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EXISTENCE OF ASYMPTOTICALLY STABLE SOLUTIONS TO A NONLINEAR INTEGRAL EQUATION OF MIXED TYPE

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Abstract. In the present note an existence result of asymptotically stable solutions to a nonlinear integral equation of mixed (Volterra-Hammerstein) type is presented. The proof is based on the application of a fixed point theorem of Schaefer's type on Fréchet spaces.
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1. Introduction

In [2] Banás and Rzepka study a very interesting property for the solutions of some functional equations. This property was also researched by Burton and Zhang in [5], in a more general case. Let $F : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ be an operator, where $BC(\mathbb{R}_+)$ consists of all bounded and continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, $\mathbb{R}_+ := [0, \infty)$, $d \geq 1$. Let $|\cdot|$ be a norm in $\mathbb{R}^d$.

The following definition is given in [2, 5], for the solutions $x \in BC(\mathbb{R}_+)$ of the equation

$$x = Fx. \tag{1.1}$$

Definition 1.1. A function x is said to be an *asymptotically stable solution* of (1.1) iff for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that for every $t \geq T$ and for every solution y of (1.1), we have

$$|x(t) - y(t)| \leq \varepsilon.$$

A sufficient condition for the existence of asymptotically stable solutions is given by the following proposition (see [2, 5]).

Proposition 1.1. *Assume that there exist a constant* $k \in [0, 1)$ *and a continuous function* $a : R_+ \to R_+$ *with* $\lim_{t\to\infty} a(t) = 0$*, such that*

$$|(Fx)(t) - (Fy)(t)| \leq k|x(t) - y(t)| + a(t), \quad \forall t \in \mathbb{R}_+,\ \forall x, y \in BC(\mathbb{R}_+). \tag{1.2}$$

Then every solution of (1.1) *is asymptotically stable.*

Let us remark that basically the property of the asymptotic stability is a property of the fixed points of the operator F. Actually, in [1, 5], the proof of the existence of an asymptotically stable solution is done by applying the Schauder's fixed point theorem. So, it is enough to require that Definition 1.1 is only fulfilled on the closed, bounded, and convex set on which the Schauder's theorem is applied.

Another remark concerning Proposition 1.1 is that if (1.2) is fulfilled then **every** solution of (1.1) is asymptotically stable. Moreover, by (1.2) we deduce that the result of Proposition 1.1 is appropriate for the case when $F = A + B$, where A is contraction and $\lim_{t\to\infty}(Bx)(t) = 0$, for every x belonging to the set on which the fixed point theorem is applied. On the other hand, the set of the fixed points of F should be "big" enough such that Definition 1.1 is consistent. In this direction, in the case when Schauder's fixed point Theorem is used an interesting result has been obtained by Zamfirescu in [11], stating that if B_ρ is the closed ball of radius $\rho > 0$ from a Banach space and $F : B_\rho \to B_\rho$ is a compact operator, then for most functions F, the set of solutions of (1.1) is homeomorphic to the Cantor set ("most" means "all" except those in a first category set).

Finally, let us remark that in order to fulfil Definition 1.1 it is not necessary that the solutions of (1.1) be bounded on $\mathbb{R}_+$.

In [6] Islam and Adivar prove the existence of asymptotically stable solutions to a nonlinear Volterra integral equation, by applying Schauder's fixed point theorem on the space C_l of all functions of $BC(\mathbb{R}_+)$, having finite limits at ∞. In [5] the authors proved that, under certain hypotheses, the nonlinear Volterra integral equation

$$x(t) = f(t, x(t)) + \int_0^t u(t, s, x(s))ds, \quad t \geq 0 \tag{1.3}$$

has at least one solution, and every solution is asymptotically stable and converges to the unique continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}^d$ satisfying

$$\psi(t) = f(t, \psi(t)), \quad t \geq 0.$$

Actually, in our note [1], we proved a similar theorem without using hypothesis (i*) from [5]. Let us also remark that in (1.3) one has $F = A+B$, where A is a contraction in $BC(\mathbb{R}_+)$ and B is a compact operator which in the admitted hypotheses fulfills the property

$$\lim_{t\to\infty}(Bx)(t) = 0, \tag{1.4}$$

the limit being uniform with respect to $x \in BC(\mathbb{R}_+)$. The second result in our Note [1] is obtained in the absence of condition (1.4).

In this paper we will consider the following nonlinear integral equation of mixed (Volterra-Hammerstein) type

$$x(t) = q(t) + F(t, x(t)) + \int_0^t K(t, s, x(s))\, ds + \int_0^\infty G(t, s, x(s))\, ds,\ t \in \mathbb{R}_+, \tag{1.5}$$

where $q : \mathbb{R}_+ \to \mathbb{R}^d$, $F : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $K : \Delta \times E \to E$, $G : \Delta \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous functions, and

$$\Delta = \left\{(t, s, x),\ t, s \in \mathbb{R}_+,\ 0 \leq s \leq t,\ x \in \mathbb{R}^d\right\}.$$

We will prove the existence of asymptotically stable solutions to Eq. (1.5), without requiring the boundedness of solutions. Our result needs a more sophisticated argument than the one used in [1] and it is mainly based on the application of a fixed point theorem of Schaefer's type on Fréchet spaces.

2. Fixed point approach

Two main results of the fixed point theory are Banach contraction principle and Schauder's fixed point theorem. Krasnoselskii combined them into the following result (see [7], [8], [10], [12]).

Theorem 2.1. *(Krasnoselskii). Let M be a closed convex non-empty subset of a Banach space X. Suppose that A and B maps M into X, such that the following hypotheses are fulfilled:*

(i) $Ax + By \in M$, $\forall x, y \in M$;

(ii) A is continuous and AM is contained in a compact set;

(iii) B is a contraction with constant $\alpha < 1$.

Then, there is a $x \in M$, with $Ax + Bx = x$.

This is a captivating result and it has many interesting applications. In recent years much attention has been paid to this result. T.A. Burton (see [3]) remarked that in practice it is difficult to check hypothesis (i) and he proposed to replace it by the condition

$$(x = Bx + Ay,\ y \in M) \Longrightarrow x \in M. \tag{i$'$}$$

Following the improvement of hypothesis (i), Burton and Kirk (see [4]) proved the following variant of Theorem 2.1.

Theorem 2.2. (Burton & Kirk). *Let X be a Banach space, A, $B : X \to X$, B a contraction and A a compact operator. Then either*

(a) $x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax$ has a solution for $\lambda = 1$

or

(b) the set $\left\{x \in X,\ x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax,\ \lambda \in (0,1)\right\}$ is unbounded.

We mention that through compact operator one understands a continuous operator which transforms bounded sets into relatively compact sets.

The proof of Theorem 2.2 is based on the remark that $\lambda B\left(\frac{x}{\lambda}\right)$, $\lambda \in (0,1)$ is a contraction, too, with the same contraction constant and therefore

$$x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax \Longleftrightarrow x = \lambda\left(I - B\right)^{-1} Ax$$

and it uses the following fundamental result due to Schaefer (see [8], [9], [10], [12]).

Theorem 2.3. (Schaefer). *Let E be a linear locally convex space and let $H : B \to B$ be a compact operator. Then either*

(α) the equation $x = \lambda Hx$ has a solution for $\lambda = 1$

or

(β) the set $\{x \in X,\ x = \lambda Hx,\ \lambda \in (0,1)\}$ is unbounded.

Since we research the existence of solutions to Eq. (1.5), that is defined on the noncompact interval $\mathbb{R}_+$, we remark that the application either of Theorem 2.1 or Theorem 2.2 is very difficult. Therefore, in the proof of our main result, we will apply the following extension of Theorem 2.2 from Banach spaces to Fréchet spaces (i.e locally convex spaces that are completely metrizable).

Theorem 2.4. *Let Y be a Fréchet space and A, $B : Y \to Y$ be two operators. Admit that:*

(a) *A is contraction on Y;*
(b) *B is compact operator on Y;*
(c) *The following set is bounded*

$$\left\{x \in Y, \quad x = \lambda A(\frac{x}{\lambda}) + \lambda Bx, \quad \lambda \in (0,1)\right\}. \tag{2.1}$$

Then the operator $A + B$ admits fixed points.

Proof. The proof of this theorem is immediate. Indeed, hypothesis (a) ensures us the existence and the continuity of the operator $(I - B)^{-1}$. By applying to the operator $x \to \lambda (I - B)^{-1} Ax$ the Schaefer's Theorem 2.3, from hypothesis (c) the conclusion follows, since $U_1 x = Bx + Ax$.

3. The existence of solutions

Consider the function space

$$X = C_c\left(\mathbb{R}_+, \mathbb{R}^d\right) := \left\{x : \mathbb{R}_+ \to \mathbb{R}^d,\, x \text{ continuous}\right\},$$

that, endowed with the countable familiy of seminorms

$$|x|_n := \sup_{t \in [0,n]} \{|x(t)|\}, \quad n \geq 1, \tag{3.1}$$

becomes a a Fréchet space. The most natural metric that can be defined on X is

$$d(x,y) := \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|x-y|_n}{1+|x-y|_n}.$$

In addition, we will use another countable family of seminorms,

$$\|x\|_n := \|x\|_{\gamma_n} + \|x\|_{\lambda_n}, \quad n \geq 1, \tag{3.2}$$

where

$$\|x\|_{\gamma_n} := \sup_{t \in [0,\gamma_n]} \{|x(t)|\},\ \ \|x\|_{\lambda_n} := \sup_{t \in [\gamma_n, n]} \left\{e^{-\lambda_n (t-\gamma_n)} |x(t)|\right\},$$

$\gamma_n \in (0,n)$, $\lambda_n > 0$ are arbitrary numbers.

Remark 3.1. *Since, obviously,*

$$e^{-\lambda_n (n-\gamma_n)} |x|_n \leq \|x\|_n \leq 2\,|x|_n\,,\ \forall n \in \mathbb{N}^*,$$

the families (3.1) *and* (3.2) *are equivalent and define the same topology on X, i.e. the topology of the uniform convergence on compact subsets of $\mathbb{R}_+$. Consequently, a family* *in X is relatively compact if and only if it is equicontinuous and uniformly bounded on compact subsets of $\mathbb{R}_+$.*

The following general hypotheses will be required:
(H1) there exists a continuous function $f : \mathbb{R}_+ \to [0,1)$, such that

$$|F(t,x) - F(t,y)| \le f(t)\,|x-y|\,,\ \forall t \in \mathbb{R}_+,\ \forall x,y \in \mathbb{R}^d;$$

(H2) there exists a continuous function $k : \mathbb{R}_+ \to \mathbb{R}_+$ and $\alpha \in [0,1)$, such that

$$|K(t,s,x) - K(t,s,y)| \le \frac{k(t)}{t^\alpha}\,|x-y|\,,\ \forall\,(t,s,x)\,,\ \ (t,s,y) \in \Delta,\ t>0; \tag{3.3}$$

(H3) there exists a continuous function $\psi : \Delta \to \mathbb{R}_+$, such that the integral

$$\int_0^\infty \psi(t,s)\,ds$$

is convergent, uniformly with respect to t on compact subsets of $\mathbb{R}_+$ and

$$|G(t,s,x)| \le \psi(t,s)\,,\ \forall\,(t,s) \in \Delta,\ x \in \mathbb{R}^d.$$

We can state and prove the following lemma.

Lemma 3.1. *Suppose that hypotheses (H1) and (H2) are fulfilled. Then there is a unique continuous function $\xi : \mathbb{R}_+ \to \mathbb{R}^d$, such that*

$$\xi(t) = q(t) + F(t,\xi(t)) + \int_0^t K(t,s,\xi(s))\,ds, \quad \forall t \in \mathbb{R}_+.$$

Proof. We define the operator $A : X \to X$ through the equality

$$(Ax)(t) := q(t) + F(t,x(t)) + \int_0^t K(t,s,x(s))\,ds,$$

$\forall t \in \mathbb{R}_+$, $\forall x \in X$. By taking into account hypotheses (H1) and (H2), it follows that $\forall t > 0$, $\forall x,\ y \in X$,

$$|(Ax)(t) - (Ay)(t)| \le f(t)\,|x(t)-y(t)| + \frac{k(t)}{t^\alpha}\int_0^t |x(s)-y(s)|\,ds. \tag{3.4}$$

We show that A is contraction with respect to the family of seminorms (3.2), i.e. there exists $\delta_n \in [0,1)$ such that for any $x,\ y \in X$,

$$\|Ax - Ay\|_n \le \delta_n \|x-y\|_n\,, \forall n \in \mathbb{N}^*. \tag{3.5}$$

Let $n \ge 1$ be fixed. Consider $\gamma_n \in (0,n)$ and $\lambda_n > 0$ that will be specified later.
Let $t \in (0,\gamma_n]$ be arbitrary. From (3.4) we deduce

$$\begin{aligned} |(Ax)(t) - (Ay)(t)| &\le f_n\,|x(t)-y(t)| + \frac{k_n}{t^\alpha}\int_0^t |x(s)-y(s)|\,ds \\ &\le \left(f_n + k_n\gamma_n^{1-\alpha}\right)\|x-y\|_{\gamma_n}\,, \end{aligned}$$

where

$$f_n := \sup_{t\in[0,n]}\{f(t)\}\,,\ k_n := \sup_{t\in[0,n]}\{k(t)\}\,,$$

In addition, since

$$\lim_{t\to 0}\frac{1}{t^{\alpha}}\int_0^t |x(s)-y(s)|\,ds=\lim_{t\to 0}t^{1-\alpha}\,|x(0)-y(0)|=0,$$

and A is continuous, we get

$$\begin{aligned}|(Ax)(0)-(Ay)(0)| &\leq \lim_{t\to 0}\left(f(t)\,|x(t)-y(t)|\right)+\lim_{t\to 0}\frac{k(t)}{t^{\alpha}}\int_0^t |x(s)-y(s)|\,ds\\ &= f(0)\,[x(0)-y(0)]\,.\end{aligned}$$

Therefore,

$$\|Ax-Ay\|_{\gamma_n}\leq \left(f_n+k_n\gamma_n^{1-\alpha}\right)\|x-y\|_{\gamma_n}\,. \tag{3.6}$$

Let $t\in[\gamma_n,n]$ be arbitrary. From (3.4) after easy estimates, we get

$$\begin{aligned}|(Ax)(t)-(Ay)(t)| &\leq f_n\,|x(t)-y(t)|+\frac{k_n}{t^{\alpha}}\left(\int_0^{\gamma_n}|x(s)-y(s)|\,ds\right.\\ &\quad\left.+\int_{\gamma_n}^t |x(s)-y(s)|\,e^{-\lambda_n(s-\gamma_n)}\cdot e^{\lambda_n(s-\gamma_n)}ds\right)\\ &< f_n\,|x(t)-y(t)|+k_n\gamma_n^{-\alpha}\left(\gamma_n\,\|x-y\|_{\gamma_n}\right.\\ &\quad\left.+\|x-y\|_{\lambda_n}\frac{e^{\lambda_n(t-\gamma_n)}}{\lambda_n}\right).\end{aligned}$$

It follows that

$$\begin{aligned}|(Ax)(t)-(Ay)(t)|\,e^{-\lambda_n(t-\gamma_n)} &< f_n\,|x(t)-y(t)|\,e^{-\lambda_n(t-\gamma_n)}+k_n\gamma_n^{1-\alpha}\,\|x-y\|_{\gamma_n}\\ &\quad+\frac{k_n}{\lambda_n}\gamma_n^{-\alpha}\,\|x-y\|_{\lambda_n}\end{aligned}$$

and therefore

$$\begin{aligned}\|Ax-Ay\|_{\lambda_n} &\leq f_n\sup_{t\in[\gamma_n,n]}\left\{|x(t)-y(t)|\,e^{-\lambda_n(t-\gamma_n)}\right\}\\ &\quad+k_n\gamma_n^{1-\alpha}\,\|x-y\|_{\gamma_n}+\frac{k_n}{\lambda_n}\gamma_n^{-\alpha}\,\|x-y\|_{\lambda_n}\\ &\leq \left(f_n+\frac{k_n}{\lambda_n}\gamma_n^{-\alpha_n}\right)\|x-y\|_{\lambda_n}+k_n\gamma_n^{1-\alpha_n}\,\|x-y\|_{\gamma_n}\,.\end{aligned} \tag{3.7}$$

By (3.6) and (3.7) we obtain

$$\|Ax-Ay\|_n\leq \left(f_n+2k_n\gamma_n^{1-\alpha}\right)\|x-y\|_{\gamma_n}+\left(f_n+\frac{k_n}{\lambda_n}\gamma_n^{-\alpha}\right)\|x-y\|_{\lambda_n}\,. \tag{3.8}$$

Since $f_n\in[0,1)$, for $\gamma_n\in\left(0,\left(\frac{1-f_n}{2k_n}\right)^{\frac{1}{1-\alpha}}\right)$ we deduce $f_n+\frac{k_n}{\lambda_n}\gamma_n^{1-\alpha}<1$ and for $\lambda_n>\frac{k_n}{1-f_n}\gamma^{-\alpha}$ we deduce $\gamma_n+\frac{k_n}{\lambda_n}\gamma_n^{-\alpha}<1$. Let

$$\delta_n:=\max\left\{f_n+\frac{k_n}{\lambda_n}\gamma_n^{1-\alpha},\ \gamma_n+\frac{k_n}{\lambda_n}\gamma_n^{-\alpha}\right\}.$$

It follows that $\delta_n < 1$ and, since (3.8),

$$\|Ax - Ay\|_n \leq \delta_n \left(\|x-y\|_{\gamma_n} + \|x-y\|_{\lambda_n}\right) = \delta_n \|x-y\|_n, \forall n \in \mathbb{N}^*.$$

Hence, A is contraction.

From now on, the existence and the uniqueness of the fixed point to A follows by applying the Banach Contraction Principle on Fréchet spaces. The proof is now complete. □

Remark 3.2. *The family of seminorms* (3.2) *is essential in deriving our estimates on the operator A, while the classical family of seminorms* (3.1) *is useless when trying to prove that A is contraction, because condition* (3.3) *is not fulfilled at* $t = 0$ *and so the function* $k(t)/t^\alpha$ *could be unbounded around* 0.

We state and prove now an existence result of the solutions to Eq. (1.5).

Theorem 3.1. *If hypotheses (H1)–(H3) are fulfilled, then Eq.* (1.5) *admits solutions.*

Proof. The proof will be made in several steps and it is based on the application of Theorem 2.4.

Step 1. The transformation of Eq. (1.5). In Eq. (1.5) we set $x = y + \xi(t)$, where ξ is the function obtained through Lemma 3.1. Then (1.5) becomes

$$y = A_1 y + B_1 y, \tag{3.9}$$

where

$$(A_1 y)(t) := q(t) + F(t, y(t) + \xi(t)) - \xi(t) + \int_0^t K(t, s, y(s) + \xi(s))\, ds,$$
$$(B_1 y)(t) := \int_0^\infty G(t, s, y(s) + \xi(s))\, ds.$$

Clearly, if y is a solution to Eq. (3.9), then $x = y + \xi(t)$ is a solution to Eq. (1.5), and conversely.

Step 2. The properties of the operator A_1. One has obviously $\forall y_1, y_2 \in X$, $\forall t \in \mathbb{R}_+$,

$$|(A_1 y_1)(t) - (A_1 y_2)(t)| \leq f(t)|y_1(t) - y_2(t)| + \frac{k(t)}{t^\alpha} \int_0^t |y_1(s) - y_2(s)|\, ds,$$

i.e. the operator A_1 fulfills an inequality of type (3.4) and, following the proof of Lemma 3.1, we deduce that

$$\|A_1 y_1 - A_1 y_2\|_n \leq \delta_n \|y_1 - y_2\|_n, \forall y_1,\ y_2 \in X,\ \forall n \in \mathbb{N}^*.$$

Let us also remark that $A_1 0 = 0$ and $\|A_1 y\|_n \leq \delta_n \|y\|_n$, $\forall y \in X$, $\forall n \in \mathbb{N}^*$.

Step 3. The operator $B_1 : X \to X$ *is compact.* By hypothesis (H3), the convergence of the integral

$$\int_0^\infty G(t, s, y(s) + \xi(s))\, ds$$

is uniform with respect to t on each compact subset of $\mathbb{R}_+$, and so $(B_1 y)(t)$ is a continuous function of t.

(a) *We prove that B_1 is a continuous operator*. Indeed, let us consider $\{y_m\}_m \subset X$, $y_m \to y$ in X, that is, $\forall \varepsilon > 0$, $\forall n \geq 1$, $\exists N = N(\varepsilon, n)$, $\forall m \geq N$, $|y_m - y|_n < \varepsilon$.

Let us fix $n \geq 1$. From the convergence of $\{y_m\}_m$ and the continuity of ξ, there is $r \geq 0$ such that $|y_m + \xi|_n \leq r$, $|y + \xi|_n \leq r$, $\forall m$.

Consider $\varepsilon > 0$. By hypothesis (H3), there is $T > 0$, such that

$$\int_T^\infty \psi(t,s)\,ds < \frac{\varepsilon}{3},\ \forall t \in [0,n]. \tag{3.10}$$

Since G is uniformly continuous on the set $[0,n] \times [0,T] \times \overline{B(r)}$, where

$$\overline{B(r)} := \left\{x \in \mathbb{R}^d,\ \ |x| \leq r\right\},$$

it follows that for all $t \in [0,n]$, $s \in [0,T]$, and $m \geq N$,

$$|G(t,s,y_m(s) + \xi(s)) - G(t,s,y(s) + \xi(s))| < \frac{\varepsilon}{3T}.$$

Therefore, for every $t \in [0,n]$ and $m \geq N$, we have

$$\begin{aligned} |(B_1 y_m)(t) - (B_1 y)(t)| \leq &\int_0^T |G(t,s,y_m(s) + \xi(s)) - G(t,s,y(s) + \xi(s))|\,ds \\ &+ 2\int_T^\infty \psi(t,s)\,ds. \end{aligned}$$

Hence,

$$|B_1 y_m - B_1 y|_n \leq \varepsilon,\ \forall m \geq N,$$

and the continuity of B_1 is proved.

It remains to show that B_1 maps bounded sets into relatively compact sets. Let $\mathcal{S}$ be a bounded subset of X. We have to prove that for each $n \geq 1$ the family $\left\{B_1 y|_{[0,n]},\ y \in \mathcal{S}\right\}$ is uniformly bounded and equicontinuous.

Let $n \geq 1$ be fixed. Then $\exists p_n > 0$, $\forall y \in \mathcal{S}$, $|y|_n \leq p_n$.

Due to hypothesis (H3), there is a $c_n \geq 0$, such that for all $t \in [0,n]$ and $y \in \mathcal{S}$, we have

$$|(B_1 y)(t)| \leq \int_0^\infty \psi(t,s)\,ds \leq c_n.$$

So, $\left\{B_1 y|_{[0,n]},\ y \in \mathcal{S}\right\}$ is uniformly bounded.

Let $\varepsilon > 0$ be arbitrarily fixed and $T > 0$ given by (3.10). By hypothesis (H3), it follows that $G(t,s,x)$ is uniformly continuous on the set $[0,n] \times [0,T] \times \overline{B(R)}$, where

$$R := p_{[T]+1} + \xi_n \text{ and } \xi_n := \sup_{t \in [0,n]} \{|\xi(t)|\}.$$

Hence, there is a $\delta > 0$ such that for all $t_1, t_2 \in [0,n]$ with $|t_1 - t_2| < \delta$, all $s \in [0,n]$, and all $y \in \mathcal{S}$,

$$|G(t_1,s,y(s) + \xi(s)) - G(t_2,s,y(s) + \xi(s))| < \frac{\varepsilon}{3T}.$$

Then it follows that for all all $t_1,\, t_2 \in [0,n]$ with $|t_1 - t_2| < \delta$ and all $y \in \mathcal{S}$,

$$\begin{aligned} \left|(B_1 y)(t_1) - (B_1 y)(t_2)\right| \le & \int_0^T \left|G(t_1, s, y(s) + \xi(s)) - G(t_2, s, y(s) + \xi(s))\right| ds \\ & + \int_T^\infty \psi(t_1, s)\, ds + \int_T^\infty \psi(t_2, s)\, ds < \varepsilon. \end{aligned}$$

Hence the set $\left\{B_1 y|_{[0,n]},\ y \in \mathcal{S}\right\}$ is equicontinuous.

By Remark 3, we deduce that B_1 is compact operator.

Step 4. Eq. (3.9) *admits solutions.* It suffices now to show that the set (2.1) is bounded. We recall a general result stating that if a set is bounded with respect to a family of seminorms, then it will be bounded with respect to every other equivalent family of seminorms. So, let us consider $y \in X$, such that

$$y = \lambda A_1\left(\frac{y}{\lambda}\right) + \lambda B_1 y,\ \ \lambda \in (0,1).$$

Therefore,

$$\|y\|_n \le \lambda \left\|A_1\left(\frac{y}{\lambda}\right)\right\|_n + \|B_1 y\|_n \le \delta_n \|y\|_n + \|B_1 y\|_n\,. \tag{3.11}$$

By hypothesis (H3),

$$\left|(B_1 y)(t)\right| \le \int_0^\infty \psi(t,s)\, ds, \forall t \ge 0$$

and it is readily seen that

$$\|B_1 y\|_n \le b_n, \tag{3.12}$$

where $b_n := b_n^1 + b_n^2$,

$$b_n^1 := \sup_{t\in[0,\gamma_n]} \left\{\int_0^\infty \psi(t,s)\, ds\right\} \text{ and } b_n^2 := \sup_{t\in[\gamma_n, n]} \left\{e^{-\lambda_n (t-\gamma_n)} \int_0^\infty \psi(t,s)\, ds\right\}.$$

From relations (3.11) and (3.12), it follows that

$$\|y\|_n \le \frac{b_n}{1-\delta_n},\ \ \forall n \ge 1,$$

which allows us to conclude that the set (2.1) is bounded and so the proof of Theorem 3.1 is now complete. □

4. Asymptotically stable solutions

Let x_1, x_2 be two solutions to Eq. (1.5). Then $y_i = x_i + \xi,\ i \in \overline{1,2}$ are solutions to (3.9). From the definitions of the function ξ and of the operator A_1, we get for all $t \in \mathbb{R}_+$

$$\begin{aligned} \left|(A_1 y_i)(t)\right| \le & \left|F(t, y_i(t) + \xi(t)) - F(t, \xi(t))\right| \\ & + \int_0^t \left|K(t, s, y_i(s) + \xi(s)) - K(t, s, \xi(s))\right| ds, \end{aligned}$$

Thus

$$|y_i(t)| \le f(t)\, |y_i(t)| + \frac{k(t)}{t^\alpha} \int_0^t |y_i(s)|\, ds + \int_0^\infty \psi(t,s)\, ds,\ \ \forall t > 0$$

or, equivalently,

$$|y_i(t)| \leq \frac{k(t)}{t^\alpha (1 - f(t))} \int_0^t |y_i(s)| \, ds + \frac{1}{1 - f(t)} \int_0^\infty \psi(t,s) \, ds, \ \forall t > 0. \tag{4.1}$$

Setting

$$w_i(t) := \int_0^t |y(s)| \, ds, \ i \in \overline{1,2}, \ \beta(t) := \frac{k(t)}{t^\alpha (1 - f(t))},$$

and

$$\gamma(t) := \frac{1}{1 - f(t)} \int_0^\infty \psi(t,s) \, ds,$$

since (4.1) we obtain

$$w_i(0) = 0, \ w_i'(t) = |y_i(t)| \leq \beta(t) \, w_i(t) + \gamma(t), \ \forall t > 0, \ \forall i \in \overline{1,2}. \tag{4.2}$$

By (4.2) classical estimates lead us to

$$\begin{aligned} |y_i(t)| &\leq \beta(t) \, e^{\int_0^t \beta(s) ds} \cdot \int_0^t \gamma(s) \, e^{-\int_0^s \beta(u) du} ds + \gamma(t) \\ &= : h(t), \ \forall t > 0, \ \forall i \in \overline{1,2}. \end{aligned} \tag{4.3}$$

Let us suppose that $\lim_{t \to \infty} h(t) = 0$. From (4.3) on a hand we have

$$|x_1(t) - x_2(t)| = |y_1(t) - y_2(t)| \leq 2h(t), \ \forall t > 0$$

and on the other hand, for every solution y to (3.9) we obtain $\lim_{t \to \infty} y(t) = 0$ and so for every solution x to (1.5), we obtain

$$\lim_{t \to \infty} |x(t) - \xi(t)| = 0.$$

Therefore we have proved the following theorem.

Theorem 4.1. *Assume that hypotheses (H1)–(H3) are fulfilled and*

$$\lim_{t \to \infty} h(t) = 0. \tag{4.4}$$

Then every solution $x(t)$ to Eq. (1.5) *is asymptotically stable and*

$$\lim_{t \to \infty} |x(t) - \xi(t)| = 0.$$

In the sequel we present an example when condition (4.4) holds. If the following conditions are satisfied:

(1) $\lim_{t \to \infty} \beta(t) = 0$;

(2) $\int_0^\infty \beta(t) \, dt < \infty$;

(3) $\lim_{t \to \infty} \gamma(t) = 0$;

(4) $\int_0^\infty \gamma(t) \, dt < \infty$,

then $\lim\limits_{t\to\infty} h(t) = 0$.
For instance, if we consider

$$f : \mathbb{R}_+ \to [0,1),\ f(t) = \frac{1}{1+t^2},$$
$$k : \mathbb{R}_+ \to \mathbb{R}_+,\ k(t) := \frac{t^\alpha \sin^2 t}{1+t^2},$$
$$\psi : \Delta \to \mathbb{R}_+,\ \psi(t,s) = \frac{t^2 e^{-t^2-s^2}}{1+t^2},$$

and $\alpha \in [0,1)$ arbitrary, then the conditions $(1)-(4)$ are fulfilled.

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