are bifunctions such that F is continuous in the first argument, and B is continuous in both the arguments. We consider the problem of finding $u \in K_r$ such that

$$F(u, v) + B(u, v - u) + \varphi(v) - \varphi(u) + \gamma\|v - u\|^2 \geq 0, \quad \forall v \in K_r, \tag{3.1}$$

where $\gamma = \frac{1}{2r}$. It is called *regularized nonconvex mixed equilibrium problem* (RNMEP).

By considering different choices of F, B and φ, we can easily obtain the mixed equilibrium problem, the nonsmooth variational inequality problem, the hemivariational inequality problem, etc., studied in [6, 7, 10, 11, 17, 21, 22, 24, 25] and the references therein. We denote by RNMEP(F, B, φ, K_r) the set of solutions of problem (3.1).

For a given $u \in K_r$, consider the auxiliary regularized nonconvex mixed equilibrium problem of finding $w \in K_r$ such that

$$\begin{aligned} \rho F(w, v) + \rho B(w, v - w) + \langle w - u, v - w\rangle + \rho\varphi(v) - \rho\varphi(w) \\ + \rho\gamma\|v - w\|^2 \geq 0, \quad \forall v \in K_r, \end{aligned} \tag{3.2}$$

where $\rho > 0$ is a constant. If $w = u$, then obviously w is a solution of RNMEP (3.1). This fact enables us to suggest the following predictor-corrector method for solving RNMEP (3.1).

Algorithm 3.1. For a given $u_0 \in K_r$, compute $\{u_n\} \in K_r$ by the following iterative scheme:

$$\rho F(u_{n+1}, v) + \rho B(u_{n+1}, v - u_{n+1}) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ + \rho\varphi(v) - \rho\varphi(u_{n+1}) + \rho\gamma\|v - u_{n+1}\|^2 \geq 0, \quad \forall v \in K_r, \quad (3.3)$$

where $\rho > 0$ is a constant and $n = 0, 1, 2 \ldots$.

In order to prove the convergence of the iterative sequence generated by Algorithm 3.1, we need the following definition.

Definition 3.2. Let $\gamma > 0$. The bifunctions $F, B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ are said to be *jointly pseudomonotone* with respect to the function φ with constant γ if

$$F(u_1, u_2) + B(u_1, u_2 - u_1) + \varphi(u_2) - \varphi(u_1) + \gamma\|u_2 - u_1\|^2 \geq 0$$

implies that

$$F(u_2, u_1) + B(u_2, u_1 - u_2) + \varphi(u_1) - \varphi(u_2) + \gamma\|u_2 - u_1\|^2 \leq 0, \forall u_1, u_2 \in K_r.$$

The following proposition plays a key role in the study of convergence analysis of the iterative sequence generated by Algorithm 3.1.

Proposition 3.3. *Let $\{u_n\}$ be a sequence generated by Algorithm* 3.1 *and $u \in K_r$ be a solution of* RNMEP (3.1). *If F and B are jointly pseudomonotone with respect to the function φ with constant γ, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2, \quad \forall n \geq 0. \quad (3.4)$$

Proof. Since $u \in K_r$ is a solution of RNMEP (3.1), we have

$$\rho F(u, v) + \rho B(u, v - u) + \rho\varphi(v) - \rho\varphi(u) + \rho\gamma\|v - u\|^2 \geq 0, \quad \forall v \in K_r,$$

where $\rho > 0$ is an arbitrary real constant as in Algorithm 3.1. Taking $v = u_{n+1}$ in the above inequality, we obtain

$$\rho F(u, u_{n+1}) + \rho B(u, u_{n+1} - u) + \rho\varphi(u_{n+1}) - \rho\varphi(u) + \rho\gamma\|u_{n+1} - u\|^2 \geq 0.$$

Since F and B are jointly pseudomonotone with respect to the function φ with constant γ, we have

$$\rho F(u_{n+1}, u) + \rho B(u_{n+1}, u - u_{n+1}) + \rho\varphi(u) - \rho\varphi(u_{n+1}) + \rho\gamma\|u_{n+1} - u\|^2 \leq 0. \quad (3.5)$$

Taking $v = u$ in (3.3), we get

$$\rho F(u_{n+1}, u) + \rho B(u_{n+1}, u - u_{n+1}) + \langle u_{n+1} - u_n, u - u_{n+1}\rangle \\ + \rho\varphi(u) - \rho\varphi(u_{n+1}) + \rho\gamma\|u - u_{n+1}\|^2 \geq 0. \quad (3.6)$$

By combining (3.5) and (3.6), we obtain

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq -\rho F(u_{n+1}, u) - \rho B(u_{n+1}, u - u_{n+1}) \\ + \rho\varphi(u_{n+1}) - \rho\varphi(u) - \rho\gamma\|u - u_{n+1}\|^2 \geq 0. \quad (3.7)$$

On the other hand, by utilizing well known property of the inner product, we have

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u_{n+1} - u_n\|^2 - \|u - u_{n+1}\|^2. \quad (3.8)$$

Applying (3.7) and (3.8), it follows that

$$\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2 - \|u_n-u_{n+1}\|^2,$$

which is the required result (3.4). This completes the proof. □

We now establish the convergence of iterative sequence generated by Algorithm 3.1 to a solution of RNMEP (3.1).

Theorem 3.4. *Let $\mathcal{H}$ be a finite dimensional real Hilbert space,* RNMEP(F,B,φ,K_r) *be nonempty and all the conditions of Proposition* 3.3 *hold. Then, the iterative sequence $\{u_n\}$ generated by Algorithm* 3.1 *converges to a solution $\hat{u} \in K_r$ of* RNMEP (3.1)*.*

Proof. Let $u \in K_r$ be a solution of RNMEP (3.1). It follows from (3.4) that the sequence $\{\|u_n-u\|\}$ is nonincreasing and so the sequence $\{u_n\}$ is bounded. Further, applying (3.4), we obtain

$$\sum_{n=0}^{\infty} \|u_{n+1}-u_n\|^2 \leq \|u_0-u\|^2,$$

which implies that $\|u_{n+1}-u_n\| \to 0$ as $n \to \infty$. Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$. Since $\{u_n\}$ is a bounded sequence, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \to \hat{u}$ as $i \to \infty$. By (3.3), for any subsequence $\{u_{n_{i_k}}\}$ of $\{u_{n_i}\}$, we have

$$\begin{aligned}\rho F(u_{n_{i_k}+1},v) + \rho B(u_{n_{i_k}+1}, v-u_{n_{i_k}+1}) + \langle u_{n_{i_k}+1}-u_{n_{i_k}}, v-u_{n_{i_k}+1}\rangle \\ + \rho\varphi(v) - \rho\varphi(u_{n_{i_k}+1}) + \rho\gamma\|v-u_{n_{i_k}+1}\|^2 \geq 0, \quad \forall v \in K_r. \qquad (3.9)\end{aligned}$$

Letting $k \to \infty$ in relation (3.9) and by using the continuity of F, B and φ, we get

$$F(\hat{u},v) + B(\hat{u},v-\hat{u}) + \varphi(v) - \varphi(\hat{u}) + \gamma\|v-\hat{u}\|^2 \geq 0, \quad \forall v \in K_r,$$

that is, $\hat{u} \in K_r$ is a solution of RNMEP (3.1). Accordingly, Proposition 3.3 implies that

$$\|u_{n+1}-\hat{u}\| \leq \|u_n-\hat{u}\|, \quad \forall n \geq 0. \qquad (3.10)$$

By virtue of inequality (3.10), it follows that $u_n \to \hat{u}$ as $n \to \infty$. Thus, the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$. This completes the proof. □

It is well known that to implement the proximal point method, one has to calculate the approximate solution implicitly, which is itself a difficult task. To overcome with this drawback, we consider the following auxiliary nonconvex mixed equilibrium problem and iterative algorithm for solving RNMEP (3.1).

For a given $u \in K_r$, consider the auxiliary regularized nonconvex mixed equilibrium problem of finding $w \in K_r$ such that

$$\begin{aligned}\rho F(u,v) + \rho B(u,v-u) + \langle w-u, v-w\rangle + \rho\varphi(v) - \rho\varphi(w) \\ + \rho\gamma\|v-w\|^2 \geq 0, \quad \forall v \in K_r, \qquad (3.11)\end{aligned}$$

where $\rho > 0$ is a constant. Clearly, problems (3.2) and (3.11) are quite different. If $w = u$, then w is a solution of RNMEP (3.1). We apply this fact to suggest the following predictor-corrector algorithm for solving RNMEP (3.1).

Algorithm 3.5. For a given $u_0 \in K_r$, define the iterative sequence $\{u_n\}$ in K_r by the following iterative scheme:

$$\begin{aligned}\rho F(u_n, v) + \rho B(u_n, v - u_n) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ + \rho\varphi(v) - \rho\varphi(u_{n+1}) + \rho\gamma\|v - u_{n+1}\|^2 \geq 0, \quad \forall v \in K_r, \end{aligned}\tag{3.12}$$

where $\rho > 0$ is a constant and $n = 0, 1, 2 \ldots$.

We need the following definitions to establish the convergence of iterative sequence generated by Algorithm 3.5.

Definition 3.6. A bifunction $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to be

(a) *monotone* if

$$B(u_1, u_2 - u_1) + B(u_2, u_1 - u_2) \leq 0, \quad \forall u_1, u_2 \in K_r;$$

(b) *partially (ϱ, θ)-mixed relaxed and strongly monotone of type* (I) if there exist constants $\varrho, \theta > 0$ such that

$$B(u_1, u_2 - u_1) + B(u_2, z - u_2) \leq \varrho\|z - u_1\|^2 - \theta\|z - u_2\|^2, \quad \forall u_1, u_2, z \in K_r.$$

Definition 3.7. The bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to be

(a) *monotone* if

$$F(u_1, u_2) + F(u_2, u_1) \leq 0, \quad \forall u_1, u_2 \in K_r;$$

(b) *partially ζ-relaxed monotone of type* (I) if there exists a constant $\zeta > 0$ such that

$$F(u_1, u_2) + F(u_2, z) \leq \zeta\|z - u_1\|^2, \quad \forall u_1, u_2, z \in K_r.$$

The following proposition is the main tool to study the convergence analysis of the iterative sequence generated by Algorithm 3.5.

Proposition 3.8. *Let $\{u_n\}$ be a sequence generated by Algorithm* 3.5 *and $u \in K_r$ be a solution of* RNMEP (3.1). *If F is partially β-relaxed monotone of type* (I)*, and B is partially $(\alpha, 2\gamma)$-mixed relaxed and strongly monotone of type* (I)*, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - (1 - 2(\alpha + \beta)\rho)\|u_{n+1} - u_n\|^2, \quad \forall n \geq 0. \tag{3.13}$$

Proof. Since $u \in K_r$ is a solution of RNMEP (3.1), we have

$$\rho F(u, v) + \rho B(u, v - u) + \rho\varphi(v) - \rho\varphi(u) + \rho\gamma\|v - u\|^2 \geq 0, \quad \forall v \in K_r,$$

where $\rho > 0$ is the same as in Algorithm 3.5. Taking $v = u_{n+1}$ in the above inequality, we get

$$\rho F(u, u_{n+1}) + \rho B(u, u_{n+1} - u) + \rho\varphi(u_{n+1}) - \rho\varphi(u) + \rho\gamma\|u_{n+1} - u\|^2 \geq 0. \tag{3.14}$$

Letting $v = u$ in (3.12), we obtain

$$\rho F(u_n, u) + \rho B(u_n, u - u_n) + \langle u_{n+1} - u_n, u - u_{n+1}\rangle + \rho\varphi(u) - \rho\varphi(u_{n+1}) + \rho\gamma\|u - u_{n+1}\|^2 \geq 0. \tag{3.15}$$

By combining (3.14) and (3.15) and considering the facts that F is partially β-relaxed monotone of type (I), and B is partially $(\alpha, 2\gamma)$-mixed relaxed and strongly monotone of type (I), we obtain

$$\begin{aligned}
&\langle u_{n+1} - u_n, u - u_{n+1}\rangle \\
&\geq -\rho F(u_n, u) - \rho B(u_n, u - u_n) + \rho\varphi(u_{n+1}) - \rho\varphi(u) - \rho\gamma\|u - u_{n+1}\|^2 \\
&\geq -\rho\Big(F(u_n, u) + F(u, u_{n+1}) + B(u_n, u - u_n) + B(u, u_{n+1} - u)\Big) \\
&\quad - 2\rho\gamma\|u - u_{n+1}\|^2 \\
&\geq -\beta\rho\|u_{n+1} - u_n\|^2 - \alpha\rho\|u_{n+1} - u_n\|^2 \\
&= -(\alpha + \beta)\rho\|u_{n+1} - u_n\|^2.
\end{aligned} \tag{3.16}$$

Making use of (3.8) and (3.16), we yield

$$\|u - u_n\|^2 - \|u - u_{n+1}\|^2 - \|u_{n+1} - u_n\|^2 \geq -2(\alpha + \beta)\rho\|u_{n+1} - u_n\|^2,$$

which leads to

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - (1 - 2(\alpha + \beta)\rho)\|u_{n+1} - u_n\|^2,$$

the required result (3.13). $\square$

Now we discuss the convergence of iterative sequence generated by Algorithm 3.5 to a solution of RNMEP (3.1).

Theorem 3.9. *Let $\mathcal{H}$ be a finite dimensional real Hilbert space,* RNMEP(F, B, φ, K_r) *be nonempty and all the conditions of Proposition* 3.8 *hold. If $\rho \in \Big(0, \frac{1}{2(\alpha+\beta)}\Big)$, then the iterative sequence $\{u_n\}$ generated by Algorithm* 3.5 *converges to a solution $\hat{u} \in K_r$ of* RNMEP (3.1).

Proof. Let $u \in K_r$ be a solution of RNMEP (3.1). From (3.13), it follows that the sequence $\{\|u_n - u\|\}$ is nonincreasing and so the sequence $\{u_n\}$ is bounded. Moreover, inequality (3.13) implies that

$$\sum_{n=0}^{\infty}(1 - 2(\alpha + \beta)\rho)\|u_{n+1} - u_n\|^2 \leq \|u - u_0\|^2,$$

whence we deduce that $\|u_{n+1} - u_n\| \to 0$ as $n \to \infty$. Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$. As in the proof of Theorem 3.4, one can establish that $\hat{u} \in K_r$ is a solution of RNMEP (3.1) and the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$. $\square$

4. Some extra remarks

The main focus of this section is to investigate and analyze the main results presented in [24, 25] and to point out some errors in these papers. We present the correct version of the corresponding results given in [24, 25].

For a given bifunction $B(.,.): \mathcal{H}\times\mathcal{H}\to\mathbb{R}$, Noor et al. [24] considered the problem of finding $u\in K_r$ such that

$$B(u,v-u)+\gamma\|v-u\|^2\geq 0,\quad \forall v\in K_r, \tag{4.1}$$

where $\gamma=\frac{1}{2r}$. They called it *nonconvex bifunction variational inequality problem.* If we consider the convex case, then problem (4.1) is called *nonsmooth variational inequality problem* as it provides the necessary and sufficient condition for a solution of a nonsmooth minimization problem. A comprehensive study on nonsmooth variational inequality problems can be found in [5]. Therefore, rest of the paper, problem (4.1) shall be called *regularized nonconvex nonsmooth variational inequality problem* (RNNVIP) as it is commonly used in the literature. We denote by RNNVIP(B,K_r) the set of solutions of RNNVIP (4.1).

Noor et al. [24] considered the following *auxiliary regularized nonconvex variational inequality problem*: For a given $u\in K_r$ satisfying (4.1), find $w\in K_r$ such that

$$\rho B(w,v-w)+\langle w-u,v-w\rangle+\rho\gamma\|v-w\|^2\geq 0,\quad \forall v\in K_r, \tag{4.2}$$

where $\rho>0$ is a constant.

It should be noted that the auxiliary problem (4.2) is considered for finding a solution of problem (4.1). But at the same time, it is assumed that "$u\in K_r$ satisfying (4.1)". So once we know the solution of problem (4.1) then there is no need to consider auxiliary problem (4.2). It shows that there is a drawback in the formulation of auxiliary problem (4.2). Also, the condition "$u\in K_r$ satisfying (4.1)" cannot be assumed and it should be omitted.

Noor et al. [24] proposed the following predictor-corrector method for solving problem (4.1).

Algorithm 4.1. [24, Algorithm 3.1] For a given $u_0\in K_r$, compute the approximate solution u_{n+1} in K_r by the following iterative scheme:

$$\rho B(u_{n+1},v-u_{n+1})+\langle u_{n+1}-u_n,v-u_{n+1}\rangle+\rho\gamma\|v-u_{n+1}\|^2\geq 0,\quad \forall v\in K_r,$$

where $\rho>0$ is a constant and $n=0,1,2\ldots.$

Definition 4.2. Let $\gamma>0$. A bifunction $B:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ is said to be *pseudomonotone* with constant γ if for all $u_1,u_2\in K_r$

$$B(u_1,u_2-u_1)+\gamma\|u_2-u_1\|^2\geq 0 \quad\text{implies}\quad B(u_2,u_1-u_2)+\gamma\|u_2-u_1\|^2\leq 0.$$

By taking $F\equiv\varphi\equiv 0$ in Proposition 3.3 and $F\equiv\varphi\equiv 0$ in Theorem 3.4, we achieve the following main results given in [24].

Corollary 4.3. [24, Theorem 3.1] *Let $B(.,.):\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ be pseudomonotone. If $\{u_{n+1}\}$ is a sequence generated by Algorithm* 4.1 *and $u\in K_r$ is a solution of* (4.1), *then*

$$\|u-u_{n+1}\|^2\leq\|u-u_n\|^2-\|u_n-u_{n+1}\|^2,\quad \forall n\geq 0.$$

Corollary 4.4. [24, Theorem 3.2] *Let $\mathcal{H}$ be a finite dimensional real Hilbert space and $\{u_{n+1}\}$ be a sequence generated by Algorithm* 4.1. *Let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be pseudomonotone and continuous in both the arguments. If $u \in K_r$ is a solution of* (4.1)*, then $\lim\limits_{n\to\infty} u_n = u$.*

Remark 4.5. It should be pointed out that there is a small error in Theorem 3.1 in [24]. In fact, $B(.,.) : K \times K \to \mathcal{H}$ must be replaced by $B(.,.) : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$. Also, in Theorem 3.2 in [24] the bifunction $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ must be pseudomonotone and continuous in both the arguments.

Noor et al. [24] considered the following auxiliary regularized nonconvex variational inequality problem: For a given $u \in K_r$ satisfying (4.1), find $w \in K_r$ such that

$$\rho B(u, v-w) + \langle w-u, v-w\rangle + \gamma\|v-w\|^2 \geq 0, \quad \forall v \in K_r, \tag{4.3}$$

where $\rho > 0$ is a constant.

They claimed that if $w = u$, then clearly w is a solution of RNNVIP (4.1). Based on this fact, they suggested the following predictor-corrector method for solving RNNVIP (4.1).

Algorithm 4.6. [24, Algorithm 3.3] For a given $u_0 \in K_r$, compute u_{n+1} in K_r by the following iterative scheme:

$$\rho B(u_n, v-u_{n+1}) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle + \gamma\|v-u_{n+1}\|^2 \geq 0, \quad \forall v \in K_r.$$

Noor et al. [24] asserted that using essentially the technique of Theorems 3.1 and 3.2 in [24], one can study the convergence analysis of Algorithm 4.6. However, we now show that for various reasons, this claim is not true in general. By an argument analogous to the previous one, the condition "$u \in K_r$ satisfying (4.1)" in auxiliary problem (4.3) cannot be assumed. If we remove this condition, then w is not necessarily a solution of RNNVIP (4.1) when $w = u$. In fact, if $w = u$, then auxiliary problem (4.3) reduces to the following regularized nonconvex nonsmooth variational inequality:

$$\rho B(u, v-u) + \gamma\|v-u\|^2 \geq 0, \quad \forall v \in K_r. \tag{4.4}$$

However, the following example shows that, for an arbitrary real constant $\rho \in (0,1)$, a solution of problem (4.4) need not be a solution of RNNVIP (4.1).

Example 4.7. Let $\mathcal{H} = \mathbb{R}$ and $K_r = [0,\alpha]\cup[\beta,\delta]$ be the union of two disjoint intervals $[0,\alpha]$ and $[\beta,\delta]$ where $0 < \alpha < \beta < \delta$. Then, K_r is a uniformly r-prox-regular set with $r = \frac{\beta-\alpha}{2}$, and so we have $\gamma = \frac{1}{2r} = \frac{1}{\beta-\alpha}$. Let the bifunction $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be defined by

$$B(x,y) = \begin{cases} \varrho(e^{sx} + x^t)y, & \text{if } x, y \in K_r, \\ \mu\sqrt[q]{x^l}y, & \text{otherwise,} \end{cases}$$

where $q \in \mathbb{N}\backslash\{1\}$ and $l \in \mathbb{N}$ are arbitrary but fixed natural numbers, $s, t \in \mathbb{R}$, $\mu < \frac{\alpha-\delta}{(\beta-\alpha)\sqrt[q]{\alpha^l}}$ and $\frac{\mu\sqrt[q]{\alpha^l}}{e^{s\alpha}+\alpha^t} < \varrho < \frac{\alpha-\delta}{(\beta-\alpha)(e^{s\alpha}+\alpha^t)}$ are arbitrary but fixed real numbers. Let us take $u = \alpha$ and let $\rho > 0$ be an arbitrary real constant belonging to

$\Big(-\frac{1}{\mu\sqrt[q]{\alpha^l}}, -\frac{1}{\varrho(e^{s\alpha}+\alpha^t)}\Big]$. Then for all $v \in K_r$, we have

$$\rho B(u, v-u) + \gamma\|v-u\|^2 = (v-\alpha)\Big(\rho\varrho(e^{s\alpha}+\alpha^t) + \frac{1}{\beta-\alpha}(v-\alpha)\Big).$$

If $v \in [0, \alpha]$, then $\varrho < 0$ implies that

$$(v-\alpha)\Big(\rho\varrho\left(e^{s\alpha}+\alpha^t\right) + \frac{1}{\beta-\alpha}(v-\alpha)\Big) \geq 0.$$

For the case when $v \in [\beta, \delta]$, considering the facts that $\frac{1}{\beta-\alpha}(v-\alpha) \geq 1$ for all $v \in [\beta, \delta]$, and $0 < \rho \leq -\frac{1}{\varrho(e^{s\alpha}+\alpha^t)}$, it follows that

$$(v-\alpha)\Big(\rho\varrho\left(e^{s\alpha}+\alpha^t\right) + \frac{1}{\beta-\alpha}(v-\alpha)\Big) \geq 0.$$

Therefore, $\rho B(u, v-u) + \gamma\|v-u\|^2 \geq 0$ for all $v \in K_r$. Whereas, for all $v \in (\alpha, \beta)$, we obtain

$$\rho B(u, v-u) + \gamma\|v-u\|^2 = (v-\alpha)\Big(\rho\mu\sqrt[q]{\alpha^l} + \frac{1}{\beta-\alpha}(v-\alpha)\Big).$$

Since $\frac{1}{\beta-\alpha}(v-\alpha) \in (0, 1)$ for all $v \in (\alpha, \beta)$, the facts that $\mu < 0$ and $\rho > -\frac{1}{\mu\sqrt[q]{\alpha^l}}$ imply that

$$(v-\alpha)\Big(\rho\mu\sqrt[q]{\alpha^l} + \frac{1}{\beta-\alpha}(v-\alpha)\Big) < 0, \quad \forall v \in (\alpha, \beta),$$

that is,

$$\rho B(u, v-u) + \gamma\|v-u\|^2 < 0, \quad \forall v \in (\alpha, \beta).$$

Hence, $\rho B(u, v-u) + \gamma\|v-u\|^2 \geq 0$ cannot hold for all $v \in \mathcal{H}$, but it holds only for all $v \in K_r$.

On the other hand, for all $v \in K_r$, we have

$$B(u, v-u) + \gamma\|v-u\|^2 = (v-\alpha)\Big(\varrho\left(e^{s\alpha}+\alpha^t\right) + \frac{1}{\beta-\alpha}(v-\alpha)\Big).$$

Thanks to the facts that $\varrho < \frac{\alpha-\delta}{(\beta-\alpha)(e^{s\alpha}+\alpha^t)}$ and $\frac{1}{\beta-\alpha}(v-\alpha) \in \Big[1, \frac{\delta-\alpha}{\beta-\alpha}\Big]$ for all $v \in [\beta, \delta]$, we deduce

$$(v-\alpha)\Big(\varrho\left(e^{s\alpha}+\alpha^t\right) + \frac{1}{\beta-\alpha}(v-\alpha)\Big) < 0, \quad \forall v \in [\beta, \delta].$$

Therefore, $B(u, v-u) + \gamma\|v-u\|^2 \geq 0$ cannot hold for all $v \in K_r$. Thus a solution of problem (4.4) need not be a solution of RNNVIP (4.1).

Algorithm 4.6 is constructed based on the fact that in auxiliary problem (4.3), if $w = u$ for some $u \in K_r$ satisfying (4.1), then w is a solution of RNNVIP (4.1), but it is not true. Thus, Algorithm 4.6 cannot be used to solve RNNVIP (4.1). In order to overcome with these difficulties, we need to present the correct version of auxiliary problem (4.3) and Algorithm 4.6.

Let B and γ be the same as in RNNVIP (4.1). For a given $u \in K_r$, we consider the auxiliary regularized nonconvex variational inequality problem of finding $w \in K_r$ such that

$$\rho B(u, v-u) + \langle w-u, v-w\rangle + \rho\gamma\|v-w\|^2 \geq 0, \quad \forall v \in K_r, \tag{4.5}$$

where $\rho > 0$ is a constant. Note that problems (4.2) and (4.5) are quite different. In special case, if $w = u$ then clearly w is a solution of RNNVIP (4.1). This fact enables us to construct the following iterative algorithm for solving RNNVIP (4.1).

Algorithm 4.8. For a given $u_0 \in K_r$, compute $u_n \in K_r$ by the following iterative scheme:

$$\rho B(u_n, v-u_n) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle + \rho\gamma\|v-u_{n+1}\|^2 \geq 0, \quad \forall v \in K_r,$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \ldots$.

The next proposition is obtained by taking $F \equiv \varphi \equiv 0$ in Proposition 3.8 which plays a crucial role in the study of convergence analysis of the iterative sequence generated by Algorithm 4.8.

Proposition 4.9. *Let $\{u_n\}$ be a sequence generated by Algorithm* 4.8 *and $u \in K_r$ be a solution of* RNNVIP (4.1)*. If B is partially $(\alpha, 2\gamma)$-mixed relaxed and strongly monotone of type* (I)*, then*

$$\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2 - (1-2\alpha\rho)\|u_n-u_{n+1}\|^2, \quad \forall n \geq 0.$$

By taking $F \equiv \varphi \equiv 0$ in Theorem 3.9, we have the convergence of iterative sequence generated by Algorithm 4.8.

Theorem 4.10. *Let $\mathcal{H}$ be a finite dimensional real Hilbert space. Assume that $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is continuous in both the arguments,* RNNVIP$(B, K_r) \neq \emptyset$ *and all the conditions of Proposition* 4.9 *hold. If $\rho \in \left(0, \frac{1}{2\alpha}\right)$, then the sequence $\{u_n\}$ generated by Algorithm* 4.8 *converges to a solution $\hat{u} \in K_r$ of* RNNVIP (4.1)*.*

We now investigate and analyze the algorithm and results given in [25].

For given bifunctions $F, B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, Noor et al. [25] considered the problem of finding $u \in K_r$ such that

$$F(u, v) + B(u, v-u) + \gamma\|v-u\|^2 \geq 0, \quad \forall v \in K_r, \tag{4.6}$$

where $\gamma = \frac{1}{2r}$. They called it *nonconvex bifunction equilibrium variational inequality problem*. Rest of the paper, we shall call it *regularized nonconvex mixed equilibrium problem* (RNMEP) as it is commonly used in the literature. We denote by RNMEP(F, B, K_r) the set of solutions of RNMEP (4.6).

Noor et al. [25] considered the following auxiliary regularized nonconvex mixed equilibrium problem: For a given $u \in K_r$ satisfying (4.6), find $w \in K_r$ such that

$$\begin{aligned} &\rho F(w, v) + \rho B(w, v-w) + \langle w-u-\alpha(u-u), v-w\rangle \\ &\qquad + \rho\gamma\|v-w\|^2 \geq 0, \quad \forall v \in K_r, \end{aligned} \tag{4.7}$$

where $\rho > 0$ and $\alpha \geq 0$ are constants. Obviously, if $w = u$, then w is a solution of RNMEP (4.6). Using this observation, they proposed the following inertial proximal point method for solving RNMEP (4.6).

Algorithm 4.11. [25, Algorithm 3.1] For a given $u_0 \in K_r$, compute $u_{n+1} \in K_r$ by the following iterative scheme:

$$\rho F(u_{n+1}, v) + \rho B(u_{n+1}, v - u_{n+1}) + \langle u_{n+1} - u_n - \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle + \rho\gamma\|v - u_{n+1}\|^2 \geq 0, \quad \forall v \in K_r. \tag{4.8}$$

By taking $\alpha = 0$ in Algorithm 4.11, they also suggested the following proximal point algorithm for solving RNMEP (4.6).

Algorithm 4.12. [25, Algorithm 3.3] For a given $u_0 \in K_r$, compute $u_{n+1} \in K_r$ by the following iterative scheme:

$$\rho F(u_{n+1}, v) + \rho B(u_{n+1}, v - u_{n+1}) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \rho\gamma\|v - u_{n+1}\|^2 \geq 0, \quad \forall v \in K_r. \tag{4.9}$$

They have repeated the similar mistake as in [24] by considering "$u \in K_r$ satisfying (4.6)". Once we know the solution of problem (4.6), then there is no need to consider auxiliary problem (4.7). Therefore, the condition "$u \in K_r$ satisfying (4.6)" should be removed from auxiliary problem (4.7).

As a special case when $\alpha = 0$, problem (4.7) reduces to the auxiliary nonconvex mixed equilibrium problem of finding $w \in K_r$ such that

$$\rho F(w, v) + \rho B(w, v - w) + \langle w - u, v - w\rangle + \rho\gamma\|v - w\|^2 \geq 0, \quad \forall v \in K_r, \tag{4.10}$$

where $\rho > 0$ is a constant. Again, if $w = u$, then w is a solution of problem (4.6). With the help of this fact, one can also suggest Algorithm 4.12 for solving RNMEP (4.6).

The next theorem played a crucial role in the study of convergence analysis of the iterative sequence generated by Algorithm 4.12.

Theorem 4.13. [25, Theorem 3.8] *Let $F(.,.), B(.,.) : K_r \times K_r \to \mathbb{R}$ be monotone. If $\{u_{n+1}\}$ is a sequence generated by Algorithm* 4.12 *and $u \in K_r$ is a solution of* (4.6), *then*

$$(1 - 4\gamma\rho)\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2, \quad \forall n \geq 0. \tag{4.11}$$

By a careful reading of the proof of Theorem 4.13, we have the following observations. By assuming that $u \in K_r$ is a solution (4.6), Noor et al. [25] deduced the following inequality by using the monotonicity of the bifunctions F and B:

$$-F(v, u) - B(v, u - v) + \gamma\|v - u\|^2 \geq 0, \quad \forall v \in K_r. \tag{4.12}$$

Taking $v = u_{n+1}$ in (4.12), they obtained (which is inequality (3.10) in [25])

$$-F(u_{n+1}, u) - B(u_{n+1}, u - u_{n+1}) + \gamma\|u - u_{n+1}\|^2 \geq 0. \tag{4.13}$$

Then, setting $v = u$ in (4.9) and by virtue of (4.13), they deduced

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq -\rho F(u_{n+1}, u) - \rho B(u_{n+1}, u - u_{n+1}) - \rho\gamma\|u - u_{n+1}\|^2 \geq 0. \tag{4.14}$$

At the end of the proof, they claimed that the required inequality (4.11) follows from (4.14).

However, by taking $v = u$ in (4.9) and with the help of inequality (4.13), we cannot derive inequality (4.14). Indeed, setting $v = u$ in (4.9), we have

$$\rho F(u_{n+1}, u)+\rho B(u_{n+1}, u-u_{n+1})+\langle u_{n+1}-u_n, u-u_{n+1}\rangle+\rho\gamma\|u-u_{n+1}\|^2 \geq 0, \quad \forall n \geq 0,$$

which implies that

$$\langle u_{n+1}-u_n, u-u_{n+1}\rangle \geq -\rho F(u_{n+1}, u)-\rho B(u_{n+1}, u-u_{n+1})-\rho\gamma\|u-u_{n+1}\|^2, \quad \forall n \geq 0.$$

But, inequality (4.13) does not imply the following inequality:

$$F(u_{n+1}, u) + B(u_{n+1}, u - u_{n+1}) + \gamma\|u - u_{n+1}\|^2 \leq 0. \tag{4.15}$$

This fact is illustrated in the following example.

Example 4.14. Let $\mathcal{H}$ and K_r be the same as in Example 4.7.
Define $F, B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$F(x, y) = \begin{cases} \eta(e^{ly} + y^q)(x - y), & \text{if } x, y \in K_r, \\ \omega(x^2y - xy^2), & \text{otherwise,} \end{cases}$$

and

$$B(x, y) = \begin{cases} \sigma xy, & \text{if } x, y \in K_r, \\ \varsigma x^m y, & \text{otherwise,} \end{cases}$$

where $l, q, m \in \mathbb{R}$, $0 < \omega < \frac{1}{\alpha^2}$, $\varsigma < \frac{-1+\omega\alpha^2}{\alpha^m}$ and $-\frac{1}{\delta} < \sigma < 0 < \eta \leq \frac{1+\sigma\delta}{e^{l\alpha}+\alpha^q}$ are arbitrary but fixed real numbers. Taking $u = \alpha$, we have for all $v \in K_r$

$$-F(v, u)-B(v, u-v)+\gamma\|v-u\|^2 = (v-\alpha)\Big(-\eta(e^{l\alpha}+\alpha^q)+\sigma v+\frac{1}{\beta-\alpha}(v-\alpha)\Big).$$

If $v \in [0, \alpha]$, then $\sigma < 0 < \eta$ implies that

$$(v-\alpha)\Big(-\eta(e^{l\alpha}+\alpha^q)+\sigma v+\frac{1}{\beta-\alpha}(v-\alpha)\Big) \geq 0.$$

Since $\frac{1}{\beta-\alpha}(v-\alpha) \geq 1$ for all $v \in [\beta, \delta]$, the fact that $\sigma < 0 < \eta \leq \frac{1+\sigma\delta}{e^{l\alpha}+\alpha^q}$ implies that

$$(v-\alpha)\Big(-\eta(e^{l\alpha}+\alpha^q)+\sigma v+\frac{1}{\beta-\alpha}(v-\alpha)\Big) \geq 0.$$

Consequently,

$$-F(v, u) - B(v, u - v) + \gamma\|v - u\|^2 \geq 0, \quad \forall v \in K_r. \tag{4.16}$$

On the other hand, for all $v \in (\alpha, \beta)$, we get

$$-F(v, u) - B(v, u - v) + \gamma\|v - u\|^2 = (v - \alpha)\Big(-\omega\alpha v + \varsigma v^m + \frac{1}{\beta-\alpha}(v-\alpha)\Big).$$

Relying on the facts that $\frac{1}{\beta-\alpha}(v-\alpha) \in (0, 1)$ for all $v \in (\alpha, \beta)$, $\omega > 0$ and $\varsigma < \frac{-1+\omega\alpha^2}{\alpha^m}$, it follows that

$$(v-\alpha)\Big(-\omega\alpha v+\varsigma v^m+\frac{1}{\beta-\alpha}(v-\alpha)\Big) < 0, \quad \forall v \in (\alpha, \beta).$$

Hence, $-F(v, u) - B(v, u - v) + \gamma\|v - u\|^2 \geq 0$ cannot hold for all $v \in \mathcal{H}$, but it holds only for all $v \in K_r$.

However, $\sigma < 0 < \eta$ and the fact that $\frac{1}{\beta-\alpha}(v-\alpha) \in [1, \frac{\delta-\alpha}{\beta-\alpha}]$ for all $v \in [\beta, \delta]$, imply that

$$\begin{aligned} &F(v,u) + B(v,u-v) + \gamma\|v-u\|^2 \\ &= (v-\alpha)\Big(\eta(e^{l\alpha}+\alpha^q) - \sigma v + \frac{1}{\beta-\alpha}(v-\alpha)\Big) > 0, \quad \forall v \in [\beta,\delta], \end{aligned}$$

that is,

$$F(v,u) + B(v,u-v) + \gamma\|v-u\|^2 > 0, \quad \forall v \in [\beta,\delta]. \tag{4.17}$$

Hence, $F(v,u) + B(v,u-v) + \gamma\|v-u\|^2 \leq 0$ cannot hold for all $v \in K_r$. Now, taking $v = u_{n+1}$ in (4.16) and (4.17), we deduce that inequality (4.13) does not imply inequality (4.15) in general.

Even without considering the above mentioned fact, it is easy to see that contrary to the claim in [25], by using inequality (4.14), we do not obtain inequality (4.11) as an estimation of $\|u-u_{n+1}\|^2$ for all $n \geq 0$. In fact, by means of (4.14) and taking into account that

$$\langle u_{n+1}-u_n, u-u_{n+1}\rangle = \|u-u_n\|^2 - \|u_{n+1}-u_n\|^2 - \|u-u_{n+1}\|^2, \quad \forall n \geq 0,$$

it follows that

$$\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2 - \|u_n-u_{n+1}\|^2, \quad \forall n \geq 0. \tag{4.18}$$

Accordingly, in view of (4.14) and without considering the fatal error existing in it, we get inequality (4.18) not (4.11) as an estimation of $\|u-u_{n+1}\|^2$ for all $n \geq 0$. In the light of the above mentioned argument, the statement of Theorem 4.13 is not valid in its present form.

Noor et al. [25] established the following theorem.

Theorem 4.15. [25, Theorem 3.9] *Let $\mathcal{H}$ be a finite dimensional real Hilbert space, and $\{u_{n+1}\}$ be a sequence generated by Algorithm* 4.12. *If $u \in K_r$ is a solution of* (4.6) *and $\rho < \frac{1}{4\gamma}$, then $\lim\limits_{n\to\infty} u_n = u$.*

Now we analyze the proof of Theorem 4.15.

Theorem 4.13 plays a crucial role in the proof of Theorem 4.15. However, as we have seen Theorem 4.13 is not valid in its present form. Beside this fact, we also observe that there are two other fatal errors in the proof of Theorem 4.15. First, it is claimed that inequality (4.9) guarantees the boundedness of the sequence $\{u_n\}$. In fact, Noor et al. [25] asserted that by using inequality (4.11), it follows that

$$\|u-u_{n+1}\| \leq \|u-u_n\|, \quad \forall n \geq 0, \tag{4.19}$$

that is, the sequence $\{u-u_n\}$ is nonincreasing, and then by utilizing inequality (4.19), they concluded the boundedness of the sequence $\{u_n\}$. But, by means of inequality (4.11), we obtain

$$\|u-u_{n+1}\| \leq \frac{1}{\sqrt{1-4\gamma\rho}}\|u-u_n\|, \tag{4.20}$$

not inequality (4.19). It is clear that inequality (4.20) does not imply that the sequence $\{\|u-u_n\|\}$ is nonincreasing. Hence, under the assumptions of Theorem 4.15, the sequence $\{u_n\}$ is not necessarily bounded.

Noor et al. [25] also claimed that by virtue of inequality (4.11), one can deduce the following inequality:

$$\sum_{n=0}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u - u_0\|^2, \tag{4.21}$$

which implies that

$$\lim_{n\to\infty} \|u_n - u_{n+1}\| = 0. \tag{4.22}$$

However, by employing inequality (4.11), we obtain

$$\sum_{n=0}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u - u_0\|^2 + \sum_{n=0}^{\infty} 4\rho\gamma\|u - u_{n+1}\|^2, \tag{4.23}$$

not inequality (4.21). Obviously, inequality (4.23) does not imply inequality (4.21). Taking into consideration the facts that the boundedness of the sequence $\{u_n\}$ and relation (4.22) have key roles in proving Theorem 4.15, it follows that Theorem 4.15 is not true in its present form.

By taking $\varphi \equiv 0$ in Proposition 3.3 and $\varphi \equiv 0$ in Theorem 3.4, we obtain the correct version of Theorems 4.13 and 4.15, respectively.

Theorem 4.16. *Let $\{u_n\}$ be a sequence generated by Algorithm* 4.12 *and $u \in K_r$ be a solution of* RNMEP (4.6)*. If the bifunctions F and B are jointly pseudomonotone with constant γ, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2, \quad \forall n \geq 0.$$

Theorem 4.17. *Let $\mathcal{H}$ be a finite dimensional real Hilbert space. Assume that $B : K_r \times K_r \to \mathbb{R}$ is continuous in both the arguments, $F : K_r \times K_r \to \mathbb{R}$ is continuous in the first argument,* RNMEP$(F, B, K_r) \neq \emptyset$ *and all the conditions of Theorem* 4.16 *hold. Then, the sequence $\{u_n\}$ generated by Algorithm* 4.12 *converges to a solution $\hat{u} \in K_r$ of* RNMEP (4.6).

Noor et al. [25] considered the following auxiliary regularized nonconvex mixed equilibrium problem: For a given $u \in K_r$ satisfying (4.6), find $w \in K_r$ such that

$$\rho F(u, v) + \rho B(u, v - w) + \langle w - u, v - w\rangle + \gamma\|v - w\|^2 \geq 0, \quad \forall v \in K_r, \tag{4.24}$$

where $\rho > 0$ is a constant. They claimed that if $w = u$, then clearly w is a solution of problem (4.6). With the help of this fact, they proposed the following predictor-corrector method for solving RNMEP (4.6).

Algorithm 4.18. [25, Algorithm 3.10] For a given $u_0 \in K_r$, compute $u_{n+1} \in K_r$ by the following iterative scheme:

$$\rho F(u_n, v) + \rho B(u_n, v - u_{n+1}) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \gamma\|v - u_{n+1}\|^2 \geq 0, \quad \forall v \in K_r.$$

As we argued, the condition "$u \in K_r$ satisfying (4.6)" should be removed from problem (4.24). But, if we omit the condition "$u \in K_r$ satisfying (4.6)" from problem (4.24), then w would not be a solution of RNMEP (4.6) when $w = u$. In fact, if $w = u$

then problem (4.24) reduces to the following regularized nonconvex mixed equilibrium problem:

$$\rho F(u,v)+\rho B(u,v-u)+\gamma\|v-u\|^2\geq 0,\quad \forall v\in K_r. \tag{4.25}$$

However, the following example illustrates that a solution of problem (4.25) need not be a solution of RNMEP (4.6).

Example 4.19. Suppose that $\mathcal{H}$ and K_r are the same as in Example 4.7. Define $F,B:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ by

$$F(x,y)=\begin{cases}\varsigma(e^{mx}+x^n)(y-x), & \text{if } x,y\in K_r,\\ \theta\sqrt[p]{x^q}(x-y), & \text{otherwise,}\end{cases}$$

and

$$B(x,y)=\begin{cases}\xi\sqrt[s]{x^t}y, & \text{if } x,y\in K_r,\\ \lambda xy, & \text{otherwise,}\end{cases}$$

respectively, where $p,s\in\mathbb{N}\backslash\{1\}$ and $q,t\in\mathbb{N}$ are arbitrary but fixed natural numbers, $m,n\in\mathbb{R}$, $0<\theta<\frac{\delta-\alpha}{(\beta-\alpha)\sqrt[p]{\alpha^q}}$, $\lambda<\frac{\alpha-\delta+\theta(\beta-\alpha)\sqrt[p]{\alpha^q}}{\alpha(\beta-\alpha)}$, $\frac{\alpha-\delta}{(\beta-\alpha)\sqrt[s]{\alpha^t}}<\xi<0$ and $\frac{\lambda\alpha-\theta\sqrt[p]{\alpha^q}-\xi\sqrt[s]{\alpha^t}}{e^{m\alpha}+\alpha^n}<\varsigma<\frac{\alpha-\delta-(\beta-\alpha)\xi\sqrt[s]{\alpha^t}}{(\beta-\alpha)(e^{m\alpha}+\alpha^n)}$ are arbitrary but fixed real numbers. Take $u=\alpha$ and let $\rho\in\Big(-\frac{1}{\lambda\alpha-\theta\sqrt[p]{\alpha^q}},-\frac{1}{\varsigma(e^{m\alpha}+\alpha^n)+\xi\sqrt[s]{\alpha^t}}\Big]$ be a positive real constant. Then, for all $v\in K_r$, we have

$$\rho F(u,v)+\rho B(u,v-u)+\gamma\|v-u\|^2=(v-\alpha)\Big(\rho\varsigma(e^{m\alpha}+\alpha^n)+\rho\xi\sqrt[s]{\alpha^t}+\frac{1}{\beta-\alpha}(v-\alpha)\Big).$$

In the case when $v\in[0,\alpha]$, the fact that $\varsigma,\xi<0$ implies that

$$(v-\alpha)\Big(\rho\varsigma(e^{m\alpha}+\alpha^n)+\rho\xi\sqrt[s]{\alpha^t}+\frac{1}{\beta-\alpha}(v-\alpha)\Big)\geq 0.$$

If $v\in[\beta,\delta]$, from the facts that $\frac{1}{\beta-\alpha}(v-\alpha)\geq 1$ for all $v\in[\beta,\delta]$, $\varsigma,\xi<0$ and $0<\rho\leq-\frac{1}{\varsigma(e^{m\alpha}+\alpha^n)+\xi\sqrt[s]{\alpha^t}}$, we deduce that

$$(v-\alpha)\Big(\rho\varsigma(e^{m\alpha}+\alpha^n)+\rho\xi\sqrt[s]{\alpha^t}+\frac{1}{\beta-\alpha}(v-\alpha)\Big)\geq 0.$$

Thus, $\rho F(u,v)+\rho B(u,v-u)+\gamma\|v-u\|^2\geq 0$ for all $v\in K_r$. In the meanwhile, for all $v\in(\alpha,\beta)$, we get

$$\rho F(u,v)+\rho B(u,v-u)+\gamma\|v-u\|^2=(v-\alpha)\Big(-\rho\theta\sqrt[p]{\alpha^q}+\rho\lambda\alpha+\frac{1}{\beta-\alpha}(v-\alpha)\Big).$$

Then $\lambda<0<\theta$ and $\rho>\frac{1}{\theta\sqrt[p]{\alpha^q}-\lambda\alpha}$ together with the fact that $\frac{1}{\beta-\alpha}(v-\alpha)\in(0,1)$ for all $v\in(\alpha,\beta)$, imply that

$$(v-\alpha)\Big(-\rho\theta\sqrt[p]{\alpha^q}+\rho\lambda\alpha+\frac{1}{\beta-\alpha}(v-\alpha)\Big)<0,\quad\forall v\in(\alpha,\beta),$$

that is, $\rho F(u,v)+\rho B(u,v-u)+\gamma\|v-u\|^2<0,\quad\forall v\in(\alpha,\beta)$.
Hence, $\rho F(u,v)+\rho B(u,v-u)+\gamma\|v-u\|^2\geq 0$ cannot hold for all $v\in\mathcal{H}$, but it holds only for all $v\in K_r$.

On the other hand, for all $v \in K_r$, one has

$$F(u,v)+B(u,v-u)+\gamma\|v-u\|^2 = (v-\alpha)\Big(\varsigma(e^{m\alpha}+\alpha^n)+\xi\sqrt[s]{\alpha^t}+\frac{1}{\beta-\alpha}(v-\alpha)\Big).$$

In light of the facts that $\frac{1}{\beta-\alpha}(v-\alpha) \in [1, \frac{\delta-\alpha}{\beta-\alpha}]$ for all $v \in [\beta, \delta]$, $\frac{\alpha-\delta}{(\beta-\alpha)\sqrt[s]{\alpha^t}} < \xi$ and $\varsigma < \frac{\alpha-\delta-(\beta-\alpha)\xi\sqrt[s]{\alpha^t}}{(\beta-\alpha)(e^{m\alpha}+\alpha^n)}$, it follows that

$$(v-\alpha)\Big(\varsigma(e^{m\alpha}+\alpha^n)+\xi\sqrt[s]{\alpha^t}+\frac{1}{\beta-\alpha}(v-\alpha)\Big) < 0, \quad \forall v \in [\beta,\delta].$$

Hence, $F(u,v)+B(u,v-u)+\gamma\|v-u\|^2 \geq 0$ cannot hold for all $v \in K_r$. Thus, a solution of problem (4.25) may not be a solution of RNMEP (4.6).

Algorithm 4.12 is constructed based on the fact that in auxiliary problem (4.24), if $w = u$ for some given $u \in K_r$ satisfying (4.6), then w is a solution of RNMEP (4.6). But, it is not true, and hence Algorithm 4.18 is not applicable. To overcome with these difficulties, we consider the following auxiliary problem and algorithm.

Let F, B and γ be the same as in RNMEP (4.6). For a given $u \in K_r$, we consider the auxiliary regularized nonconvex mixed equilibrium problem of finding $w \in K_r$ such that

$$\rho F(u,v)+\rho B(u,v-u)+\langle w-u, v-w\rangle+\rho\gamma\|v-w\|^2 \geq 0, \quad \forall v \in K_r, \tag{4.26}$$

where $\rho > 0$ is a constant. Clearly, problems (4.10) and (4.26) are quite different. In special case, if $w = u$ then obviously w is a solution of problem (4.6). This fact allows us to suggest the following iterative algorithm for solving RNMEP (4.6).

Algorithm 4.20. For a given $u_0 \in K_r$, compute $\{u_n\} \in K_r$ by the following iterative scheme:

$$\rho F(u_n,v)+\rho B(u_n,v-u_n)+\langle u_{n+1}-u_n, v-u_{n+1}\rangle+\rho\gamma\|v-u_{n+1}\|^2 \geq 0, \quad \forall v \in K_r,$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \ldots$.

The following proposition is obtained by taking $\varphi \equiv 0$ in Proposition 3.8 which is a main tool to establish the convergence of the iterative sequence generated by Algorithm 4.20

Proposition 4.21. *Let $\{u_n\}$ be a sequence generated by Algorithm* 4.20 *and $u \in K_r$ be a solution of* RNMEP (4.6)*. If F is partially β-relaxed monotone of type* (I)*, and B is partially $(\alpha, 2\gamma)$-mixed relaxed and strongly monotone of type* (I)*, then*

$$\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2-(1-2(\alpha+\beta)\rho)\|u_{n+1}-u_n\|^2, \quad \forall n \geq 0.$$

By taking $\varphi \equiv 0$ in Theorem 3.9, we obtain the following theorem which provides the convergence of the iterative sequence generated by Algorithm 4.20 to a solution of RNMEP (4.6).

Theorem 4.22. *Let $\mathcal{H}$ be a finite dimensional real Hilbert space, $F : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ be continuous in the first argument and $B : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ be continuous in both the arguments. Suppose that all conditions of Proposition* 4.21 *hold and* RNMEP$(F, B, K_r) \neq$

$\emptyset$. *Then, the iterative sequence* $\{u_n\}$ *generated by Algorithm* 4.20 *converges to a solution* $\hat{u} \in K_r$ *of* RNMEP (4.6).

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