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ON THE EXTENDED MULTIVALUED SUZUKI TYPE CONTRACTIONS VIA A TOPOLOGICAL PROPERTY

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Abstract. In this paper, by using the idea of Suzuki, we present some new results on absolute retractivity of the common fixed points set of multivalued Suzuki type contractions.
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1. Introduction

In 1970, Schirmer proved some results on topological properties of the fixed points set of multifunctions ([13]). Later, some authors continued the study by providing different conditions ([7]). In 2008, Sintamarian proved some results on absolute retractivity of the common fixed points set of two multivalued operators ([14] and [15]). On the other hand, Suzuki generalized the notion of contractive mappings in [16] and [17]. Since then, there have appeared some results on Suzuki method for mappings and multifunctions (see [3], [9], [6], [11], [12] and the references cited). Afshari, Rezapour and Shahzad proved some results about absolute retractivity of the fixed points set of multifunctions, (see [1], [2], [4] and [5]).
Let X be a nonempty set. We denote by $P(X)$ the set of all nonempty subsets of X. Let $F : X \to P(X)$ be a multivalued operator. We denote by $\mathcal{F}_F$ the fixed points set of F, i.e. $\mathcal{F}_F = \{x \in X | x \in F(x)\}$.
Let $F_1, F_2 : X \to P(X)$ be two multivalued operators. We denote by $(\mathcal{CF})_{F_1,F_2}$ the common fixed points set of F_1 and F_2, i.e. $(\mathcal{CF})_{F_1,F_2} = \{x \in X | x \in F_1(x) \cap F_2(x)\}$.
In this paper, we present some new results on absolute retractivity of the fixed points set of Suzuki type contractive multifunctions.
Let X and Y be two nonempty sets, $P(Y)$ be the set of all nonempty subsets of Y and $F : X \to P(Y)$ be a multifunction. A mapping $\varphi : X \to Y$ is called a selection

of F whenever $\varphi(x) \in Fx$ for all $x \in X$. Throughout the paper, the set of all nonempty closed and bounded subsets of X is denoted by $P_{b,cl}(X)$. Let (X,d) be a metric space and $B(x_0,r) = \{x \in X : d(x_0,x) < r\}$. For $x \in X$ and $A, B \subseteq X$, set $d(x,A) = \inf_{y \in A} d(x,y)$ and

$$H(A,B) = \max\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\}.$$

It is known that H is a metric on closed bounded subsets of X which is called the Pompieu-Hausdorff metric (for more details, see [8], [14] and [15]). We say that a topological space X is an absolute retract for metric spaces whenever for each metric space Y, $A \in P_{cl}(Y)$ and each continuous function $\psi : A \to X$, there exists a continuous function $\varphi : Y \to X$ such that $\varphi|_A = \psi$. Let $\mathcal{M}$ be the set of all metric spaces, $X \in \mathcal{M}$, $\mathcal{D} \in P(\mathcal{M})$ and $F : X \to P_{b,cl}(X)$ be a lower semi-continuous multifunction. We say that F has the selection property with respect to $\mathcal{D}$ if for each $Y \in \mathcal{D}$, each continuous function $f : Y \to X$ and each continuous functional $g : Y \to (0,\infty)$ such that $G(y) := \overline{F(f(y)) \cap B(f(y), g(y))} \neq \emptyset$ for all $y \in Y$, $A \in P_{cl}(Y)$, every continuous selection $\psi : A \to X$ of $G|_A$ admits a continuous extension $\varphi : Y \to X$, which is a selection of G. If $\mathcal{D} = \mathcal{M}$, then we say that F has the selection property and we denote this by $F \in Sp(X)$ (for more details, see [14] and [15]).
In 2008, Suzuki [16] proved the following remarkable generalization of the Banach contraction principle that characterizes the metric completeness of X.
Define $\theta : [0,1) \to (\frac{1}{2}, 1]$ as

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ (1+r)^{-1} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

The above function θ is nonincreasing and continuous on $[0,1)$.

Theorem 1.1. [16] *Let (X,d) be a metric space. Then X is complete if and only if every mapping T on X satisfying the following:*
there exists $r \in [0,1)$ such that

$$\forall x,y \in X,\ \theta(r)d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq rd(x,y),$$

has a fixed point.

Take $\mathcal{R}$ be the set of continuous functions $e : [0,\infty)^5 \longrightarrow [0,\infty)$ satisfying the following conditions:
a) $e(1,1,1,2,0) = e(1,1,1,0,2) = h \in (0,1)$,
b) e is sub-homogeneous, that is, $e(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5) \leq \alpha e(x_1,x_2,x_3,x_4,x_5)$ for all $\alpha \geq 0$ and all $(x_1,x_2,x_3,x_4,x_5) \in [0,\infty)^5$,
c) If $x_i, y_i \in [0,\infty)$ and $x_i < y_i$ for $i = 1,...,4$, then $e(x_1,x_2,x_3,x_4,0) < e(y_1,y_2,y_3,y_4,0)$ and $e(x_1,x_2,x_3,0,x_4) < e(y_1,y_2,y_3,0,y_4)$ (see [6]).
We present the following two illustrated examples.

Example 1.2. For $t_i \geq 0$, $i = 1,...,5$, define $e(t_1,t_2,t_3,t_4,t_5) = h\max\{t_1,t_2,t_3\}$ where $h \in (0,1)$. It is clear that $e \in \mathcal{R}$.

Example 1.3. For $t_i \geq 0$, $i = 1, ..., 5$, define $e(t_1, t_2, t_3, t_4, t_5) = at_1 + bt_2 + ct_3$ where $a, b, c \in (0, 1)$ such that $a + b + c = h \in (0, 1)$. It is clear that $e \in \mathcal{R}$.

We appeal the following results in the sequel.

Proposition 1.4. [6] *If $e \in \mathcal{R}$ and $u, v \in [0, \infty)$ are such that*

$$u \leq \max\{e(v, v, u, v + u, 0), e(v, u, v, v + u, 0), e(v, u, v, 0, v + u)\},$$

then $u \leq hv$.

Theorem 1.5. [7] *Let (X, d) be a complete metric space and $F_1, F_2 : X \to P_{b,cl}(X)$ be two multifunctions. Suppose that there exist $\alpha \in (0, \infty)$ and $e \in \mathcal{R}$ such that $\alpha(h + 1) \leq 1$ and $\alpha d(x, F_1x) \leq d(x, y)$ or $\alpha d(x, F_2x) \leq d(x, y)$ implies*

$$H(F_1x, F_2y) \leq e(d(x, y), d(x, F_1x), d(y, F_2y), d(x, F_2y), d(y, F_1x))$$

for all $x, y \in X$. Then $\mathcal{F}(F_1) = \mathcal{F}(F_2)$ and $(\mathcal{CF})_{F_1,F_2}$ is non-empty.

Theorem 1.6. [7] *Let (X, d) be a complete metric space and $F : X \to P_{b,cl}(X)$ be a multifunction. Assume that there exists $r \in [2^{\frac{-1}{2}}, 1)$ such that $\theta(r)d(x, Fx) \leq d(x, y)$ implies $H(Fx, Fy) \leq r \max\{d(x, y), d(x, Fx), d(y, Fy)\}$ for all $x, y \in X$. Then $\mathcal{F}_F$ is non-empty.*

Theorem 1.7. [7] *Let (X, d) be a complete metric space and $F : X \to P_{b,cl}(X)$ be a multifunction. Assume that there exist $\beta, \gamma \in [0, 1)$ such that*

$$\frac{1}{2\beta + \gamma + 1} d(x, Fx) \leq d(x, y)$$

implies $H(Fx, Fy) \leq \gamma d(x, y) + \beta d(x, Fx) + \beta d(y, Fy)$ for all $x, y \in X$. Then $\mathcal{F}_F$ is non-empty.

In this paper, we are concerned to provide some results on absolute retractivity of fixed points set involving Suzuki type contractive multifunctions.

2. Main result

Our first result is stated as follows.

Theorem 2.1. *Let (X, d) be a complete metric space and absolute retract for metric spaces, $F_1, F_2 : X \to P_{b,cl}(X)$ are multifunctions and $F_1, F_2 \in Sp(X)$. Suppose there exist $\alpha \in (0, \infty)$ and $e \in \mathcal{R}$ such that $\alpha(h + 1) \leq 1$ and the conditions $\alpha d(x, F_1x) \leq d(x, y)$ and $\alpha d(x, F_2x) \leq d(x, y)$ imply*

$$H(F_1(x), F_2(y)) \leq e(d(x, y), d(x, F_1(x)), d(y, F_2(y)), d(x, F_2(y)), d(y, F_1(x))),$$

for all $x, y \in X$. Then $(\mathcal{CF})_{F_1,F_2}$ is an absolute retract for metric spaces.

Proof. Let Y be a metric space, $A \in P_{cl}(Y)$ and $q > 1$. Let $\psi : A \to (\mathcal{CF})_{F_1,F_2}$ be a continuous function. Since X is an absolute retract for metric spaces, there exists a continuous function $\varphi_0 : Y \to X$ such that $\varphi_0|_A = \psi$. Define the function $g_0 : Y \to (0, \infty)$ by

$$g_0(y) = \sup\{d(\varphi_0(y), z) | z \in F_1(\varphi_0(y))\} + 1,$$

for all $y \in Y$. It is not difficult to see that g_0 is continuous and

$$F_1(\varphi_0(y)) \cap B(\varphi_0(y), g_0(y)) = F_1(\varphi_0(y)),$$

for all $y \in A$ (see [14]). Also, we observe that the function $\psi : A \to (\mathcal{CF})_{F_1,F_2}$ has the property $\psi(y) \in F_1(\varphi_0(y))$ $(y \in A)$, so it is a continuous selection of the multivalued mapping. Since $F_1 \in Sp(X)$, there exists a continuous function $\varphi_1 : Y \to X$ such that $\varphi_1|_A = \psi$ and $\varphi_1(y) \in F_1(\varphi_0(y))$ for all $y \in Y$. Since $\alpha d(\varphi_0(y), F_1(\varphi_0(y))) \leq \alpha d(\varphi_0(y), \varphi_1(y)) \leq d(\varphi_0(y), \varphi_1(y))$, we have

$$\begin{aligned}&H(F_1(\varphi_0(y)), F_2(\varphi_1(y))) \leq e(d(\varphi_0(y), \varphi_1(y))\\&, d(\varphi_0(y), F_1(\varphi_0(y))), d(\varphi_1(y), F_2(\varphi_1(y))), d(\varphi_0(y), F_2(\varphi_1(y))), d(\varphi_1(y), F_1(\varphi_0(y)))).\end{aligned}$$

Hence

$$\begin{aligned}d(\varphi_1(y), F_2(\varphi_1(y))) &\leq H(F_1(\varphi_0(y)), F_2(\varphi_1(y)))\\&\leq e(d(\varphi_0(y), \varphi_1(y)), d(\varphi_0(y), F_1(\varphi_0(y))), d(\varphi_1(y), F_2(\varphi_1(y)))\\&, d(\varphi_0(y), F_2(\varphi_1(y))), d(\varphi_1(y), F_1(\varphi_0(y))))\\&\leq e(d(\varphi_0(y), \varphi_1(y)), d(\varphi_0(y), \varphi_1(y)), d(\varphi_1(y), F_2(\varphi_1(y)))\\&, d(\varphi_0(y), \varphi_1(y)) + d(\varphi_1(y), F_2(\varphi_1(y))), 0).\end{aligned}$$

Fix $h < r < 1$. By using Proposition 1.4, we get

$$d(\varphi_1(y), F_2(\varphi_1(y))) \leq hd(\varphi_0(y), \varphi_1(y)) < rd(\varphi_0(y), \varphi_1(y)) < rd(\varphi_0(y), \varphi_1(y)) + q^{-1}.$$

Thus, $G_2(y) := F_2(\varphi_1(y)) \cap B(\varphi_1(y), rd(\varphi_0(y), \varphi_1(y)) + q^{-1}) \neq \emptyset$ for all $y \in Y$. Due to the fact that $F_2 \in Sp(X)$, there exists a continuous function $\varphi_2 : Y \to X$ such that $\varphi_2|_A = \psi$ and $\varphi_2(y) \in \overline{G_2(y)}$ for all $y \in Y$. But, note that

$$\alpha d(\varphi_1(y), F_2(\varphi_1(y))) \leq \alpha d(\varphi_1(y), \varphi_2(y)) \leq d(\varphi_1(y), \varphi_2(y)).$$

By assumption of theorem, we have

$$\begin{aligned}&d(\varphi_2(y), F_1(\varphi_2(y)))\\&\leq H(F_2(\varphi_1(y)), F_1(\varphi_2(y)))\\&\leq e(d(\varphi_1(y), \varphi_2(y)), d(\varphi_1(y), F_2(\varphi_1(y))), d(\varphi_2(y), F_1(\varphi_2(y))),\\&d(\varphi_1(y), F_1(\varphi_2(y))), d(\varphi_2(y), F_2(\varphi_1(y))))\\&\leq e(d(\varphi_1(y), \varphi_2(y)), d(\varphi_1(y), \varphi_2(y)), d(\varphi_2(y), F_1(\varphi_2(y)))\\&, d(\varphi_1(y), \varphi_2(y)) + d(\varphi_2(y), F_1(\varphi_2(y))), 0).\end{aligned}$$

Again, by using Proposition 1.4, we obtain

$$\begin{aligned}d(\varphi_2(y), F_1(\varphi_2(y))) &\leq hd(\varphi_1(y), \varphi_2(y)) < rd(\varphi_1(y), \varphi_2(y))\\&\leq r(rd(\varphi_0(y), \varphi_1(y))) = r^2 d(\varphi_0(y), \varphi_1(y))\\&< r^2 d(\varphi_0(y), \varphi_1(y)) + q^{-2}.\end{aligned}$$

So $G_3(y) := F_1(\varphi_2(y)) \cap B(\varphi_2(y)), r^2(d(\varphi_0(y), \varphi_1(y)) + q^{-2}) \neq \emptyset$.
Since $F_1 \in Sp(X)$, there exists a continuous function $\varphi_3 : Y \to X$ such that

$\varphi_3(y) \in \overline{G_3(y)}$ for all $y \in Y$ and $\varphi_3|_A = \psi$. Also, we have

$$d(\varphi_2(y), \varphi_3(y)) < r^2 d(\varphi_0(y), \varphi_1(y)) + q^{-2}$$

and $\varphi_3(y) \in F_1(\varphi_2(y))$ for all $y \in Y$. By continuing this process, we obtain that $\{\varphi_n : Y \to X\}_{n\geq 0}$ is a sequence of continuous functions such that
(i) $\varphi_{2n-1}(y) \in F_1(\varphi_{2n-2}(y))$;
(ii) $\varphi_{2n}(y) \in F_2(\varphi_{2n-1}(y))$;
(iii) $\varphi_n|_A = \psi$;
(iv) $d(\varphi_{n-1}(y), \varphi_n(y)) < r^n d(\varphi_0(y), \varphi_1(y)) + q^{-(n-1)}$ for all $y \in Y$ and $n \geq 1$.
For each $\lambda > 0$, put $Y_\lambda := \{y \in Y | d(\varphi_0(y), \varphi_1(y)) < \lambda\}$. The family $\{Y_\lambda | \lambda > 0\}$ is an open covering for Y (see [14]). Since

$$d(\varphi_{n-1}(y), \varphi_n(y)) < r^{n-1} d(\varphi_0(y), \varphi_1(y)) + q^{-(n-1)}$$

for all $y \in Y$ and $n \geq 1$, $r < 1$ and X is complete, the sequence $\{\varphi_n\}_{n\geq 0}$ converges uniformly on Y_λ for all $\lambda > 0$. Let $\varphi : Y \to X$ be the point-wise limit of $\{\varphi_n\}_{n\geq 0}$. Then φ is continuous and $\varphi|_A = \psi$ because $\varphi_n|_A = \psi$ for all $n \geq 0$. We have $\varphi_{2n-1}(y) \in F_1(\varphi_{2n-2}(y))$, and $\varphi_{2n}(y) \in F_2(\varphi_{2n-1}(y))$ for each $n \in \mathbb{N}$. Since F is lower semi-continuous, $\varphi(y) \in F_1(\varphi(y))$ and $\varphi(y) \in F_2(\varphi(y))$ for all $y \in Y$. Thus $\varphi : Y \to (\mathcal{CF})_{F_1,F_2}$ is a continuous extension of ψ. Therefore, $(\mathcal{CF})_{F_1,F_2}$ is an absolute retract for metric spaces. □

Our second main result is

Theorem 2.2. *Let (X, d) be a complete metric space and absolute retract for metric spaces. Let $F : X \to P_{b,cl}(X)$ be a multifunction and $F \in Sp(X)$. Suppose that there exist $\alpha \in (0, \infty)$ and $e \in \mathcal{R}$ such that $\alpha(h + 1) \leq 1$ and $\alpha d(x, Fx) \leq d(x, y)$ imply*

$$H(Fx, Fy) \leq e(d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx))$$

for all $x, y \in X$. Then $\mathcal{F}_F$ is an absolute retract for metric spaces.

Proof. Let Y be a metric space, $A \in P_{cl}(Y)$ and $q > 1$. Take $\psi : A \to \mathcal{F}_F$ be a continuous function.
Via a similar argument used in the proof of Theorem 2.1, we obtain a sequence $\{\varphi_n\}_{n\geq 0}$ such that $\varphi_n : Y \to X$ is a continuous function for each $n \geq 0$ with $\varphi_0|_A = \psi$ and satisfies the following properties:
(a) $\varphi_n(y) \in F(\varphi_{n-1}(y))$;
(b) $\varphi_n|_A = \psi$;
(c) $d(\varphi_{n-1}(y), \varphi_n(y)) < r^n d(\varphi_0(y), \varphi_1(y)) + q^{-(n-1)}$ for all $y \in Y$ and $n \geq 1$.
The sequence $\{\varphi_n\}_{n\geq 0}$ converges pointwise in Y to a continuous function $\varphi : Y \to X$ with $\varphi|_A = \psi$. Also, it can be written that $\varphi : Y \to \mathcal{F}_F$. □

Corollary 2.3. *Let (X, d) be a complete metric space and absolute retract for metric spaces. Let $F : X \to P_{b,cl}(X)$ be a multifunction and $F \in Sp(X)$. Assume that there exists $r \in [2^{\frac{-1}{2}}, 1)$ such that $\theta(r)d(x, Fx) \leq d(x, y)$ implies $H(Fx, Fy) \leq r \max\{d(x, y), d(x, Fx), d(y, Fy)\}$ for all $x, y \in X$. Then $\mathcal{F}_F$ is an absolute retract for metric spaces.*

Proof. Take $e \in \mathcal{R}$ as $e(x_1, x_2, x_3, x_4, x_5) = r \max\{x_1, x_2, x_3\}$ and $\alpha = \theta(r)$. Since $h = r$ and $\alpha(1 + h) \leq 1$, by using Theorem 2.2, $\mathcal{F}_F$ is an absolute retract for metric spaces. □

Example 2.4. Let $X = [0, \infty)$ be equipped with $d(x, y) = |x-y|$. (X, d) is a complete metric space. Consider $F : X \to P_{b,cl}(X)$ be defined by $F(x) = [2^{\frac{-1}{2}}x, \infty)$ for all $x \in X$. For $r \in [2^{\frac{-1}{2}}, 1)$ and for all $x, y \in X$, we have $\theta(r)d(x, Fx) = 0 \leq d(x, y)$. Also

$$H(Fx, Fy) = 2^{\frac{-1}{2}}|x - y| \leq r \max\{d(x, y), d(x, Fx), d(y, Fy)\}.$$

By using Corollary 2.3, $\mathcal{F}_F$ is an absolute retract for metric spaces.
Here, $\mathcal{F}_F = [0, \infty] = X$.

Corollary 2.5. *Let (X, d) be a complete metric space and absolute retract for metric spaces. Let $F : X \to P_{b,cl}(X)$ be a multifunction and $F \in Sp(X)$. Assume that there exist $\beta, \gamma \in [0, 1)$ such that $\frac{1}{2\beta+\gamma+1} d(x, Fx) \leq d(x, y)$ implies*

$$H(Fx, Fy) \leq \gamma d(x, y) + \beta d(x, Fx) + \beta d(y, Fy)$$

for all $x, y \in X$. Then $\mathcal{F}_F$ is an absolute retract for metric spaces.

Proof. Consider $e \in \mathcal{R}$ as $e(x_1, x_2, x_3, x_4, x_5) = \gamma x_1 + \beta x_2 + \beta x_3$ and $\alpha = \frac{1}{2\beta+\gamma+1}$. Since $h = 2\beta + \gamma$ and $\alpha(1 + h) \leq 1$, by using Theorem 2.2, $\mathcal{F}_F$ is an absolute retract for metric spaces. □

Let $\phi : [0, \infty)^5 \to [0, \infty)$ be continuous (or upper semi-continuous) and increasing in each coordinate variable and $\phi(t, t, t, at, bt) \leq t$ for every $t \in [0, \infty)$, where $a+b = 2$, $a, b \in \{0, 1, 2\}$. Also let $\phi : [0, 1) \to (0, 1]$ be non-increasing function defined by

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{2}, \\ (1 - r) & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Theorem 2.6. [10] *Let (X, d) be a complete metric space and let $F_1, F_2 : X \to P_{b,cl}(X)$. Assume that there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$\theta(r) \min\{d(x, F_1x), d(y, F_2y)\} \leq d(x, y),$$

implies

$$H(F_1(x), F_2(y)) \leq r\phi(d(x, y), d(x, F_1(x)), d(y, F_2(y)), d(x, F_2(y)), d(y, F_1(x))),$$

Then $(\mathcal{CF})_{F_1,F_2}$ is non-empty.

Theorem 2.7. *Let (X, d) be a complete metric space and absolute retract for metric spaces, $F_1, F_2 : X \to P_{b,cl}(X)$ are multifunctions and $F_1, F_2 \in Sp(X)$. Assume that there exists $r \in [0, 1)$ such that fore every $x, y \in X$,*

$$\theta(r) \min\{d(x, F_1x), d(y, F_2y)\} \leq d(x, y),$$

implies

$$H(F_1(x), F_2(y)) \leq r\phi(d(x, y), d(x, F_1(x)), d(y, F_2(y)), d(x, F_2(y)), d(y, F_1(x))),$$

for all $x, y \in X$. Then $(\mathcal{CF})_{F_1,F_2}$ is an absolute retract for metric spaces.

Proof. Let Y be a metric space, $A \in P_{cl}(Y)$ and $q > 1$. Let $\psi : A \to (\mathcal{CF})_{F_1,F_2}$ be a continuous function. Since X is an absolute retract for metric spaces, there exists a continuous function $\varphi_0 : Y \to X$ such that $\varphi_0|_A = \psi$. Define the function $g_0 : Y \to (0, \infty)$ by

$$g_0(y) = \sup\{d(\varphi_0(y), z)|z \in F_1(\varphi_0(y))\} + 1,$$

for all $y \in Y$. It is not difficult to see that g_0 is continuous and

$$F_1(\varphi_0(y)) \cap B(\varphi_0(y), g_0(y)) = F_1(\varphi_0(y)),$$

for all $y \in A$ (see [14]). Also, we observe that the function $\psi : A \to (\mathcal{CF})_{F_1,F_2}$ has the property $\psi(y) \in F_1(\varphi_0(y))$ $(y \in A)$, so it is a continuous selection of the multivalued mapping. Since $F_1 \in Sp(X)$, there exists a continuous function $\varphi_1 : Y \to X$ such that $\varphi_1|_A = \psi$ and $\varphi_1(y) \in F_1(\varphi_0(y))$ for all $y \in Y$. Since

$$\begin{aligned}
&\theta(r) \min\{d(\varphi_0(y), F_1(\varphi_0(y))), d(\varphi_1(y), F_2(\varphi_1(y)))\}\\
&\leq \theta(r)d(\varphi_0(y), F_1(\varphi_0(y)))\\
&\leq \theta(r)d(\varphi_0(y), \varphi_1(y))\\
&\leq d(\varphi_0(y), \varphi_1(y)),
\end{aligned}$$

we have

$$\begin{aligned}
&H(F_1(\varphi_0(y)), F_2(\varphi_1(y))) \leq r\phi(d(\varphi_0(y), \varphi_1(y))\\
&, d(\varphi_0(y), F_1(\varphi_0(y))), d(\varphi_1(y), F_2(\varphi_1(y))), d(\varphi_0(y), F_2(\varphi_1(y))), d(\varphi_1(y), F_1(\varphi_0(y)))).
\end{aligned}$$

Hence

$$\begin{aligned}
d(\varphi_1(y), F_2(\varphi_1(y))) &\leq H(F_1(\varphi_0(y)), F_2(\varphi_1(y)))\\
&\leq r\phi(d(\varphi_0(y), \varphi_1(y)), d(\varphi_0(y), F_1(\varphi_0(y))), d(\varphi_1(y), F_2(\varphi_1(y)))\\
&, d(\varphi_0(y), F_2(\varphi_1(y))), d(\varphi_1(y), F_1(\varphi_0(y))))\\
&\leq r\phi(d(\varphi_0(y), \varphi_1(y)), d(\varphi_0(y), \varphi_1(y)), d(\varphi_1(y), F_2(\varphi_1(y)))\\
&, d(\varphi_0(y), \varphi_1(y)) + d(\varphi_1(y), F_2(\varphi_1(y))), 0).
\end{aligned}$$

If $\max\{d(\varphi_0(y), \varphi_1(y)) + d(\varphi_1(y), F_2(\varphi_1(y)))\} = d(\varphi_1(y), F_2(\varphi_1(y)))$, then by properties of ϕ we obtain,

$$\begin{aligned}
d(\varphi_1(y), F_2(\varphi_1(y))) &\leq r\phi(d(\varphi_0(y), \varphi_1(y)), d(\varphi_0(y), \varphi_1(y)), d(\varphi_1(y), F_2(\varphi_1(y)))\\
&, d(\varphi_0(y), \varphi_1(y)) + d(\varphi_1(y), F_2(\varphi_1(y))), 0)\\
&\leq r\phi(d(\varphi_1(y), F_2(\varphi_1(y))), d(\varphi_1(y), F_2(\varphi_1(y))), d(\varphi_1(y), F_2(\varphi_1(y)))\\
&, 2d(\varphi_1(y), F_2(\varphi_1(y))), 0)\\
&\leq rd(\varphi_1(y), F_2(\varphi_1(y))),
\end{aligned}$$

which is contradiction.
So $\max\{d(\varphi_0(y), \varphi_1(y)) + d(\varphi_1(y), F_2(\varphi_1(y)))\} = d(\varphi_0(y), \varphi_1(y))$, hence

$$d(\varphi_1(y), F_2(\varphi_1(y)) \leq rd(\varphi_0(y), \varphi_1(y)) < rd(\varphi_0(y), \varphi_1(y)) + q^{-1}.$$

Thus, $G_2(y) := F_2(\varphi_1(y)) \cap B(\varphi_1(y), rd(\varphi_0(y), \varphi_1(y)) + q^{-1}) \neq \emptyset$ for all $y \in Y$. Due to the fact that $F_2 \in Sp(X)$, there exists a continuous function $\varphi_2 : Y \to X$ such that $\varphi_2|_A = \psi$ and $\varphi_2(y) \in \overline{G_2(y)}$ for all $y \in Y$. But, note that

$$\begin{aligned}
&\theta(r)\min\{d(\varphi_1(y), F_2(\varphi_1(y))), d(\varphi_2(y), F_1(\varphi_2(y)))\} \\
&\leq \theta(r)d(\varphi_1(y), F_2(\varphi_1(y))) \\
&\leq \theta(r)d(\varphi_1(y), \varphi_2(y)) \\
&\leq d(\varphi_1(y), \varphi_2(y)),
\end{aligned}$$

By assumption of theorem, we have

$$\begin{aligned}
&d(\varphi_2(y), F_1(\varphi_2(y))) \\
&\leq H(F_2(\varphi_1(y)), F_1(\varphi_2(y))) \\
&\leq r\phi(d(\varphi_1(y), \varphi_2(y)), d(\varphi_1(y), F_2(\varphi_1(y))), d(\varphi_2(y), F_1(\varphi_2(y))), \\
&d(\varphi_1(y), F_1(\varphi_2(y))), d(\varphi_2(y), F_2(\varphi_1(y)))) \\
&\leq r\phi(d(\varphi_1(y), \varphi_2(y)), d(\varphi_1(y), \varphi_2(y)), d(\varphi_2(y), F_1(\varphi_2(y))) \\
&, d(\varphi_1(y), \varphi_2(y)) + d(\varphi_2(y), F_1(\varphi_2(y))), 0).
\end{aligned}$$

Again, by properties of ϕ, we obtain

$$\begin{aligned}
d(\varphi_2(y), F_1(\varphi_2(y))) &\leq rd(\varphi_1(y), \varphi_2(y)) \\
&\leq r(rd(\varphi_0(y), \varphi_1(y))) = r^2 d(\varphi_0(y), \varphi_1(y)) \\
&< r^2 d(\varphi_0(y), \varphi_1(y)) + q^{-2}.
\end{aligned}$$

So $G_3(y) := F_1(\varphi_2(y)) \cap B(\varphi_2(y)), r^2(d(\varphi_0(y), \varphi_1(y)) + q^{-2}) \neq \emptyset$.
Since $F_1 \in Sp(X)$, there exists a continuous function $\varphi_3 : Y \to X$ such that $\varphi_3(y) \in \overline{G_3(y)}$ for all $y \in Y$ and $\varphi_3|_A = \psi$. Also, we have

$$d(\varphi_2(y), \varphi_3(y)) < r^2 d(\varphi_0(y), \varphi_1(y)) + q^{-2}$$

and $\varphi_3(y) \in F_1(\varphi_2(y))$ for all $y \in Y$. By continuing this process, we obtain that $\{\varphi_n : Y \to X\}_{n \geq 0}$ is a sequence of continuous functions such that
(i) $\varphi_{2n-1}(y) \in F_1(\varphi_{2n-2}(y))$;
(ii) $\varphi_{2n}(y) \in F_2(\varphi_{2n-1}(y))$;
(iii) $\varphi_n|_A = \psi$;
(iv) $d(\varphi_{n-1}(y), \varphi_n(y)) < r^n d(\varphi_0(y), \varphi_1(y)) + q^{-(n-1)}$ for all $y \in Y$ and $n \geq 1$.
Similar to the proof of Theorem 2.1, $(\mathcal{CF})_{F_1,F_2}$ is an absolute retract for metric spaces. □

Corollary 2.8. *Let (X, d) be a complete metric space and absolute retract for metric spaces, $F : X \to P_{b,cl}(X)$ be a multifunction and $F \in Sp(X)$. Assume that there exists $r \in [0, 1)$ such that fore every $x, y \in X$,*

$$\theta(r)\min\{d(x, Fx), d(y, Fy)\} \leq d(x, y),$$

implies

$$H(F(x), F(y)) \leq r\phi(d(x, y), d(x, F(x)), d(y, F(y)), d(x, F(y)), d(y, F(x))),$$

for all $x, y \in X$. Then $(\mathcal{F})_F$ is an absolute retract for metric spaces.

If $\phi\{x_1, x_2, x_3, x_4, x_5\} = \max\{x_1, x_2, x_3, \frac{x_4+x_5}{2}\}$, then we get the following corollary.

Corollary 2.9. *Let (X, d) be a complete metric space and absolute retract for metric spaces, $F_1, F_2 : X \to P_{b,cl}(X)$ are multifunctions and $F_1, F_2 \in Sp(X)$. Assume that there exists $r \in [0, 1)$ such that fore every $x, y \in X$,*

$$\theta(r) \min\{d(x, F_1x), d(y, F_2y)\} \leq d(x, y),$$

implies

$$H(F_1(x), F_2(y)) \leq r \max\left\{d(x, y), d(x, F_1(x)), d(y, F_2(y)), \frac{d(x, F_2(y)) + d(y, F_1(x))}{2}\right\},$$

for all $x, y \in X$. Then $(\mathcal{CF})_{F_1,F_2}$ is an absolute retract for metric spaces.

Theorem 2.10. [10] *Let (X, d) be a complete metric space and let $F_1, F_2 : X \to P_{b,cl}(X)$. Assume that there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$\theta(r) \min\{d(x, F_1x), d(y, F_2y)\} \leq d(x, y),$$

where

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{\sqrt{5}-1}{2}, \\ (1-r) & \text{if } \frac{\sqrt{5}-1}{2} \leq r < 1. \end{cases}$$

implies

$$H(F_1(x), F_2(y)) \leq r \max\{d(x, F_1(x)), d(y, F_2(y))\},$$

Then $(\mathcal{CF})_{F_1,F_2}$ is non-empty.

Corollary 2.11. *Let (X, d) be a complete metric space and absolute retract for metric spaces, $F_1, F_2 : X \to P_{b,cl}(X)$ are multifunctions and $F_1, F_2 \in Sp(X)$. Assume that there exists $r \in [0, 1)$ such that fore every $x, y \in X$,*

$$\theta(r) \min\{d(x, F_1x), d(y, F_2y)\} \leq d(x, y),$$

where

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{\sqrt{5}-1}{2}, \\ (1-r) & \text{if } \frac{\sqrt{5}-1}{2} \leq r < 1. \end{cases}$$

implies

$$H(F_1(x), F_2(y)) \leq r \max\{d(x, F_1(x)), d(y, F_2(y))\},$$

for all $x, y \in X$. Then $(\mathcal{CF})_{F_1,F_2}$ is an absolute retract for metric spaces.

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