Fixed Point Theory, 20(2019), No. 2, 391-406
DOI: 10.24193/fpt-ro.2019.2.25
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

HILFER AND HADAMARD RANDOM FRACTIONAL DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES

SAID ABBAS*, MOUFFAK BENCHOHRA**, YONG ZHOU***,[1] AND AHMED ALSAEDI****

*Laboratory of Mathematics, University of Saïda
P.O. Box 138, Saïda 20000, Algeria
E-mail: abbasmsaid@yahoo.fr

**Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes
P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
E-mail: benchohra@yahoo.com

***Faculty of Mathematics and Computational Science, Xiangtan University
Hunan 411105, P.R. China
NAAM Research Group, Faculty of Science, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail: yzhou@xtu.edu.cn

****NAAM Research Group, Faculty of Science, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail: aalsaedi@hotmail.com

Abstract. This paper deals with some existence and Ulam stability results for Hilfer and Hilfer-Hadamard type fractional random differential equations in Fréchet spaces. A random fixed point theorem is applied to prove the existence of random solutions. Also it is shown that the solutions to our problems are generalized Ulam-Hyers-Rassias stable.

Key Words and Phrases: Functional random differential equation, Riemann-Liouville integral of fractional order, Hadamard integral of fractional order, Hilfer fractional derivative, Hilfer-Hadamard fractional derivative, Fréchet space, Ulam stability, random fixed point.
2010 Mathematics Subject Classification: 26A33, 34A08, 34K37, 47H10.

1. Introduction

Fractional order differential equations appear in various areas of engineering, mathematics, physics, bio-engineering, etc. [15, 29]. For theoretical development of fractional calculus and fractional differential equations, we refer the reader to the monographs by Abbas *et al.* [5, 6], Kilbas *et al.* [21] and Zhou [35, 36], and a series of papers [3, 7, 8, 38, 39, 41, 40, 37], and the references therein.

The nature of a dynamic system in engineering or natural sciences depends on the accuracy of values of parameters describing the system. Further, a precisely described dynamic system gives rise to a deterministic dynamical system. Unfortunately, in

[1]Corresponding author.

most cases, the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system involves certain uncertainties. In case the parameters of a dynamic system are of statistical nature (probabilistic), the common approach to model such systems is based on random or stochastic differential equations. Random differential equations, regarded as natural extension of deterministic ones, arise in many applications and have been investigated by many researchers, for instance, see the monographs [9, 22, 31].

The issue of stability of functional equations was originally raised by Ulam [32]), followed by Hyers [16] and Rassias [25]. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation to the equation. Several authors have discussed Ulam-Hyers and Ulam-Hyers-Rassias stabilities for functional equations, for example, see [6, 17], and the articles by Abbas *et al.* [1, 2, 3, 4, 7], Petru *et al.* [23], etc. Rus [26, 27] discussed the Ulam-Hyers stability for operatorial equations and inclusions. For the historical and recent developments of such stabilities, for instance, see [18, 26].

In recent years, initial and boundary value problems of fractional differential equations involving Hilfer fractional derivative have been studied by many authors [11, 15, 19, 30, 33].

In this article, we discuss the existence and the Ulam stability of solutions for the following problem of Random Hilfer fractional differential equations:

$$\begin{cases} (D_0^{\alpha,\beta}u)(t,w) = f(t,u(t,w),w);\ t\in\mathbb{R}_+ := [0,+\infty), \\ \\ (I_0^{1-\gamma}u)(t,w)|_{t=0} = \phi(w), \end{cases} \quad w\in\Omega, \tag{1.1}$$

where $\alpha\in(0,1),\ \beta\in[0,1],\ \gamma=\alpha+\beta-\alpha\beta$, $(\Omega,\mathcal{A})$ is a measurable space, $\phi:\Omega\to\mathbb{R}$ is a measurable function, $f:\mathbb{R}_+\times\mathbb{R}\times\Omega\to\mathbb{R}$ is a given function, $I_0^{1-\gamma}$ is the left-sided mixed Riemann-Liouville integral of order $1-\gamma$, and $D_0^{\alpha,\beta}$ is the Hilfer fractional derivative of order α and type β.

Next, we consider the following problem of random Hilfer-Hadamard fractional differential equations of the form

$$\begin{cases} ({}^HD_1^{\alpha,\beta}u)(t,w) = g(t,u(t,w),w);\ t\in[1,+\infty), \\ \\ ({}^HI_1^{1-\gamma}u)(1,w) = \phi_0(w), \end{cases} \quad w\in\Omega, \tag{1.2}$$

where $\alpha\in(0,1),\ \beta\in[0,1],\ \gamma=\alpha+\beta-\alpha\beta,\ \phi_0:\Omega\to\mathbb{R}$ is a measurable function, $g:[1,+\infty)\times\mathbb{R}\times\Omega\to\mathbb{R}$ is a given function, ${}^HI_1^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1-\gamma$, and ${}^HD_1^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order α and type β.

The present paper initiates the Ulam stability for random differential equations involving Hilfer and Hilfer-Hadamard fractional derivatives in in Fréchet spaces.

2. Preliminaries

Let C be the Banach space of all continuous functions v from $I := [0,T];\ T > 0$ into $\mathbb{R}$ with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t\in I} |v(t)|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}$. By $L^1(I)$, we denote the space of Lebesgue-integrable functions $v : I \to \mathbb{R}$ with the norm

$$\|v\|_1 = \int_0^T |v(t)|dt.$$

Let $L^\infty(I)$ be the Banach space of measurable functions $u : I \to \mathbb{R}$ which are essentially bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : |u(t)| \le c,\ a.e.\ t \in I\}.$$

By $C_\gamma(I)$ and $C^1_\gamma(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \{w : (0,T] \to \mathbb{R} : t^{1-\gamma}w(t) \in C\},$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t\in I} |t^{1-\gamma}w(t)|,$$

and

$$C^1_\gamma(I) = \{w \in C : \frac{dw}{dt} \in C_\gamma\},$$

with the norm

$$\|w\|_{C^1_\gamma} := \|w\|_\infty + \|w'\|_{C_\gamma}.$$

Throughout this paper, we denote $\|w\|_{C_\gamma}$ by $\|w\|_C$.

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}^*}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \dots \quad \text{for every } x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \le \overline{M}_n \quad \text{for all } y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for $x, y \in X$. We denote by $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote by $[x]_n$ the equivalence class of x of subset X^n and we define $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more details about the properties in Fréchet spaces see [10].

For each $p \in \mathbb{N}\backslash\{0\}$, we set $I_p := [0,p]$, we consider the following set, $C_{p,\gamma} = C_\gamma([0,p])$, and we define in C_γ the semi-norms by

$$\|u\|_p = \sup_{t\in[0,p]} |t^{1-\gamma}u(t)|.$$

Then C_γ is a Fréchet space with the family of semi-norms $\{\|u\|_p\}$. Also $C_{\gamma,\ln}$ is a Fréchet space with the family of semi-norms $\{\|v\|_p\}_{p\in\mathbb{N}\setminus\{0,1\}}$, such that

$$\|v\|_p = \sup_{t\in[1,p]} |(\ln t)^{1-\gamma} v(t)|.$$

Let X be a Fréchet space. A mapping $T : \Omega \times X \to X$ is called a random operator if $T(w, u)$ is measurable in w for all $x \in X$ and it expressed as $T(w)x = T(w, x)$. In this case we say that $T(w)$ is a random operator on X. A random operator $T(w)$ on X is called continuous (resp. compact, totally bounded and completely continuous) if $T(w, x)$ is continuous (resp. compact, totally bounded and completely continuous) in x for all $w \in \Omega$. The details of completely continuous random operators in Fréchet spaces and their properties appear in Goudarzi [12].

Definition 2.1. Let β_X be the σ-algebra of Borel subsets of X. A mapping $v : \Omega \to X$ is said to be measurable if for any $B \in \beta_X$, one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

A mapping $T : \Omega \times X \to X$ is called jointly measurable if for any $B \in \beta_X$, one has

$$T^{-1}(B) = \{(w, x) \in \Omega \times X : T(w, x) \in B\} \subset \mathcal{A} \times \beta_X,$$

where $\mathcal{A} \times \beta_B$ is the direct product of the σ-algebras $\mathcal{A}$ and β_X those defined in Ω and X respectively.

Definition 2.2. Let D be a nonempty convex subset of X. A self mapping T on D is said to be affine if

$$T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y),$$

for all $x, y \in D$ and $\lambda \in (0, 1)$.

As in [14], we can show the following results about the measurability and the random Carathéodory mappings.

Lemma 2.3. *Let $T : \Omega \times X \to X$ be a mapping such that $T(\cdot, x)$ is measurable for all $x \in X$, and $T(w, \cdot)$ is continuous for all $w \in \Omega$. Then the map $(w, x) \mapsto T(w, x)$ is jointly measurable.*

Definition 2.4. A function $f : I \times X \times \Omega \to X$ is called random Carathéodory if the following conditions are satisfied:

(i) The map $(t, x, w) \to f(t, x, w)$ is jointly measurable for all $x \in X$, and
(ii) The map $x \to f(t, x, w)$ is continuous for almost all $t \in I$ and $w \in \Omega$.

Now, we give some results and properties of fractional calculus.

Definition 2.5. [5, 21, 28] The Riemann-Liouville integral of order $r > 0$ of a function $w \in L^1(I)$ is defined by

$$(I_0^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s) ds; \ \textit{for a.e. } t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt;\ \xi > 0.$$

Notice that for all $r, r_1, r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t);\ for\ a.e.\ t \in I.$$

Definition 2.6. [5, 21, 28] The Riemann-Liouville fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned}(D_0^r w)(t) &= \left(\frac{d}{dt} I_0^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} w(s)ds;\ \text{for a.e. } t \in I.\end{aligned}$$

Let $r \in (0,1],\ \gamma \in [0,1)$ and $w \in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator as follows.

$$(D_0^r I_0^r w)(t) = w(t);\ for\ all\ t \in (0,T].$$

Moreover, if $I_0^{1-r} w \in C^1_{1-\gamma}(I)$, then the following composition is proved in [28]

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r} w)(0^+)}{\Gamma(r)} t^{r-1};\ for\ all\ t \in (0,T].$$

Definition 2.7. [5, 21, 28] The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in AC(I)$ is defined by

$$\begin{aligned}({}^c D_0^r w)(t) &= \left(I_0^{1-r} \frac{d}{dt} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} w'(s)ds;\ for\ a.e.\ t \in I.\end{aligned}$$

In [15], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and the Caputo derivatives as specific cases (see also [19, 30]).

Definition 2.8. (Hilfer derivative). Let $\alpha \in (0,1),\ \beta \in [0,1],\ w \in L^1(I)$, $I_0^{(1-\alpha)(1-\beta)} \in AC(I)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{(1-\alpha)(1-\beta)} w\right)(t);\ for\ a.e.\ t \in I. \tag{2.1}$$

Properties. Let $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.
1) The operator $(D_0^{\alpha,\beta} w)(t)$ can be written as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{1-\gamma} w\right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^\gamma w\right)(t);\ for\ a.e.\ t \in I.$$

Moreover, the parameter γ satisfies

$$\gamma \in (0,1],\ \gamma \geq \alpha,\ \gamma > \beta,\ 1-\gamma < 1 - \beta(1-\alpha).$$

2) The generalization (2.1) for $\beta = 0$, coincides with the Riemann-Liouville derivative and for $\beta = 1$ with the Caputo derivative.

$$D_0^{\alpha,0} = D_0^{\alpha}, \; and \; D_0^{\alpha,1} = \, ^cD_0^{\alpha}.$$

3) If $D_0^{\beta(1-\alpha)}w$ exists and in $L^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = (I_0^{\beta(1-\alpha)} D_0^{\beta(1-\alpha)} w)(t); \; for \; a.e. \; t \in I.$$

Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)}w \in C_\gamma^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = w(t); \; for \; a.e. \; t \in I.$$

4) If $D_0^{\gamma}w$ exists and in $L^1(I)$, then

$$(I_0^{\alpha} D_0^{\alpha,\beta} w)(t) = (I_0^{\gamma} D_0^{\gamma} w)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1}; \; for \; a.e. \; t \in I.$$

Corollary 2.9. *Let $h \in C_\gamma(I)$. The linear problem*

$$\begin{cases} (D_0^{\alpha,\beta}u)(t) = h(t); \; t \in I, \\ \\ (I_0^{1-\gamma}u)(t)|_{t=0} = \phi, \end{cases}$$

has a unique solution $u \in L^1(I)$ given by

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} h)(t).$$

From the above corollary, we concluded the following lemma.

Lemma 2.10. *Let $f : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ be such that $f(\cdot, u(\cdot, w), w) \in C_\gamma$ for all $w \in \Omega$, and any $u(w) \in C_\gamma$. Then problem (1.1) is equivalent to the problem of the solutions of the integral equation*

$$u(t,w) = \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} f(\cdot, u(\cdot, w), w)(t); \; w \in \Omega.$$

Now, we consider the Ulam stability for the problem (1.1). Let $\epsilon > 0$ and $\Phi : I \times \Omega \to [0, \infty)$ be a measurable and bounded function. We consider the following inequalities

$$|(D_0^{\alpha,\beta}u)(t,w) - f(t, u(t,w), w)| \le \epsilon; \; t \in I_p, \; w \in \Omega, \tag{2.2}$$

$$|(D_0^{\alpha,\beta}u)(t,w) - f(t, u(t,w), w)| \le \Phi(t,w); \; t \in I_p, \; w \in \Omega, \tag{2.3}$$

$$|(D_0^{\alpha,\beta}u)(t,w) - f(t, u(t,w), w)| \le \epsilon\Phi(t,w); \; t \in I_p, \; w \in \Omega. \tag{2.4}$$

Definition 2.11. [5, 26] The problem *(1.1)* is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.2)* there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t,w) - v(t,w)| \le \epsilon c_f; \; t \in I_p, \; w \in \Omega.$$

Definition 2.12. [5, 26] The problem *(1.1)* is generalized Ulam-Hyers stable if there exists $c_f : C([0,\infty),[0,\infty))$ with $c_f(0) = 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.2)* there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t,w) - v(t,w)| \le c_f(\epsilon);\ t \in I_p,\ w \in \Omega.$$

Definition 2.13. [5, 26] The problem *(1.1)* is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.4)* there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t,w) - v(t,w)| \le \epsilon c_{f,\Phi}\Phi(t,w);\ t \in I_p,\ w \in \Omega.$$

Definition 2.14. [5, 26] The problem *(1.1)* is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.3)*, there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t,w) - v(t,w)| \le c_{f,\Phi}\Phi(t,w);\ t \in I_p,\ w \in \Omega.$$

Remark 2.15. It is clear that

(i) Definition 2.11 $\Rightarrow$ Definition 2.12,
(ii) Definition 2.13 $\Rightarrow$ Definition 2.14,
(iii) Definition 2.13 for $\Phi(\cdot,\cdot) = 1 \ \Rightarrow$ Definition 2.11.

One can have similar remarks for the inequalities (2.2) and (2.4).

In pure and applied aspects of Fixed point Theory, there are useful results such as Markov-Kakutani and Krasnoselski theorems. We need the following stochastic analogue fixed point of these theorems in the case of a Fréchet space.

Theorem 2.16. [12] *Let K be a compact convex subset of a Fréchet space X and $T : \Omega \times K \to K$ be a continuous affine random operator. Then T has a random fixed point.*

We recall now an integral inequality which based on an iteration argument.

Lemma 2.17. [34] *Suppose $\beta > 0,\ a(t)$ is a nonnegative function locally integrable on $0 \le t < T$ (some $T \le +\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \le t < T,\ g(t) \le M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \le t < T$ with*

$$u(t) \le a(t) + g(t)\int_0^t (t-s)^{\beta-1}u(s)ds$$

on this interval. Then

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right]ds,\ 0 \le t < T.$$

From the above lemma, we concluded the following lemma.

Lemma 2.18. *Suppose* $\beta > 0$, $a(t,w)$ *is a nonnegative function locally integrable on* $[0,T)\times\Omega$ *(some* $T \leq +\infty$*) and* $g(t,w$ *is a nonnegative, nondecreasing continuous function with respect to* t *defined on* $[0,T)\times\Omega$, $g(t,w) \leq M$ *(constant), and suppose* $u(t,w)$ *is nonnegative and locally integrable with respect to* t *on* $[0,T)\times\Omega$ *with*

$$u(t,w) \leq a(t,w) + g(t,w)\int_0^t (t-s)^{\beta-1}u(s,w)ds$$

on $[0,T)\times\Omega$. *Then*

$$u(t,w) \leq a(t,w) + \int_0^t \left[\sum_{n=1}^{\infty}\frac{(g(t,w)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s,w)\right]ds,\ (t,w)\in[0,T)\times\Omega.$$

3. Hilfer fractional random differential equations

In this section, we are concerned with the existence and Ulam-Hyers-Rassias stability for problem (1.1). Let us start by defining what we mean by a random solution of the problem (1.1).

Definition 3.1. By a random solution of the problem (1.1) we mean a measurable function $u : \Omega \to C_\gamma$ that satisfies the condition $(I_0^{1-\gamma}u)(0^+,w) = \phi(w)$, and the equation $(D_0^{\alpha,\beta}u)(t,w) = f(t,u(t,w),w)$ on $I\times\Omega$.

The following hypotheses will be used in the sequel.

- (H_1) The function $f : I_p\times\mathbb{R}\times\Omega \mapsto f(t,u,w)\in\mathbb{R}$ is random Carathéodory on $I_p\times\mathbb{R}\times\Omega$, and affine with respect to u,
- (H_2) There exists a measurable and bounded function $l : \Omega \to L^\infty(I_p,[0,\infty))$, such that

$$|f(t,u,w) - f(t,v,w)| \leq l(t,w)|u-v|;\ \textit{for a.e. } t\in I_p,\ \textit{and each } u,v\in\mathbb{R},\ w\in\Omega,$$

- (H_3) There exists $\lambda_\Phi > 0$ such that for each $t\in I_p$, and $w\in\Omega$, we have

$$\int_0^t\left[\sum_{n=1}^{\infty}\frac{(l_p^*)^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}\Phi(s,w)\right]ds \leq \lambda_\Phi\Phi(t,w),$$

where $l_p^* = \sup_{w\in\Omega}\|l(w)\|_{L^\infty(I_p)}$.

For any $p\in\mathbb{N}\backslash\{0\}$, set

$$f_p^* = \sup_{w\in\Omega}\|f(\cdot,0,w)\|_{L^\infty(I_p)},\ \textit{and}\ \phi^* = \sup_{w\in\Omega}|\phi(w)|.$$

Now, we shall prove the following theorem concerning the existence of random solutions of problem (1.1).

Theorem 3.2. *Assume that the hypotheses* (H_1) *and* (H_2) *hold. If*

$$\frac{l_p^* p^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} < 1, \tag{3.1}$$

then problem (1.1) has at least one random solution in the space C_γ. *Furthermore, if the hypothesis* (H_3) *holds, then problem (1.1) is generalized Ulam-Hyers-Rassias stable.*

Proof. Define a mapping $N:\Omega\times C_\gamma\to C_\gamma$ by:

$$(N(w)u)(t)=\frac{\phi(w)}{\Gamma(\gamma)}t^{\gamma-1}+\int_0^t(t-s)^{\alpha-1}\frac{f(s,u(s,w),w)}{\Gamma(\alpha)}ds. \tag{3.2}$$

The map ϕ is measurable for all $w\in\Omega$. Again, as the indefinite integral is continuous on I, then $N(w)$ defines a mapping $N:\Omega\times C_\gamma\to C_\gamma$. Thus u is a random solution for the problem (1.1) if and only if $u=N(w)u$.

For each $p\in\mathbb{N}\backslash\{0\}$ and any $w\in\Omega$, we can show that $N(w)$ transforms the ball $B_\eta:=\{u\in C_\gamma:\|u\|_p\le\eta_p\}$ into itself, where

$$\eta_p\ge\frac{\phi^*\Gamma(1+\alpha)+\Gamma(\gamma)f_p^*p^{1-\gamma+\alpha}}{\Gamma(\gamma)(\Gamma(1+\alpha)-l_p^*p^{1-\gamma+\alpha})}.$$

Indeed, for any $w\in\Omega$, and each $u\in B_\eta$ and $t\in\tilde{I}_p$, we have

$$\begin{aligned}
|t^{1-\gamma}(N(w)u)(t)| &\le \frac{|\phi(w)|}{\Gamma(\gamma)}+\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,u(s,w),w)|ds\\
&\le \frac{|\phi(w)|}{\Gamma(\gamma)}+\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,0,w)|ds\\
&+ \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,u(s,w),w)-f(s,0,w)|ds\\
&\le \frac{|\phi(w)|}{\Gamma(\gamma)}+\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,0,w)|ds\\
&+ \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}l(s,w)|u(s,w)|ds\\
&\le \frac{|\phi(w)|}{\Gamma(\gamma)}+\frac{f_p^*T^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\\
&+ \frac{l_p^*\eta_pT^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\\
&\le \frac{\phi^*}{\Gamma(\gamma)}+\frac{(f_p^*+l_p^*\eta_p)p^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\\
&\le \eta_p.
\end{aligned}$$

Thus

$$\|N(w)u\|_p\le\eta_p. \tag{3.3}$$

We shall show that the operator $N:\Omega\times B_\eta\to B_\eta$ satisfies all the assumptions of Theorem 2.16. The proof will be given in several steps.

Step 1. $N(w)$ is a random operator on $\Omega\times B_\eta$ into B_η.

Since $f(t,u,w)$ is random Carathéodory, the map $w\to f(t,u,w)$ is measurable in view of Definition 2.1. Similarly, the product $(t-s)^{\alpha-1}f(s,u(s,w),w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w\mapsto\frac{\phi(w)}{\Gamma(\gamma)}t^{\gamma-1}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s,w),w)ds,$$

is measurable. As a result, $N(w)$ is a random operator on $\Omega \times B_\eta$ into B_η.
Step 2. $N(w)$ *is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_η. Then, for each $t \in I_p$, and $w \in \Omega$, we have

$$\begin{aligned} &|t^{1-\gamma}(N(w)u_n)(t) - t^{1-\gamma}(N(w)u)(t)| \\ &\le \tfrac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|f(s,u_n(s,w),w) - f(s,u(s,w),w)|ds \\ &\le \tfrac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l(s,w)|u_n(s,w) - u(s,w)|ds \\ &\le \tfrac{l_p^* T^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|u_n(s,w) - u(s,w)|ds. \end{aligned} \tag{3.4}$$

Since $u_n \to u$ as $n \to \infty$, then (3.4) implies

$$\|N(w)u_n - N(w)u\|_p \to 0 \quad \text{as } n \to \infty.$$

Step 3. $N(w)$ *is affine.*
For each$u, v \in B_\eta,\ t \in I_p$, and any $\lambda \in (0,1)$ and $w \in \Omega$, we have

$$\begin{aligned} N(w)(\lambda u + (1-\lambda)v) &= \tfrac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + \int_0^t (t-s)^{\alpha-1} \tfrac{f(s,(\lambda u+(1-\lambda)v)(s,w),w)}{\Gamma(\alpha)} ds \\ &= \tfrac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + \lambda \int_0^t (t-s)^{\alpha-1} \tfrac{f(s,u(s,w),w)}{\Gamma(\alpha)} ds \\ &+(1-\lambda) \int_0^t (t-s)^{\alpha-1} \tfrac{f(s,v(s,w),w)}{\Gamma(\alpha)} ds \\ &= \tfrac{\lambda\phi}{\Gamma(\gamma)} t^{\gamma-1} + \lambda \int_0^t (t-s)^{\alpha-1} \tfrac{f(s,u(s,w),w)}{\Gamma(\alpha)} ds \\ &+ \tfrac{(1-\lambda)\phi}{\Gamma(\gamma)} t^{\gamma-1} + (1-\lambda) \int_0^t (t-s)^{\alpha-1} \tfrac{f(s,v(s,w),w)}{\Gamma(\alpha)} ds \\ &= \lambda N(w)(u) + (1-\lambda)N(w)(v). \end{aligned}$$

Hence $N(w)$ is affine.

As a consequence of Steps 1 to 3, together with the Theorem 2.16, we deduce that N has a random fixed point v which is a random solution of the problem (1.1).

Step 4. *The generalized Ulam-Hyers-Rassias stability.*
Let u be a random solution of the inequality (2.3), and let us assume that v is a random solution of problem (1.1). Thus, we have

$$v(t,w) = \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} + \int_0^t (t-s)^{\alpha-1} \frac{f(s,v(s,w),w)}{\Gamma(\alpha)} ds.$$

From the inequality (2.3) for each $t \in I_p$, and $w \in \Omega$, we have

$$\left| u(t,w) - \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} - \int_0^t (t-s)^{\alpha-1} \frac{f(s,u(s,w),w)}{\Gamma(\alpha)} ds \right| \le (I_0^\alpha \Phi)(t,w).$$

From hypotheses (H_2) and (H_3), for each $t \in I_p$, and $w \in \Omega$, we get

$$\begin{aligned} |u(t,w) - v(t,w)| &\le \left| u(t,w) - \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} - \int_0^t (t-s)^{\alpha-1} \frac{f(s,u(s,w),w)}{\Gamma(\alpha)} ds \right| \\ &+ \int_0^t (t-s)^{\alpha-1} \frac{|f(s,u(s,w),w) - f(s,v(s,w),w)|}{\Gamma(\alpha)} ds \\ &\le (I_0^\alpha \Phi)(t,w) + \frac{l_p^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|u(s,w) - v(s,w)|ds. \end{aligned}$$

From Lemma 2.18, we have

$$\begin{aligned} |u(t,w)-v(t,w)| &\leq \frac{\lambda_\phi}{l_p^*}\left[\Phi(t,w)+\int_0^t\left[\sum_{n=1}^{\infty}\frac{(l_p^*)^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}\Phi(s,w)\right]ds\right] \\ &\leq \frac{\lambda_\phi}{l_p^*}(1+\lambda_\phi)\Phi(t,w) \\ &:= c_{f,\Phi}\Phi(t,w). \end{aligned}$$

Hence, the problem (1.1) is generalized Ulam-Hyers-Rassias stable.

4. Hilfer-Hadamard fractional random differential equations

Now, we are concerned with the existence and the Ulam-Hyers-Rassias stability for problem (1.2).

Set $C := C([1,T])$. Denote the weighted space of continuous functions defined by

$$C_{\gamma,\ln}([1,T]) = \{w(t) : (\ln t)^{1-\gamma}w(t)\in C\},$$

with the norm

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in[1,T]}|(\ln t)^{1-r}w(t)|.$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [13, 21] for a more detailed analysis.

Definition 4.1. [13, 21] (Hadamard fractional integral). The Hadamard fractional integral of order $q>0$ for a function $g\in L^1([1,T])$, is defined as

$$({}^HI_1^q g)(x) = \frac{1}{\Gamma(q)}\int_1^x\left(\ln\frac{x}{s}\right)^{q-1}\frac{g(s)}{s}ds,$$

provided the integral exists.

Example 4.2. Let $0<q<1$. Then

$${}^HI_1^q \ln t = \frac{1}{\Gamma(2+q)}(\ln t)^{1+q},\ for\ a.e.\ t\in[0,e].$$

Set

$$\delta = x\frac{d}{dx},\ q>0,\ n=[q]+1,$$

and

$$AC_\delta^n := \{u:[1,T]\to E : \delta^{n-1}[u(x)]\in AC(I)\}.$$

Analogous to the Riemann-Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way:

Definition 4.3. [13, 21] (Hadamard fractional derivative). The Hadamard fractional derivative of order $q>0$ applied to the function $w\in AC_\delta^n$ is defined as

$$({}^HD_1^q w)(x) = \delta^n({}^HI_1^{n-q}w)(x).$$

In particular, if $q\in(0,1]$, then

$$({}^HD_1^q w)(x) = \delta({}^HI_1^{1-q}w)(x).$$

Example 4.4. Let $0 < q < 1$. Then

$$^{H}D_1^q \ln t = \frac{1}{\Gamma(2-q)}(\ln t)^{1-q}, \ for\ a.e.\ t \in [0, e].$$

It has been proved (see e.g. Kilbas [[20], Theorem 4.8]) that in the space $L^1(I, E)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$(^{H}D_1^q)(^{H}I_1^q w)(x) = w(x).$$

From Theorem 2.3 of [21], we have

$$(^{H}I_1^q)(^{H}D_1^q w)(x) = w(x) - \frac{(^{H}I_1^{1-q}w)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

Analogous to the Hadamard fractional calculus, the Caputo-Hadamard fractional derivative is defined in the following way:

Definition 4.5. (Caputo-Hadamard fractional derivative). The Caputo-Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as

$$(^{Hc}D_1^q w)(x) = (^{H}I_1^{n-q}\delta^n w)(x).$$

In particular, if $q \in (0, 1]$, then

$$(^{Hc}D_1^q w)(x) = (^{H}I_1^{1-q}\delta w)(x).$$

From the Hadamard fractional integral, the Hilfer-Hadamard fractional derivative is defined in the following way:

Definition 4.6. (Hilfer-Hadamard fractional derivative).Let $\alpha \in (0, 1),\ \beta \in [0, 1],$ $\gamma = \alpha + \beta - \alpha\beta,\ w \in L^1(I)$, and $^{H}I_1^{(1-\alpha)(1-\beta)}w \in AC(I)$. The Hilfer-Hadamard fractional derivative of order α and type β applied to the function w is defined as

$$\begin{aligned}(^{H}D_1^{\alpha,\beta}w)(t) &= \left(^{H}I_1^{\beta(1-\alpha)}(^{H}D_1^{\gamma}w)\right)(t) \\ &= \left(^{H}I_1^{\beta(1-\alpha)}\delta(^{H}I_1^{1-\gamma}w)\right)(t);\ for\ a.e.\ t \in [1, T].\end{aligned} \tag{4.1}$$

This new fractional derivative (4.1) may be viewed as interpolating the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative. Indeed for $\beta = 0$ this derivative reduces to the Hadamard fractional derivative and when $\beta = 1$, we recover the Caputo-Hadamard fractional derivative.

$$^{H}D_1^{\alpha,0} = {}^{H}D_1^{\alpha},\ and\ {}^{H}D_1^{\alpha,1} = {}^{Hc}D_1^{\alpha}.$$

From Theorem 21 in [24], we concluded the following lemma

Lemma 4.7. *Let $g : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ be such that $g(\cdot, u(\cdot, w), w) \in C_{\gamma,\ln}([1, T])$ for any $u(., w) \in C_{\gamma,\ln}([1, T])$. Then Then problem (1.2) is equivalent to the following volterra integral equation*

$$u(t, w) = \frac{\phi_0(w)}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + (^{H}I_1^{\alpha}g(\cdot, u(\cdot, w), w))(t);\ w \in \Omega.$$

Definition 4.8. By a random solution of the problem (1.2) we mean a measurable function $u \in C_{\gamma,\ln}$ that satisfies the condition $({}^H I_1^{1-\gamma} u)(1^+, w) = \phi_0(w)$, and the equation

$$({}^H D_1^{\alpha,\beta} u)(t, w) = g(t, u(t, w), w) \text{ on } [1, T] \times \Omega.$$

For each $p \in \mathbb{N}\backslash\{0, 1\}$ we consider following set, $C_{p,\gamma,\ln} = C_\gamma([1, p])$, and we define in $C_{\gamma,\ln}$ the semi-norms by

$$\|u\|_p = \sup_{t\in[0,p]} |(\ln t)^{1-\gamma} u(t)|.$$

Then $C_{\gamma,\ln}$ is a Fréchet space with the family of semi-norms $\{\|u\|_p\}$.

Now we give (without proof) existence and Ulam stability results for problem (1.2).

The following hypotheses will be used in the sequel.

- (H_1') The function $f : [1, p] \times \mathbb{R} \times \Omega \mapsto f(t, u, w) \in \mathbb{R}$ is random Carathéodory on $[1, p] \times \mathbb{R} \times \Omega$, and affine with respect to u,
- (H_2') There exists a measurable and bounded function $\tilde{l} : \Omega \to L^\infty([1, p], [0, \infty))$, such that

$$|g(t, u, w) - g(t, v, w)| \le \tilde{l}(t, w)|u - v|; \ \textit{for a.e. } t \in [1, p], \ \textit{and each } u, v \in \mathbb{R}, \ w \in \Omega,$$

- (H_3') There exists $\lambda_\Phi > 0$ such that for each $t \in [1, p]$, and $w \in \Omega$, we have

$$\int_1^t \left[\sum_{n=1}^\infty \frac{(l_{p^*})^n}{\Gamma(n\alpha)} \left(\ln \frac{t}{s}\right)^{n\alpha-1} \Phi(s, w)\right] \frac{ds}{s} \le \lambda_\Phi \Phi(t, w),$$

where $l_{p*} = \sup_{w\in\Omega} \|\tilde{l}(w)\|_{L^\infty([1,p])}$.

Theorem 4.9. *Assume that the hypotheses (H_1') and (H_2') hold. If*

$$\frac{l_{p*}(\ln p)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} < 1, \tag{4.2}$$

then problem (1.2) has at least one random solution in the space $C_{\gamma,\ln}$. Furthermore, if the hypothesis (H_3') holds, then problem (1.2) is generalized Ulam-Hyers-Rassias stable.

5. An example

Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Given a measurable function $u : \Omega \to C_{\frac{3}{4}}([0, 1])$.

As an application of our results, we consider the following problem of Hilfer fractional differential equation of the form

$$\begin{cases} (D_0^{\frac{1}{2},\frac{1}{2}} u)(t, w) = f(t, u(t, w), w); \ t \in [0, \infty), \\ (I_0^{\frac{1}{4}} u)(t)|_{t=0} = 1, \end{cases} \quad w \in \Omega, \tag{5.1}$$

where

$$\begin{cases} f(t, u, w) = \dfrac{c_p t^{\frac{-1}{4}} u \sin t}{(1 + \sqrt{t})(1 + w^2)}; \ t \in (0, \infty) \ u \in \mathbb{R}, \\ f(0, u, w) = 0; \qquad u \in \mathbb{R}, \end{cases} \quad w \in \Omega,$$

and $0 < c_p < \frac{\sqrt{\pi}}{2}p^{-3/4};\ p \in \mathbb{N} - \{0\}$. The hypothesis (H_2) is satisfied with

$$\begin{cases} l_p(t,w) = \dfrac{c_p t^{\frac{-1}{4}}|\sin t|}{(1+\sqrt{t})(1+w^2)};\ t \in (0,p], \\ l_p(0,w) = 0, \end{cases} \quad w \in \Omega.$$

Also, the hypothesis (H_3) is satisfied with

$$\Phi(t,w) = \frac{e^3}{1+w^2},\ and\ \lambda_\Phi = \sum_{n=1}^{\infty} \frac{c_p^n}{\Gamma(1+n\alpha)}.$$

A simple computation shows that conditions of Theorem 3.2 are satisfied. Hence, problem (5.1) has at least one solution defined on $[0,1]$. Moreover, problem (5.1) is generalized Ulam-Hyers-Rassias stable.

Acknowledgement. The third author's work was supported by National Natural Science Foundation of P.R. China (11671339).

References

[1] S. Abbas, W. Albarakati, M. Benchohra, G.M. N'Guérékata, *Existence and Ulam stabilities for Hadamard fractional integral equations in Fréchet spaces*, J. Frac. Calc. Appl., **7**(2)(2016), 1-12.

[2] S. Abbas, W.A. Albarakati, M. Benchohra, S. Sivasundaram, *Dynamics and stability of Fredholm type fractional order Hadamard integral equations*, J. Nonlinear Stud., **22**(4)(2015), 673-686.

[3] S. Abbas, M. Benchohra, *Uniqueness and Ulam stabilities results for partial fractional differential equations with not instantaneous impulses*, Appl. Math. Comput., **257**(2015), 190-198.

[4] S. Abbas, M. Benchohra, *Existence and Ulam stability results for quadratic integral equations*, Libertas Math., **35**(2)(2015), 83-93.

[5] S. Abbas, M. Benchohra, G.M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[6] S. Abbas, M. Benchohra, G.M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[7] S. Abbas, M. Benchohra, S. Sivasundaram, *Ulam stability for partial fractional differential inclusions with multiple delay and impulses via Picard operators*, J. Nonlinear Stud., **20**(4)(2013), 623-641.

[8] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, *Existence results for functional differential equations of fractional order*, J. Math. Anal. Appl., **338**(2008), 1340-1350.

[9] A.T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York, 1972.

[10] M. Frigon, A. Granas, *Théorèmes d'existence pour des inclusions différentielles sans convexité*, C.R. Acad. Sci. Paris, Ser. I, **310**(1990), 819-822.

[11] K.M. Furati, M.D. Kassim, N. Tatar, *Existence and uniqueness for a problem involving Hilfer fractional derivative*, Comput. Math. Appl., **64**(2012), 1616-1626.

[12] H.R. Goudarzi, *Random fixed point theorems in Fréchet spaces with their applications*, J. Math. Ext., **8**(2)(2014), 71-81.

[13] J. Hadamard, *Essai sur l'étude des fonctions données par leur développment de Taylor*, J. Pure Appl. Math., **4**(8)(1892), 101-186.

[14] X. Han, X. Ma, G. Dai, *Solutions to fourth-order random differential equations with periodic boundary conditions*, Electron. J. Differ. Equ., **235**(2012), 1-9.

[15] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[16] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci., **27**(1941), 222-224.

[17] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[18] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

[19] R. Kamocki, C. Obczńnski, *On fractional Cauchy-type problems containing Hilfer's derivative*, Electron. J. Qual. Theory Differ. Equ., (2016), no. 50, 1-12.

[20] A.A. Kilbas, *Hadamard-type fractional calculus*, J. Korean Math. Soc., **38**(6)(2001), 1191-1204.

[21] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.

[22] G.S. Ladde, V. Lakshmikantham, *Random Differential Inequalities*, Academic Press, New York, 1980.

[23] T.P. Petru, A. Petruşel, J.-C. Yao, *Ulam-Hyers stability for operatorial equations and inclusions via nonself operators*, Taiwanese J. Math., **15**(2011), 2169-2193.

[24] M.D. Qassim, N. Tatar, *Well-posedness and stability for a differential problem with Hilfer-Hadamard fractional derivative*, Abstract Appl. Anal., Vol. 2013, Art. ID 605029, 12 pages, 2013.

[25] Th.M. Rassias, *On the stability of linear mappings in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[26] I.A. Rus, *Ulam stability of ordinary differential equations*, Stud. Univ. Babeş-Bolyai Math., **54**(2009), no. 4, 125-133.

[27] I.A. Rus, *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory, **10**(2009), 305-320.

[28] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Amsterdam, 1987.

[29] V.E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[30] Ž. Tomovski, R. Hilfer, H.M. Srivastava, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions*, Integral Transforms Spec. Funct., **21**(11)(2010), 797-814.

[31] C.P. Tsokos, W.J. Padgett, *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York, 1974.

[32] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1968.

[33] J.-R. Wang, Y. Zhang, *Nonlocal initial value problems for differential equations with Hilfer fractional derivative*, Appl. Math. Comput., **266**(2015), 850-859.

[34] H. Ye, J. Gao, Y. Ding, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl., **328**(2007), 1075-1081.

[35] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[36] Y. Zhou, *Fractional Evolution Equations and Inclusions: Analysis and Control*, Elsevier Science, 2016.

[37] Y. Zhou, B. Ahmad, A. Alsaedi, *Existence of nonoscillatory solutions for fractional neutral differential equations*, Applied Math. Letters, **72**(2017), 70-74.

[38] Y. Zhou, L. Peng, *On the time-fractional Navier-Stokes equations*, Comput. Math. Appl., **73**(6)(2017), 874-891.

[39] Y. Zhou, L. Peng, *Weak solutions of the time-fractional Navier-Stokes equations and optimal control*, Comput. Math. Appl., **73**(6)(2017), 1016-1027.

[40] Y. Zhou, V. Vijayakumar, R. Murugesu, *Controllability for fractional evolution inclusions without compactness*, Evolution Equations and Control Theory, **4**(2015), 507-524.

[41] Y. Zhou, L. Zhang, *Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems*, Comput. Math. Appl., **73**(6)(2017), 1325-1345.

Received: March 15, 2017; Accepted: May 18, 2017.