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# ON SOLVING THE VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS IN $q$-UNIFORMLY SMOOTH BANACH SPACES

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**Abstract.** In this research, we focus on two main problems, the first one is a fixed point problem of a nonexpansive semigroup and the other is a variational inequality problem for an inverse strongly accretive mapping. Passing through the modified Mann iterative method, we propose the new iterative scheme to find the common elements solving our mentioned problems. Furthermore, we aim to obtain some strong convergence theorems under certain appropriate conditions in the $q$-uniformly smooth Banach spaces. Our results improve and extend resulting outcomes in the literature.
**Key Words and Phrases**: Banach space, fixed point, inverse-strongly accretive mapping, nonexpansive semigroup, $q$-uniformly smooth, variational inequality.
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## 1. Introduction

According to our framework throughout this research, we first preview some definitions involving a Banach space $E$ as follows. Let $U = \{x \in E : \|x\| = 1\}$.

- $E$ is said to be *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$.
  It is known that a uniformly convex Banach space is reflexive and strictly convex.

- $E$ is said to be *smooth* if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$.
  It is also said to be *uniformly smooth* if the limit is attained uniformly for all $x, y \in U$. The *modulus of smoothness* of $E$ is defined by
  $$\rho(\tau) = \sup\Big\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\Big\},$$
  where $\rho : [0,\infty) \to [0,\infty)$ is a function.
  It is known that $E$ is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$.
- $E$ is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$ where $q$ is a fixed real number with $1 < q \leq 2$.

Let $E$ be a real Banach space and $E^*$ be the dual space of $E$ with norm $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ pairing between $E$ and $E^*$. For $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by
$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$
for all $x \in E$. In particular, if $q = 2$, the mapping $J_2$ is called the *normalized duality mapping* and written by $J_2 = J$ as usual. Further, we have the following properties of the generalized duality mapping $J_q$:
(i) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$;
(ii) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0,\infty)$;
(iii) $J_q(-x) = -J_q(x)$ for all $x \in E$.
Certainly, if $E$ is smooth, then $J_q$ is single-valued and can be written by $j_q$ (see also [7, 31]).

Let $C$ be a nonempty closed convex subset of a real Banach space $E$. Recall that a mapping $A : C \to C$ is said to be
(i) *Lipschitzian* with Lipschitz constant $L > 0$ if $\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C$;
(ii) *nonexpansive* if $\|Ax - Ay\| \leq \|x - y\|, \quad \forall x, y \in C$.

An operator $A : C \to E$ is said to be
(i) *accretive* if there exists $j_q(x-y) \in J_q(x-y)$ such that
$$\langle Ax - Ay, j_q(x-y)\rangle \geq 0, \quad \forall x, y \in C;$$
(ii) *β-strongly accretive* if for any $\beta > 0$ there exists $j_q(x-y) \in J_q(x-y)$ such that
$$\langle Ax - Ay, j_q(x-y)\rangle \geq \beta\|x-y\|^q, \quad \forall x, y \in C;$$
(iii) *β-inverse strongly accretive* if, for any $\beta > 0$ there exists $j_q(x-y) \in J_q(x-y)$,
$$\langle Ax - Ay, j_q(x-y)\rangle \geq \beta\|Ax - Ay\|^q, \quad \forall x, y \in C.$$

Let $D$ be a subset of $C$ and $Q : C \to D$. Then $Q$ is said to be *sunny* if
$$Q(Qx + t(x - Qx)) = Qx, \text{ whenever } Qx + t(x - Qx) \in C \text{ for } x \in C \text{ and } t \geq 0.$$
A subset $D$ of $C$ is said to be a *sunny nonexpansive retract* of $C$ if there exists a sunny nonexpansive retraction $Q$ of $C$ onto $D$ (see [32, 9, 18]). A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all $z$ are in the range of $Q$.

A family $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ of mappings of $C$ into itself is called a nonexpansive semigroup on $C$ if it satisfies the following conditions:

(i) $S(0)x = x$ for all $x \in C$;

(ii) $S(s+t) = S(s)S(t)$ for all $s, t \geq 0$;
(iii) $\|S(s)x - S(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(iv) for each $x \in C$, the mapping $S(\cdot)x$ from $[0, \infty)$ into $C$ is continuous.

Let $F(\mathcal{S})$ stands for the common fixed point set of the semigroup $\mathcal{S}$, i.e., $F(\mathcal{S}) = \{x \in C : S(s)x = x, \forall s > 0\}$. It is easy to see that $F(\mathcal{S})$ is closed and convex (see also [20, 25, 38, 11]).

In 1969, Takahashi [36] proved the first fixed point theorem for a noncommutative semigroup of nonexpansive mappings which generalizes De Marr's [8] fixed point theorem. For works related to semigroups of nonexpansive, asymptotically nonexpansive, and asymptotically nonexpansive type related to reversibility of a semigroup, we refer the reader to [13, 15, 26, 19, 24, 21, 22, 23, 37, 1, 12, 14]. In 2007, Lau et al. [22] introduced the following Mann's explicit iteration process;

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T(\mu_n)x_n, \quad \forall n \geq 1,$$

for a semigroup $\mathcal{S} = \{T(s) : s \in S\}$ of nonexpansive mappings on a compact convex subset $C$ of a smooth and strictly convex Banach space. In 2012, Wangkeeree and Preechasilp [39] introduced the iterative scheme:

$$\begin{cases} x_1 \in C, \\ z_n = \gamma_n x_n + (1 - \gamma_n)T(t_n)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)T(t_n)z_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n, n \geq 0. \end{cases}$$

They proved the strong convergence theorems by using a nonexpansive semigroup in Banach spaces.

In 2006, Aoyama et al. [3] proved a weak convergence theorem in Banach spaces by using the iterative algorithm as the following

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n A x_n), \end{cases}$$

for all $n \geq 1$. They solved the *generalized variational inequality problem* for finding a point $x \in C$ such that

$$\langle Ax, J(y - x)\rangle \geq 0 \tag{1.1}$$

for all $y \in C$. The solution set of (1.1) is denoted by $VI(C, A)$. Variational inequality has become a rich of inspiration in pure and applied mathematics. Recently, classical variational inequality problems have been extended and generalized to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences and have witnessed an explosive growth in theoretical advances, algorithmic development, etc; see e.g. [5, 6, 29].

In 2013, Song and Ceng [33] proved a strong convergence theorem in a q-uniformly smooth Banach space as the following:

$$\begin{cases} x_1 \in C, \\ z_n = Q_C(x_n - \sigma B x_n), \\ k_n = Q_C(z_n - \lambda A z_n), \\ y_n = \beta_n k_n + (1 - \alpha_n)x_n, \\ x_{n+1} = Q_C[\alpha_n \gamma f x_n + \gamma_n x_n + ((1 - \gamma_n)I - \alpha_n \mu V)T_n y_n], n \geq 0. \end{cases} \tag{1.2}$$

They introduced a general iterative algorithm for finding a common element of the set of common fixed points of an infinite family of nonexpansive mappings and the solution set of systems of variational inequalities.

Motivated and inspired by Wangkeeree and Preechasilp [39] and Song and Ceng [33]. In this paper, we introduce a new iterative scheme for finding common solutions of a variational inequality for an inverse-strongly accretive mapping and the solutions of a fixed point problem for a nonexpansive semigroup by using the modified Mann iterative method. We shall prove the strong convergence theorem in a $q$-uniformly smooth Banach spaces under some parameters controlling conditions. Our results extend and improve the recent results of Aoyama et al. [3], Wangkeeree and Preechasilp [39], Song and Ceng [33] and other authors.

## 2. Preliminaries

A Banach space $E$ is said to satisfy *Opial's condition* if for any sequence $\{x_n\}$ in $E$, $x_n \rightharpoonup x (n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \forall y \in E \quad \text{with} \quad x \neq y.$$

By [10, Theorem 1], it is well known that if $E$ admits a weakly sequentially continuous duality mapping, then $E$ satisfies Opial's condition, and $E$ is smooth.

We need the following lemmas for proving our main results.

**Proposition 2.1.** ([32]) *Let $E$ be a smooth banach space and let $C$ be a nonempty subset of $E$. Let $Q : E \to C$ be a retraction and let $J$ be the normalized duality mapping on $E$. Then the following are equivalent:*
*(i) $Q$ is sunny and nonexpansive;*
*(ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$;*
*(iii) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in E, y \in C$.*
*If $J_q$ is the generalized duality mapping on $E$ then $\langle x - Qx, J_q(y - Qx)\rangle \leq 0$, $\forall x \in E$, $y \in C$ is equivalent to this Proposition (see [33]).*

**Proposition 2.2.** ([9, 18, 16]) *Let $C$ be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $E$ and let $T$ be a nonexpansive mapping of $C$ into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of $C$.*

**Lemma 2.3.** ([3]) *Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$ and let $A$ be an accretive operator of $C$ into $E$. Then, for all $\lambda > 0$,*

$$VI(C, A) = F(Q(I - \lambda A)),$$

*where $VI(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0, \forall x \in C\}$.*

**Lemma 2.4.** ([4])*Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $E$ and $T : C \to C$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence of $C$ such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ then $x$ is a fixed point of $T$.*

**Lemma 2.5.** ([40]) *Let $r > 0$ and let $E$ be a uniformly convex Banach space. Then, there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

*for all $x, y \in B_r := \{z \in E : \|z\| \leq r\}$ and $0 \leq \lambda \leq 1$.*

**Lemma 2.6.** ([17]) *Let $E$ be a real smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \leq \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \forall x, y \in B_r,$$

*where $B_r = \{z \in E : \|z\| \leq r\}$.*

**Lemma 2.7.** ([40]) *Let $E$ be a real $q$-uniformly smooth Banach space, then there exists a constant $c_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q, \forall x, y \in E.$$

*In particular, if $E$ is real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2K\|y\|^2, \forall x, y \in E.$$

**Lemma 2.8.** ([27]) *Let $E$ be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$$

*for all $j(x+y) \in J(x+y)$ with $x \neq y$.*

**Lemma 2.9.** ([35]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose*

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$$

*for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then,*

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

**Lemma 2.10.** ([41]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \ n \geq 0$$

*where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(1) $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$

(2) $\displaystyle\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\displaystyle\sum_{n=1}^{\infty} |\delta_n| < \infty$.

*Then $\lim_{n\to\infty} a_n = 0$.*

**Lemma 2.11.** ([30, 34]) *Let $C$ be a nonempty, closed and convex subset of a real $q$-uniformly smooth Banach space $E$, $L_2 : C \to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constants $\kappa, \eta > 0$ and let*

$$0 < \mu < \left(\frac{q\eta}{c_q\kappa^q}\right)^{\frac{1}{q-1}}, \quad \tau = \mu\left(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q}\right),$$

*then for $t \in (0, \min\{1, \frac{1}{\tau}\})$, the mapping $S : C \to E$ defined by $S := (I - t\mu L_2)$ is a contraction with a constant $1 - t\tau$.*

**Lemma 2.12.** ([33]) *Let $C$ be a nonempty, closed and convex subset of a real reflexive and $q$-uniformly smooth Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J_q$ from $E$ into $E^*$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $V : C \to E$ a $k$-Lipschitzian and $\eta$-strongly accretive operator with constants $k$, $\eta > 0$. Suppose $f : C \to E$ is a $L$-Lipschitzian mapping with constant $L > 0$ and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Let*

$$0 < \mu < \left(\frac{q\eta}{c_q\kappa^q}\right)^{\frac{1}{q-1}} \text{ and } 0 \le \gamma L < \tau \text{ where } \tau = \mu\left(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q}\right).$$

*Then $\{x_t\}$ defined by $x_t = Q_C[t\gamma f x_t + (I - t\mu V)Tx_t]$ converges strongly to some point $x^* \in F(T)$ as $t \to 0$, which is the unique solution of the variational inequality:*

$$\langle \gamma f x^* - \mu V x^*, J_q(p - x^*)\rangle \le 0, \forall p \in F(T).$$

**Lemma 2.13.** ([33]) *Let $C$ be a closed convex subset of a smooth Banach space $E$. Let $\widetilde{C}$ be a nonempty subset of $C$. Let $Q : C \to \widetilde{C}$ be a retraction and let $J, J_q$ be the normalized duality mapping and generalized duality mapping on $E$, respectively. Then the following are equivalent:*
*(i) $Q$ is sunny and nonexpansive;*
*(ii) $\|Qx - Qy\|^2 \le \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$;*
*(iii) $\langle x - Qx, J(y - Qx)\rangle \le 0, \forall x \in C, y \in \widetilde{C}$;*
*(iv) $\langle x - Qx, J_q(y - Qx)\rangle \le 0, \forall x \in C, y \in \widetilde{C}$.*

**Lemma 2.14.** ([28]) *Let $q > 1$. Then the following inequality holds:*

$$ab \le \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$$

*for arbitrary positive real numbers $a, b$.*

## 3. Main results

**Theorem 3.1.** *Let $C$ be a sunny nonexpansive retract and nonempty closed convex subset of a $q$-uniformly smooth and uniformly convex Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $A : C \to E$ be an $\beta$-inverse-strongly accretive operator, $\mathcal{S} = \{S(s) : s \ge 0\}$ be a nonexpansive semigroup from $C$ into itself, $L_1 : C \to E$ be a $L$-Lipschitzian mapping with constant $L \ge 0$ and $L_2 : C \to E$*

*be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$, $\{\mu_n\} \subset (0,\infty)$ such that*

$$\{\lambda_n\} \subset [a,b] \subset (0,1),\ 0 < \mu < \left(\frac{q\eta}{c_q \kappa^q}\right)^{\frac{1}{q-1}}$$

*where $c_q$ is a positive real number,*

$$0 < a \le \lambda_n \le b < \left(\frac{q\beta}{c_q}\right)^{\frac{1}{q-1}},\ 0 \le \gamma L < \tau \text{ where } \tau = \mu\left(\eta - \frac{c_q \mu^{q-1}\kappa^q}{q}\right)$$

*and $F := F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu L_2) S(\mu_n) z_n\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S(\mu_n) y_n, \end{cases} \tag{3.1}$$

*which satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$*;*
(C2) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$, $\liminf_{n\to\infty} \lambda_n > 0$*;*
(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$*;*
(C4) $\lim_{n\to\infty} \mu_n = 0$*;*
(C5) $\lim_{n\to\infty} \sup_{x \in \widetilde{C}} \|S(\mu_{n+1})x - S(\mu_n)x\| = 0$, *$\widetilde{C}$ bounded subset of $C$;*
(C6) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0, 0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1.$

*Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:*

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \le 0, \forall z \in F. \tag{3.2}$$

*Proof.* First of all, we prove that $\{x_n\}$ is bounded. Let $p \in F$ and

$$0 < a \le \lambda_n \le b < \left(\frac{q\beta}{c_q}\right)^{\frac{1}{q-1}},$$

we have

$$\begin{aligned} \|z_n - p\|^q &= \|Q_C(x_n - \lambda_n A x_n) - Q_C(p - \lambda_n A p)\|^q \\ &\le \|(I - \lambda_n A)x_n - (I - \lambda_n A)p\|^q \\ &= \|(x_n - p) - \lambda_n(Ax_n - Ap)\|^q \\ &\le \|x_n - p\|^q - q\lambda_n \langle Ax_n - Ap, j_q(x_n - p)\rangle + c_q \lambda_n^q \|Ax_n - Ap\|^q \\ &\le \|x_n - p\|^q - q\beta\lambda_n \|Ax_n - Ap\|^2 + c_q \lambda_n^q \|Ax_n - Ap\|^q \\ &= \|x_n - p\|^q - \lambda_n(q\beta - c_q \lambda_n^{q-1})\|Ax_n - Ap\|^q \\ &\le \|x_n - p\|^q. \end{aligned} \tag{3.3}$$

Therefore $\|z_n - p\| \leq \|x_n - p\|$ and $I - \lambda_n A$ is a nonexpansive where $I$ is an identity mapping. By condition $(C1)$, we may assume, without loss of generality, that

$$\alpha_n < \min\left\{\alpha, \frac{\alpha}{\tau}\right\} \text{ where } 0 < \alpha < \liminf_{n\to\infty}(1-\gamma_n).$$

From Lemma 2.11, we conclude that $\|(1-\gamma_n)I - \alpha_n\mu L_2\| \leq (1-\gamma_n) - \alpha_n\tau$.
Since $0 \leq \gamma L < \tau$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n(x_n - p) + (1-\beta_n)(S(\mu_n)y_n - p)\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\
&= \beta_n\|x_n - p\| + (1-\beta_n)\big\|Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n \\
&\quad +((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - p\big\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)\|[(1-\gamma_n)I - \alpha_n\mu L_2][S(\mu_n)z_n - p] \\
&\quad +\alpha_n(\gamma L_1 x_n - \mu L_2 p) + \gamma_n(x_n - p)\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)(1-\gamma_n-\alpha_n\tau)\|S(\mu_n)z_n - p\| \\
&\quad +(1-\beta_n)\alpha_n\|\gamma L_1 x_n - \mu L_2 p\| + (1-\beta_n)\gamma_n\|x_n - p\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)(1-\gamma_n-\alpha_n\tau)\|x_n - p\| \\
&\quad +(1-\beta_n)\alpha_n\gamma\|L_1 x_n - L_1 p\| + (1-\beta_n)\alpha_n\|\gamma L_1 p - \mu L_2 p\| \\
&\quad +(1-\beta_n)\gamma_n\|x_n - p\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| - (1-\beta_n)\gamma_n\|x_n - p\| \\
&\quad -(1-\beta_n)\alpha_n\tau\|x_n - p\| + (1-\beta_n)\alpha_n\gamma L\|x_n - p\| \\
&\quad +(1-\beta_n)\alpha_n\|\gamma L_1 p - \mu L_2 p\| + (1-\beta_n)\gamma_n\|x_n - p\| \\
&= \|x_n - p\| - (1-\beta_n)\alpha_n\tau\|x_n - p\| \\
&\quad +(1-\beta_n)\alpha_n\gamma L\|x_n - p\| + (1-\beta_n)\alpha_n\|\gamma L_1 p - \mu L_2 p\| \\
&= \|x_n - p\| - (1-\beta_n)\alpha_n(\tau - \gamma L)\|x_n - p\| \\
&\quad +(1-\beta_n)\alpha_n(\tau-\gamma L)\frac{\|\gamma L_1 p - \mu L_2 p\|}{\tau - \gamma L}.
\end{aligned}$$

By induction, we conclude that

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma L_1 p - \mu L_2 p\|}{\tau - \gamma L}\right\}, \forall n \geq 1.$$

This implies that $\{x_n\}$ is bounded, so are $\{Ax_n\}$, $\{y_n\}$, $\{S(\mu_n)y_n\}$, $\{z_n\}$ and $\{S(\mu_n)z_n\}$.

Next, we will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and we observe that

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|Q_C(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - Q_C(x_n - \lambda_n Ax_n)\| \\
&\leq \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_n Ax_n)\| \\
&= \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n) + (\lambda_n - \lambda_{n+1})Ax_n\| \\
&\leq \|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_{n+1}A)x_n\| + |\lambda_{n+1} - \lambda_n\|Ax_n\| \\
&\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\|,
\end{aligned}$$

$$\begin{aligned}\|S(\mu_{n+1})z_{n+1}-S(\mu_n)z_n\| &\le \|S(\mu_{n+1})z_{n+1}-S(\mu_{n+1})z_n\|\\ &+\|S(\mu_{n+1})z_n-S(\mu_n)z_n\|\\ &\le \|z_{n+1}-z_n\|+\|S(\mu_{n+1})z_n-S(\mu_n)z_n\|\\ &\le \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|\\ &+\sup_{z\in\{z_n\}}\|S(\mu_{n+1})z-S(\mu_n)z\|,\end{aligned}$$

and

$$\begin{aligned}\|y_{n+1}-y_n\| &= \big\|Q_C\big[\alpha_{n+1}\gamma L_1x_{n+1}+\gamma_{n+1}x_{n+1}\\ &\quad+((1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2)S(\mu_{n+1})z_{n+1}\big]\\ &\quad-Q_C\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]\big\|\\ &\le \big\|\big[\alpha_{n+1}\gamma L_1x_{n+1}+\gamma_{n+1}x_{n+1}\\ &\quad+((1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2)S(\mu_{n+1})z_{n+1}\big]\\ &\quad-\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]\big\|\\ &= \big\|\big[\alpha_{n+1}\gamma L_1x_{n+1}+\gamma_{n+1}x_{n+1}\\ &\quad+((1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2)S(\mu_{n+1})z_{n+1}\big]\\ &\quad-\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]\\ &\quad+\alpha_{n+1}\gamma L_1x_n-\alpha_{n+1}\gamma L_1x_n+\gamma_{n+1}x_n-\gamma_{n+1}x_n\\ &\quad+((1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2)S(\mu_n)z_n\\ &\quad-((1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2)S(\mu_n)z_n\big\|\\ &\le \alpha_{n+1}\gamma\|L_1x_{n+1}-L_1x_n\|+\gamma_{n+1}\|x_{n+1}-x_n\|\\ &\quad+\big\|\big[(1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2\big]\big[S(\mu_{n+1})z_{n+1}-S(\mu_n)z_n\big]\big\|\\ &\quad+|\alpha_{n+1}-\alpha_n|\gamma\|L_1x_n\|+|\alpha_{n+1}-\alpha_n|\mu\|L_2S(\mu_n)z_n\|\\ &\quad+|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n-x_n\|\\ &\le \alpha_{n+1}\gamma L\|x_{n+1}-x_n\|+\gamma_{n+1}\|x_{n+1}-x_n\|\\ &\quad+[(1-\gamma_{n+1})I-\alpha_{n+1}\tau]\|S(\mu_{n+1})z_{n+1}-S(\mu_n)z_n\|\\ &\quad+|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\|+\mu\|L_2S(\mu_n)z_n\|\big]\\ &\quad+|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n-x_n\|\\ &\le \alpha_{n+1}\gamma L\|x_{n+1}-x_n\|+\gamma_{n+1}\|x_{n+1}-x_n\|\\ &\quad+[(1-\gamma_{n+1})I-\alpha_{n+1}\tau]\big[\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|\\ &\quad+\sup_{z\in\{z_n\}}\|S(\mu_{n+1})z-S(\mu_n)z\|\big]\\ &\quad+|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\|+\mu\|L_2S(\mu_n)z_n\|\big]\\ &\quad+|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n-x_n\|\\ &= [1-\alpha_{n+1}(\tau-\gamma L)]\|x_{n+1}-x_n\|\\ &\quad+[(1-\gamma_{n+1})I-\alpha_{n+1}\tau]\big[|\lambda_{n+1}-\lambda_n|\|Ax_n\|\end{aligned}$$

$$
\begin{aligned}
&\quad + \sup_{z\in\{z_n\}} \|S(\mu_{n+1})z - S(\mu_n)z\|\big] \\
&\quad +|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\| + \mu\|L_2S(\mu_n)z_n\|\big] \\
&\quad +|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n - x_n\| \\
&\leq \|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|\|Ax_n\| \\
&\quad + \sup_{z\in\{z_n\}} \|S(\mu_{n+1})z - S(\mu_n)z\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\| + \mu\|L_2S(\mu_n)z_n\|\big] \\
&\quad +|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n - x_n\| \\
&\leq \|x_{n+1}-x_n\| + \big[|\alpha_{n+1}-\alpha_n| + |\gamma_{n+1}-\gamma_n| + |\lambda_{n+1}-\lambda_n|\big]M \\
&\quad + \sup_{z\in\{z_n\}} \|S(\mu_{n+1})z - S(\mu_n)z\|,
\end{aligned}
$$

where

$$M = \sup_{n\geq 0}\big\{\|Ax_n\|, \gamma\|L_1x_n\| + \mu\|L_2S(\mu_n)z_n\|, \|S(\mu_n)z_n - x_n\|\big\} < \infty.$$

It follows that

$$
\begin{aligned}
\|S(\mu_{n+1})y_{n+1} - S(\mu_n)y_n\| &\leq \|S(\mu_{n+1})y_{n+1} - S(\mu_{n+1})y_n\| + \|S(\mu_{n+1})y_n - S(\mu_n)y_n\| \\
&\leq \|y_{n+1}-y_n\| + \|S(\mu_{n+1})y_n - S(\mu_n)y_n\| \\
&\leq \|x_{n+1}-x_n\| + \big[|\alpha_{n+1}-\alpha_n| + |\gamma_{n+1}-\gamma_n| + |\lambda_{n+1}-\lambda_n|\big]M \\
&\quad + \sup_{z\in\{z_n\}} \|S(\mu_{n+1})z - S(\mu_n)z\| \\
&\quad + \sup_{y\in\{y_n\}} \|S(\mu_{n+1})y - S(\mu_n)y\|. \qquad (3.4)
\end{aligned}
$$

Form the condition (C1), (C2), (C5)-(C6) and 3.4, we have

$$\limsup_{n\to\infty}\big(\|S(\mu_{n+1})y_{n+1} - S(\mu_n)y_n\| - \|x_{n+1}-x_n\|\big) \leq 0.$$

Applying Lemma 2.9, we obtain

$$\lim_{n\to\infty}\|S(\mu_n)y_n - x_n\| = 0.$$

Therefore, we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \qquad (3.5)$$

Next, we will show that

$$\lim_{n\to\infty}\|x_n - S(\mu_n)x_n\| = 0,$$

by the convexity of $\|\cdot\|^q$ for all $q>1$, Lemma 2.7 and (3.3), we have

$$\begin{aligned}
\|y_n-p\|^q &= \|Q_C\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]-p\|^q\\
&\le \|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)+\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)\|^q\\
&\le \|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\|^q\\
&\quad +q\big\langle \alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big),J_q\big(\gamma_n(x_n-p)\\
&\quad +(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\big)\big\rangle\\
&\quad +c_q\|\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)\|^q\\
&\le \gamma_n\|x_n-p\|^q+(1-\gamma_n)\|S(\mu_n)z_n-p\|^q\\
&\quad +q\alpha_n\|\gamma L_1x_n-\mu L_2S(\mu_n)z_n\|\\
&\quad \times\|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\|^{q-1}\\
&\quad +c_q\alpha_n^q\|\gamma L_1x_n-\mu L_2S(\mu_n)z_n\|^q\\
&\le \gamma_n\|x_n-p\|^q+(1-\gamma_n)\|z_n-p\|^q+\alpha_nM_1\\
&\le \gamma_n\|x_n-p\|^q+(1-\gamma_n)\Big[\|x_n-p\|^q-\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax_n-Ap\|^q\Big]\\
&\quad +\alpha_nM_1\\
&= \|x_n-p\|^q-(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax_n-Ap\|^q+\alpha_nM_1,
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= \sup_{n\ge 0}\Big\{q\|\gamma L_1x_n-\mu L_2S(\mu_n)z_n\|\|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\|^{q-1}\\
&\quad +c_q\alpha_n^{q-1}\|\gamma L_1x_n-\mu L_2S(\mu_n)z_n\|^q\Big\}<\infty.
\end{aligned}$$

By the convexity of $\|\cdot\|^q$ for all $q>1$, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^q &\le \beta_n\|x_n-p\|^q+(1-\beta_n)\|S(\mu_n)y_n-p\|^q\\
&\le \beta_n\|x_n-p\|^q+(1-\beta_n)\|y_n-p\|^q\\
&\le \beta_n\|x_n-p\|^q+(1-\beta_n)\big[\|x_n-p\|^q\\
&\quad -(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax-Ay\|^q+\alpha_nM_1\\
&= \|x_n-p\|^q-(1-\beta_n)(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax-Ay\|^q\\
&\quad +(1-\beta_n)\alpha_nM_1.
\end{aligned}$$

By the fact that $a^r-b^r\le ra^{r-1}(a-b),\forall r\ge 1$, we get

$$\begin{aligned}
&(1-\beta_n)(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax-Ay\|^q\\
&\le \|x_n-p\|^q-\|x_{n+1}-p\|^q+(1-\beta_n)\alpha_nM_1\\
&\le q\|x_n-p\|^{q-1}\big(\|x_n-p\|-\|x_{n+1}-p\|\big)+(1-\beta_n)\alpha_nM_1\\
&\le q\|x_n-p\|^{q-1}\|x_n-x_{n+1}\|+(1-\beta_n)\alpha_nM_1.
\end{aligned}$$

From $0<a\le\lambda_n\le b<(\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, the conditions (C1)-(C3), (C6) and (3.5), we conclude that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{3.6}$$

From Proposition 2.1 (ii) and Lemma 2.6, we also have

$$\begin{aligned}
\|z_n - p\|^2 &= \|Q_C(x_n - \lambda_n Ax_n) - Q_C(p - \lambda_n Ap)\|^2 \\
&\leq \big\langle (x_n - \lambda_n Ax_n) - (p - \lambda_n Ap), J(z_n - p)\big\rangle \\
&= \big\langle (x_n - p) - \lambda_n(Ax_n - Ap), J(z_n - p)\big\rangle \\
&= \big\langle x_n - p, J(z_n - p)\big\rangle - \lambda_n\big\langle Ax_n - Ap, J(z_n - p)\big\rangle \\
&\leq \frac{1}{2}\big[\|x_n - p\|^2 + \|z_n - p\|^2 - g\|x_n - z_n\|\big] + \lambda_n\|Ax_n - Ap\|\|z_n - p\|.
\end{aligned}$$

So, we get

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - g\|x_n - z_n\| + 2\lambda_n\|Ax_n - Ap\|\|z_n - p\|.$$

By Lemma 2.8, it follows that

$$\begin{aligned}
\|y_n - p\|^2 &= \|Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - p\|^2 \\
&\leq \|\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big) + \alpha_n\big(\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\big)\|^2 \\
&\leq \|\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big)\|^2 \\
&\quad +2\alpha_n\big\langle \gamma L_1 x_n - \mu L_2 S(\mu_n)z_n, J\big(\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big) \\
&\quad +\alpha_n\big(\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\big)\big)\big\rangle \\
&\leq \gamma_n\|x_n - p\|^2 + (1-\gamma_n)\|z_n - p\|^2 + \alpha_n M_2 \\
&\leq \gamma_n\|x_n - p\|^2 + (1-\gamma_n)\Big[\|x_n - p\|^2 - g\|x_n - z_n\| \\
&\quad +2\lambda_n\|Ax_n - Ap\|\|z_n - p\|\Big] + \alpha_n M_2 \\
&= \|x_n - p\|^2 - (1-\gamma_n)g\|x_n - z_n\| + 2(1-\gamma_n)\lambda_n\|Ax_n - Ap\|\|z_n - p\| \\
&\quad +\alpha_n M_2,
\end{aligned}$$

where

$$\begin{aligned}
M_2 &= \sup_{n\geq 0}\Big\{2\big\langle \gamma L_1 x_n - \mu L_2 S(\mu_n)z_n, J\big(\gamma_n(x_n - p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big) \\
&\quad +\alpha_n\big(\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\big)\big)\big\rangle\Big\} < \infty.
\end{aligned}$$

We obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|S(\mu_n)y_n - p\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\big[\|x_n - p\|^2 - (1-\gamma_n)g\|x_n - z_n\| \\
&\quad +2(1-\gamma_n)\lambda_n\|Ax_n - Ap\|\|z_n - p\| + \alpha_n M_2\big] \\
&= \|x_n - p\|^2 - (1-\beta_n)(1-\gamma_n)g\|x_n - z_n\| \\
&\quad +2(1-\beta_n)(1-\gamma_n)\lambda_n\|Ax_n - Ap\|\|z_n - p\| + (1-\beta_n)\alpha_n M_2.
\end{aligned}$$

Then we get

$$\begin{aligned}(1-\gamma_n)g\|x_n-z_n\| &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &+2(1-\beta_n)(1-\gamma_n)\lambda_n\|Ax_n-Ap\|\|z_n-p\|+(1-\beta_n)\alpha_n M_2\\ &\le \|x_n-x_{n+1}\|\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\\ &+2(1-\beta_n)(1-\gamma_n)\lambda_n\|Ax_n-Ap\|\|z_n-p\|+(1-\beta_n)\alpha_n M_2.\end{aligned}$$

By the conditions (C1)-(C3), (C6), (3.5) and (3.6), we have

$$\lim_{n\to\infty} g(\|x_n-z_n\|)=0.$$

It follows from the property of $g$ that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.7}$$

Similar to the proof of (3.7), we start by using Lemma 2.5 and Lemma 2.8

$$\begin{aligned}\|y_n-p\|^2 &= \|Q_C\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]-p\|^2\\ &\le \|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)+\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)\|^2\\ &\le \|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\|^2\\ &\quad +2\alpha_n\big\langle \gamma L_1x_n-\mu L_2S(\mu_n)z_n, J\big(\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\\ &\quad +\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)\big)\big\rangle\\ &\le \gamma_n\|x_n-p\|^2+(1-\gamma_n)\|S(\mu_n)z_n-p\|^2\\ &\quad -\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|)+\alpha_nM_2\\ &\le \gamma_n\|x_n-p\|^2+(1-\gamma_n)\|z_n-p\|^2\\ &\quad -\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|)+\alpha_nM_2\\ &\le \gamma_n\|x_n-p\|^2+(1-\gamma_n)\|x_n-p\|^2\\ &\quad -\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|)+\alpha_nM_2\\ &= \|x_n-p\|^2-\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|)+\alpha_nM_2,\end{aligned}$$

where

$$\begin{aligned}M_2 &= \sup_{n\ge 0}\Big\{2\big\langle\gamma L_1x_n-\mu L_2S(\mu_n)z_n, J\big(\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\\ &\quad +\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)\big)\big\rangle\Big\}<\infty.\end{aligned}$$

We obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|S(\mu_n)y_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\big[\|x_n-p\|^2\\ &-\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|)+\alpha_nM_2\big]\\ &= \|x_n-p\|^2-(1-\beta_n)\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|)+(1-\beta_n)\alpha_nM_2.\end{aligned}$$

Then we get

$$\begin{aligned}(1-\beta_n)\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|) &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(1-\beta_n)\alpha_nM_2\\ &\le \|x_n-x_{n+1}\|\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\\ &\quad+(1-\beta_n)\alpha_nM_2.\end{aligned}$$

By the conditions (C1), (C3), (C6) and (3.5), we have

$$\lim_{n\to\infty} g(\|x_n-S(\mu_n)z_n\|)=0.$$

It follows from the property of $g$ that

$$\lim_{n\to\infty}\|x_n-S(\mu_n)z_n\|=0. \tag{3.8}$$

Since $S(\mu_n)$ is a nonexpansive and from the proof of Lemma 2.12, we get

$$Q_CS(\mu_n)z_n=S(\mu_n)z_n$$

and observe that

$$\begin{aligned}\|y_n-S(\mu_n)z_n\| &= \big\|Q_C\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]-S(\mu_n)z_n\big\|\\ &\le \big\|\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]-S(\mu_n)z_n\big\|\\ &= \big\|\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)+\gamma_n(x_n-S(\mu_n)z_n)\big\|\\ &\le \alpha_n\|\gamma L_1x_n-\mu L_2S(\mu_n)z_n\|+\gamma_n\|x_n-S(\mu_n)z_n\|.\end{aligned}$$

It follows from the conditions (C1), (C6) and (3.8), we get

$$\lim_{n\to\infty}\|y_n-S(\mu_n)z_n\|=0. \tag{3.9}$$

Since

$$\begin{aligned}\|x_n-S(\mu_n)x_n\| &\le \|x_n-S(\mu_n)z_n\|+\|S(\mu_n)z_n-S(\mu_n)x_n\|\\ &\le \|x_n-S(\mu_n)z_n\|+\|z_n-x_n\|,\end{aligned}$$

we have

$$\lim_{n\to\infty}\|x_n-S(\mu_n)x_n\|=0.$$

Now, we show that $z\in F:=F(\mathcal{S})\cap VI(C,A)$. We can choose a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is bounded and there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to $z$. Without loss of generality, we can assume that $x_{n_k}\rightharpoonup z$.
(I) First, we show that $z\in F(\mathcal{S})$. Let $\mu_{n_k}\ge 0$ such that $\mu_{n_k}\to 0$ and

$$\frac{\|S(\mu_{n_k})x_{n_k}-x_{n_k}\|}{\mu_{n_k}}\to 0,\ k\to\infty.$$

Fix $s > 0$, we can notice that

$$\begin{aligned}\|x_{n_k} - S(s)z\| &\le \sum_{i=0}^{[s/\mu_{n_k}]-1} \big\|S\big((i+1)\mu_{n_k}\big)x_{n_k} - S(i\mu_{n_k})x_{n_k}\big\| \\ &\quad + \big\|S\big([s/\mu_{n_k}]\mu_k\big)x_{n_k} - S\big([s/\mu_{n_k}]\mu_{n_k}\big)z\big\| \\ &\quad + \big\|S\big([s/\mu_{n_k}]\mu_{n_k}\big)z - S(s)z\big\| \\ &\le [s/\mu_{n_k}]\|S(\mu_{n_k})x_{n_k} - x_{n_k}\| + \|x_{n_k} - z\| + \big\|S\big(s - [s/\mu_{n_k}]\mu_{n_k}\big)z - z\big\| \\ &\le s\frac{\|S(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} + \|x_{n_k} - z\| + \big\|S\big(s - [s/\mu_{n_k}]\mu_{n_k}\big)z - z\big\| \\ &\le s\frac{\|S(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} + \|x_{n_k} - z\| + \max\{\|S(\mu)z - z\| : 0 \le \mu \le \mu_{n_k}\}.\end{aligned}$$

For all $k \in \mathbb{N}$, we have

$$\limsup_{k\to\infty} \|x_{n_k} - S(s)z\| \le \limsup_{k\to\infty} \|x_{n_k} - z\|.$$

Since a Banach space $E$ with a weakly sequentially continuous duality mapping satisfies the Opial's condition, this implies $S(s)z = z$.
(II) Next, we show that $z \in VI(C, A)$. From the assumption, we see that the control sequence $\{\lambda_{n_k}\}$ is bounded. So, there exists a subsequence $\{\lambda_{n_{k_j}}\}$ converges to $\lambda_0$. We may assume, without loss of generality, that $\lambda_{n_k} \rightharpoonup \lambda_0$. Observe that

$$\begin{aligned}\|Q_C(x_{n_k} - \lambda_0 Ax_{n_k}) - x_{n_k}\| &\le \|Q_C(x_{n_k} - \lambda_0 Ax_{n_k}) - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \\ &\le \|(x_{n_k} - \lambda_0 Ax_{n_k}) - (x_{n_k} - \lambda_{n_k} Ax_{n_k})\| \\ &\quad + \|x_{n_k} - S(\mu_{n_k})z_{n_k}\| + \|S(\mu_{n_k})z_{n_k} - y_{n_k}\| \\ &\le M\|\lambda_{n_k} - \lambda_0\| + \|x_{n_k} - S(\mu_{n_k})z_{n_k}\| \\ &\quad + \|S(\mu_{n_k})z_{n_k} - y_{n_k}\|,\end{aligned}$$

where $M$ is as appropriate constant such that $M \ge \sup_{n\ge1}\{\|Ax_n\|\}$. It follows from (3.8), (3.9) and $\lambda_{n_k} \rightharpoonup \lambda_0$ that

$$\lim_{k\to\infty} \|Q_C(x_{n_k} - \lambda_0 Ax_{n_k}) - x_{n_k}\| = 0.$$

We know that $Q_C(I - \lambda_0 A)$ is nonexpansive and it follows from Lemma 2.4 that

$$z \in F(Q_C(I - \lambda_0 A)).$$

By using Lemma 2.3, we obtain that

$$z \in F(Q_C(I - \lambda_0 A)) = VI(C, A).$$

Therefore, from (I) and (II), we conclude that $z \in F := F(\mathcal{S}) \cap VI(C, A)$.

Next, we show that $\limsup_{n\to\infty}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle \le 0$, where $x^*$ is the solution of the variational inequality (1.1). Since the Banach space $E$ has a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$ and $y_{n_k} \rightharpoonup z$, we obtain that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle \\
&= \lim_{k\to\infty}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_{n_k} - x^*)\rangle \\
&= \langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \le 0.
\end{aligned} \tag{3.10}$$

Finally, we show that $\{x_n\}$ converges strongly to $x^*$. Setting

$$u_n = \alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n, \forall n \ge 0,$$

it follows from Lemma 2.11, 2.13 and 2.14 that

$$\begin{aligned}
\|y_n - x^*\|^q &= \langle Q_C u_n - u_n, J_q(y_n - x^*)\rangle + \langle u_n - x^*, J_q(y_n - x^*)\rangle \\
&\le \langle u_n - x^*, J_q(y_n - x^*)\rangle \\
&= \langle [(1-\gamma_n)I - \alpha_n\mu L_2][S(\mu_n)z_n - x^*], J_q(y_n - x^*)\rangle \\
&\quad + \alpha_n\langle \gamma L_1 x_n - \mu L_2 x^*, J_q(y_n - x^*)\rangle + \gamma_n\langle x_n - x^*, J_q(y_n - x^*)\rangle \\
&\le (1-\gamma_n-\alpha_n\tau)\|S(\mu_n)z_n - x^*\|\|y_n - x^*\|^{q-1} \\
&\quad + \gamma_n\|x_n - x^*\|\|y_n - x^*\|^{q-1} + \alpha_n\langle \gamma L_1 x_n - \gamma L_1 x^*, J_q(y_n - x^*)\rangle \\
&\quad + \alpha_n\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle \\
&\le (1-\gamma_n-\alpha_n\tau)\|x_n - x^*\|\|y_n - x^*\|^{q-1} \\
&\quad + \gamma_n\|x_n - x^*\|\|y_n - x^*\|^{q-1} + \alpha_n\gamma L\|x_n - x^*\|\|y_n - x^*\|^{q-1} \\
&\quad + \alpha_n\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle \\
&\le [1-\alpha_n(\tau-\gamma L)]\|x_n - x^*\|\|y_n - x^*\|^{q-1} \\
&\quad + \alpha_n\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle \\
&\le [1-\alpha_n(\tau-\gamma L)]\Big[\frac{1}{q}\|x_n - x^*\|^q + \frac{q-1}{q}\|y_n - x^*\|^q\Big] \\
&\quad + \alpha_n\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|y_n - x^*\|^q &\le \frac{1-\alpha_n(\tau-\gamma L)}{1+(q-1)\alpha_n(\tau-\gamma L)}\|x_n - x^*\|^q \\
&\quad + \frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle \\
&\le [1-\alpha_n(\tau-\gamma L)]\|x_n - x^*\|^q \\
&\quad + \frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{n+1}-x^*\|^q &\leq \beta_n\|x_n-x^*\|^q+(1-\beta_n)\|S(\mu_n)y_n-x^*\|^q \\
&\leq \beta_n\|x_n-x^*\|^q+(1-\beta_n)\|y_n-x^*\|^q \\
&\leq \beta_n\|x_n-x^*\|^q+(1-\beta_n)\Bigg[[1-\alpha_n(\tau-\gamma L)]\|x_n-x^*\|^q \\
&\quad +\frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1x^*-\mu L_2x^*, J_q(y_n-x^*)\rangle\Bigg] \\
&= [1-\alpha_n(\tau-\gamma L)(1-\beta_n)]\|x_n-p\|^q \\
&\quad +\frac{q\alpha_n(1-\beta_n)}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1x^*-\mu L_2x^*, J_q(y_n-x^*)\rangle. \qquad (3.11)
\end{aligned}$$

Now, from (C1), (3.10) and applying Lemma 2.10 to (3.11), we get $\|x_n-x^*\|\to 0$ as $n\to\infty$. Therefore, the sequence $\{x_n\}$ converges strongly to $x^*\in F$. The proof is complete. □

**Corollary 3.2.** *Let $C$ be a sunny nonexpansive retract and nonempty closed convex subset of a 2-uniformly smooth and uniformly convex Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J: E\to E^*$ with the best smooth constant $K$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $A: C\to E$ be an $\beta$-inverse-strongly accretive operator, $\mathcal{S}=\{S(s): s\geq 0\}$ be a nonexpansive semigroup from $C$ into itself, $L_1: C\to E$ be a $L$-Lipschitzian mapping with constant $L\geq 0$ and $L_2: C\to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constant $\kappa,\eta>0$. Assume $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\lambda_n\}\subset(0,1)$, $\{\mu_n\}\subset(0,\infty)$ such that $\{\lambda_n\}\subset[a,b]\subset(0,1)$, $0<\mu<\frac{\eta}{K^2\kappa^2}$, $0<a\leq\lambda_n\leq b<\frac{\beta}{K^2}$, $0\leq\gamma L<\tau$ where $\tau=\mu(\eta-K^2\mu\kappa^2)$ and $F:=F(\mathcal{S})\cap VI(C,A)\neq\emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1\in C$ and*

$$\begin{cases} z_n=Q_C(x_n-\lambda_nAx_n) \\ y_n=Q_C\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big], \\ x_{n+1}=\beta_nx_n+(1-\beta_n)S(\mu_n)y_n, \end{cases}$$

*which satisfy the conditions (C1)-(C6) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $x^*\in F$ which also solves the following variational inequality:*

$$\langle \gamma L_1x^*-\mu L_2x^*, J(z-x^*)\rangle\leq 0,\ \forall z\in F.$$

**Corollary 3.3.** *Let $C$ be a sunny nonexpansive retract and nonempty closed convex subset of a $q$-uniformly smooth and uniformly convex Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J_q: E\to E^*$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $A: C\to E$ be an $\beta$-inverse-strongly accretive operator, $\mathcal{S}=\{S(s): s\geq 0\}$ be a nonexpansive semigroup from $C$ into itself, $L_1: C\to E$ be a $L$-Lipschitzian mapping with constant $L\geq 0$ and $L_2: C\to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constant $\kappa,\eta>0$. Assume $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\lambda_n\}\subset(0,1)$, $\{\mu_n\}\subset(0,\infty)$ such that $\{\lambda_n\}\subset[a,b]\subset(0,1)$, $0<\mu<\left(\frac{q\eta}{c_q\kappa^q}\right)^{\frac{1}{q-1}}$ where $c_q$ is a positive real number, $0<a\leq\lambda_n\leq b<\left(\frac{q\beta}{c_q}\right)^{\frac{1}{q-1}}$,*

*$0 \leq \gamma L < \tau$ where $\tau = \mu\left(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q}\right)$ and $F := F(\mathcal{S}) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)\frac{1}{t_n}\displaystyle\int_0^{t_n} S(s)z_n ds\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\frac{1}{t_n}\displaystyle\int_0^{t_n} S(s)y_n ds, \end{cases}$$

*which satisfy the conditions (C1)-(C3) and (C6) in Theorem 3.1 and assume that*

$$\lim_{n\to\infty} \sup_{x\in\widetilde{C}} \left\| \frac{1}{t_{n+1}} \int_0^{t_{n+1}} S(s)x ds - \frac{1}{t_n}\int_0^{t_n} S(s)x ds \right\| = 0,$$

*$\widetilde{C}$ bounded subset of $C$, $\lim_{n\to\infty} \mu_n = \infty$ and $\lim_{n\to\infty} \frac{\mu_n}{\mu_{n+1}} = 1$. Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:*

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0, \forall z \in F.$$

**Corollary 3.4.** *Let $C$ be a sunny nonexpansive retract and nonempty closed convex subset of a q-uniformly smooth and uniformly convex Banach space $E$ which admits a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$, $A : C \to E$ be an $\beta$-inverse-strongly accretive operator, $L_1 : C \to E$ be a L-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0, 1)$ such that $\{\lambda_n\} \subset [a, b] \subset (0, 1)$, $0 < \mu < \left(\frac{q\eta}{c_q\kappa^q}\right)^{\frac{1}{q-1}}$ where $c_q$ is a positive real number,*

$$0 < a \leq \lambda_n \leq b < \left(\frac{q\beta}{c_q}\right)^{\frac{1}{q-1}}, \ 0 \leq \gamma L < \tau$$

*where*

$$\tau = \mu\left(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q}\right)$$

*and $F := VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)z_n\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \end{cases}$$

*which satisfy the conditions (C1)-(C3) and (C6) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:*

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0, \forall z \in F.$$

*Proof.* Taking $\mu_n = 0$ in Theorem 3.1, we can conclude the desired conclusion easily. The proof is complete. □

## 4. Numerical example

In this section, we illustrate a real numerical example by using main theorem.

**Example 4.1.** Let $E = \mathbb{R}$, $C = [0,1]$, $q = 2$, $j_q = I$, $\gamma = \mu = \frac{1}{2}$, $\alpha_n = \mu_n = \frac{1}{3n}$, $\beta_n = \frac{n+1}{2n}$, $\gamma_n = \frac{2n-1}{7n}$, $\lambda_n = \frac{6n-1}{14n}$ and $x_1 = 1$ which satisfy the conditions $(C1)-(C6)$ in Theorem 3.1. We define the mappings as follows:

$$Q_C x = \begin{cases} \frac{x}{|x|}, & x \in \mathbb{R} - C \\ x, & x \in C, \end{cases} \quad (S(s))(x) = xe^{-s},\ Ax = \frac{x}{2},\ L_1 x = x^2$$

and

$$L_2 x = \frac{1}{3}(x^2 + 2x),$$

where $A$ is $\frac{1}{2}$-inverse strongly accretive, $L_1$ is 1-Lipschitzian and $L_2$ is 1-Lipschitzian and $\frac{2}{3}$-strongly accretive. Then the sequence

$$\begin{cases} z_n = x_n\left(\frac{22n+1}{28n}\right), \\ y_n = \frac{x_n}{n}\left(\frac{x_n}{6} + \frac{2n-1}{7}\right) + \frac{z_n e^{-1/3n}}{n}\left(\frac{5n+1}{7n} - \frac{z_n e^{-1/3n}+2}{18}\right), \\ x_{n+1} = \frac{n+1}{2n}x_n + \frac{n-1}{2n}y_n e^{-1/3n} \end{cases}$$

converges strongly to 0 shown in Figure 1 and Table 1.

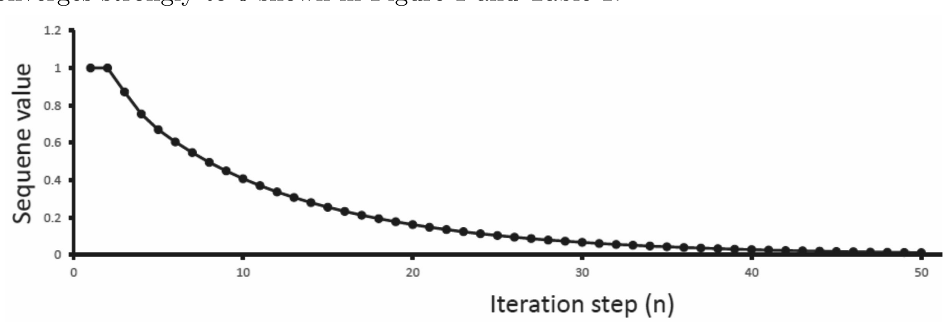


Figure 1. The iteration process.

Table 1. The value of sequence $\{x_n\}$

| Iteration step $(n)$ | Sequence value $(x_n)$ | Iteration step $(n)$ | Sequence value $(x_n)$ |
|---|---|---|---|
| 1 | 1 | 10 | 0.456821 |
| 2 | 1 | 20 | 0.182023 |
| 3 | 0.915366 | 50 | 0.013817 |
| 4 | 0.828039 | 100 | 0.000222 |
| 5 | 0.747896 | 168 | 0.0000009 |

**Example 4.2.** Let $E = \mathbb{R}^3$ and an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x^1 \cdot y^1 + x^2 \cdot y^2 + x^3 \cdot y^3, \forall \mathbf{x} = (x^1, x^2, x^3), \mathbf{y} = (y^1, y^2, y^3)$$

and the usual norm $\| \cdot \| : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\|\mathbf{x}\| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}.$$

Let $C = \{\mathbf{x} \in \mathbb{R}^3 \big| \|\mathbf{x}\| \le 1\}$, $\gamma = \mu = \frac{1}{2}$, $\alpha_n = \mu_n = \frac{1}{3n}$, $\beta_n = \frac{n+1}{2n}$, $\gamma_n = \frac{2n-1}{7n}$, $\lambda_n = \frac{6n-1}{14n}$ and $\mathbf{x}_1 = (1, 2, 3)$ which satisfy the conditions $(C1) - (C6)$ in Theorem 3.1. We define the mappings as follows:

$$Q_C \mathbf{x} = \begin{cases} \frac{\mathbf{x}}{\|\mathbf{x}\|}, & \mathbf{x} \notin C \\ \mathbf{x}, & \mathbf{x} \in C, \end{cases} \quad (S(s))(\mathbf{x}) = \mathbf{x}e^{-s},\ A\mathbf{x} = \frac{\mathbf{x}}{2},\ L_1 \mathbf{x} = \mathbf{x}^2$$

and

$$L_2 \mathbf{x} = \frac{1}{3}(\mathbf{x}^2 + 2\mathbf{x}).$$

For $n = 1, \ldots, 6$, we have the sequence

$$\begin{cases} \mathbf{z}_n = \frac{\mathbf{Z}_n}{\|\mathbf{Z}_n\|}, \\ \mathbf{y}_n = \frac{\mathbf{Y}_n}{\|\mathbf{Y}_n\|}, \\ \mathbf{x}_{n+1} = \frac{n+1}{2n}\mathbf{x}_n + \frac{n-1}{2n}\mathbf{y}_n e^{-1/3n}, \end{cases}$$

where

$$\mathbf{Z}_n = \mathbf{x}_n \left( \frac{22n+1}{28n} \right),$$

$$\mathbf{Y}_n = \frac{\mathbf{x}_n}{n} \left( \frac{\mathbf{x}_n}{6} + \frac{2n-1}{7} \right) + \frac{\mathbf{z}_n e^{-1/3n}}{n} \left( \frac{5n+1}{7n} - \frac{\mathbf{z}_n e^{-1/3n} + 2}{18} \right).$$

For $n = 7$, we have the sequence

$$\begin{cases} \mathbf{z}_n = \mathbf{x}_n \left( \frac{22n+1}{28n} \right), \\ \mathbf{y}_n = \frac{\mathbf{Y}_n}{\|\mathbf{Y}_n\|}, \\ \mathbf{x}_{n+1} = \frac{n+1}{2n}\mathbf{x}_n + \frac{n-1}{2n}\mathbf{y}_n e^{-1/3n}, \end{cases}$$

where

$$\mathbf{Y}_n = \frac{\mathbf{x}_n}{n} \left( \frac{\mathbf{x}_n}{6} + \frac{2n-1}{7} \right) + \frac{\mathbf{z}_n e^{-1/3n}}{n} \left( \frac{5n+1}{7n} - \frac{\mathbf{z}_n e^{-1/3n} + 2}{18} \right).$$

For $n = 8, 9, 10, \ldots$, we have the sequence

$$\begin{cases} \mathbf{z}_n = \mathbf{x}_n \left( \dfrac{22n+1}{28n} \right), \\ \mathbf{y}_n = \dfrac{\mathbf{x}_n}{n} \left( \dfrac{\mathbf{x}_n}{6} + \dfrac{2n-1}{7} \right) + \dfrac{\mathbf{z}_n e^{-1/3n}}{n} \left( \dfrac{5n+1}{7n} - \dfrac{\mathbf{z}_n e^{-1/3n} + 2}{18} \right), \\ \mathbf{x}_{n+1} = \dfrac{n+1}{2n} \mathbf{x}_n + \dfrac{n-1}{2n} \mathbf{y}_n e^{-1/3n}. \end{cases}$$

Then the sequence converges strongly to $\mathbf{0} = (0, 0, 0)$, shown in Figure 2 and Table 2.

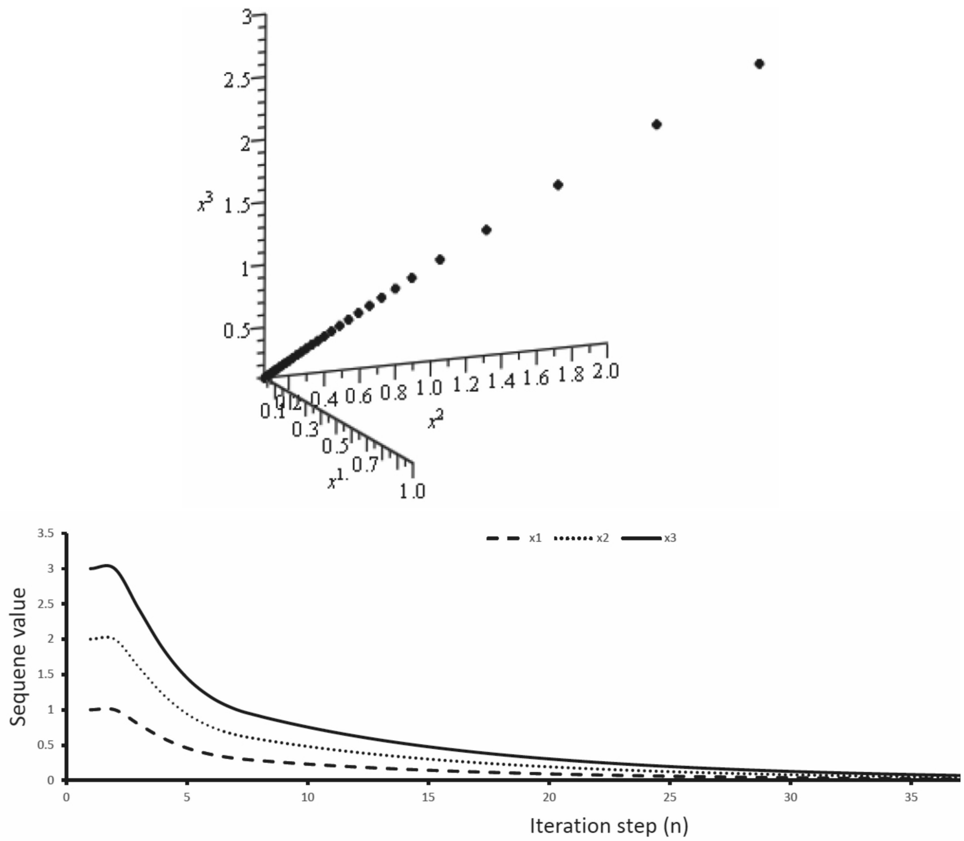


Figure 2. The iteration process.

Table 2. The value of sequence $\{\mathbf{x}_n\}$

| $n$ | $x_n^1$ | $x_n^2$ | $x_n^3$ | $n$ | $x_n^1$ | $x_n^2$ | $x_n^3$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 10 | 0.229555 | 0.480061 | 0.753976 |
| 2 | 1 | 2 | 3 | 20 | 0.090760 | 0.191436 | 0.303513 |
| 3 | 0.794057 | 1.603757 | 2.429100 | 50 | 0.006872 | 0.014536 | 0.023119 |
| 4 | 0.598144 | 1.220303 | 1.867191 | 100 | 0.000110 | 0.000233 | 0.000371 |
| 5 | 0.457286 | 0.940376 | 1.450720 | 174 | 0.0000003 | 0.0000006 | 0.0000009 |

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