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A NEW APPROACH ON COUPLED FIXED POINT THEORY IN JS-METRIC SPACES

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Abstract. In this article, we study coupled fixed point theorems in newly appeared JS-metric spaces. It is important to note that the class of JS-metric spaces includes standard metric spaces, dislocated metric spaces, b'-metric spaces, modular spaces etc. The purpose of this paper is to present several coupled fixed point results in a more general way. Moreover, the techniques used in our proofs are indeed different from the comparable existing literature. Finally, we present non-trivial examples to validate our main results.
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1. Introduction

Throughout this article, we use usual arithmetic operations in the set of (affinely) extended real number system $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ and the notations have their usual meanings. Let X be a nonempty set and $\mathcal{D} : X^2 \to [0, \infty]$ be a mapping. For every $x \in X$, we consider the set $C(\mathcal{D}, X, x)$ (see, [8]) as follows:

$$C(\mathcal{D}, X, x) = \{(x_n) \subset X : \lim_{n\to\infty} \mathcal{D}(x_n, x) = 0\}.$$

Very recently, Jleli and Samet [8] introduced an interesting generalization of a metric space in the following way.

Definition 1.1. [8] Let X be a nonempty set and $\mathcal{D} : X^2 \to [0, \infty]$ be a mapping. Then $(X, \mathcal{D})$ is said to be a generalized metric space if the following conditions are satisfied:

(D1) $\forall x, y \in X, \mathcal{D}(x, y) = 0 \Rightarrow x = y$;
(D2) $\forall x, y \in X, \mathcal{D}(x, y) = \mathcal{D}(y, x)$;
(D3) there exists $c > 0$ such that for all $(x, y) \in X^2$ and $(x_n) \in C(\mathcal{D}, X, x)$,

$$\mathcal{D}(x, y) \le c \limsup_{n\to\infty} \mathcal{D}(x_n, y).$$

If $C(\mathcal{D}, X, x) = \phi$, then $(X, \mathcal{D})$ is a generalized metric space if $\mathcal{D}$ satisfies $(D1-D2)$.

Throughout this article, we call this metric space as a '***JS-metric space***' (due to Jleli and Samet). The authors of [8] reported that different abstract spaces such as standard metric spaces, dislocated metric spaces, b'-metric spaces, modular spaces etc. can be derived from their newly introduced metric space. They also established several fixed point results for the mappings like famous Banach contraction, Ćirić quasi-contraction, Banach contraction in partially ordered metric spaces etc. Motivated by their work, Senapati et al. [13] studied and established some more important results on this structure. For the notion of convergence, Cauchy sequence, completeness and other topological details, the readers are referred to see [8] and [13].

In another direction, Bhaskar and Lakshmikantham [2] introduced the concept of coupled fixed point in the setting of partially ordered metric spaces as follows:

Definition 1.2. [2] An element $(x, y) \in X^2$ is said to be a coupled fixed point of $F : X^2 \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

They also introduced the concept of a mixed monotone operator which is given by:

Definition 1.3. [2] Let $(X, \leq)$ be a partially ordered set and $F : X^2 \to X$ be a function. Then F is said to have the mixed monotone property if F has the following properties:

$$x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y); \forall x_1, x_2, y \in X,$$

and

$$y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2); \forall x, y_1, y_2 \in X.$$

Using this concept, the authors of [2] presented the following result in support of the existence of a coupled fixed point of an operator satisfying mixed monotone property in partially ordered complete metric spaces.

Theorem 1.4. [2] *Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete partially ordered metric space. Suppose $F : X^2 \to X$ is a mixed monotone operator having the following property:*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}\{d(x, u) + d(y, v)\} \ \forall x \geq u; y \leq v. \tag{1.1}$$

Also consider that there exist $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0); y_0 \geq F(y_0, x_0)$. If

(A) *F is continuous or*

(B) *X has the following property:*

  (a) *If a non-decreasing sequence $(x_n) \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$;*

  (b) *If a non-increasing sequence $(y_n) \to y$, then $y_n \geq y$ for all $n \in \mathbb{N}$,*

then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Afterwards, in 2011, Berinde [1] generalized the contraction condition 1.1 as follows:

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k[d(x, u) + d(y, v)] \tag{1.2}$$

for all $x \geq u; y \leq v$ and established coupled fixed point for a mixed monotone operator in partially ordered complete metric spaces. For more results on fixed points and coupled fixed points, the readers may see [5, 9, 12, 11, 6, 7, 4, 3, 10].

In this article, inspired by the ideas of JS-metric spaces, we extend and improve the coupled fixed point results of Berinde [1] due to contraction condition 1.2 for a mapping satisfying mixed monotone property in complete JS-metric spaces endowed with partial order. It is notable that the triangular inequality, so called basic property of the standard metric space, is replaced by a more weaker condition in JS-metric spaces. Necessarily, the techniques used in our proofs are quite different and most remarkably some of the proofs become simpler. Finally we construct non-trivial examples to substantiate our main results.

2. Main results

In order to state our main results, we need to define some basic things regarding this structure. Let $(X, \mathcal{D})$ be a JS-metric space. Now we consider X^2 and define

$$\mathcal{D}_+((x,y),(u,v)) = \mathcal{D}(x,u) + \mathcal{D}(y,v)$$

for all $(x,y),(u,v) \in X^2$. We prove that $(X^2, \mathcal{D}_+)$ is a $\mathcal{D}_+$-JS-metric space induced by the metric $\mathcal{D}$.

(D') Let $\mathcal{D}_+((x,y),(u,v)) = 0$. It implies that $\mathcal{D}(x,u) + \mathcal{D}(y,v) = 0$. It is possible only when both $\mathcal{D}(x,u) = 0$ and $\mathcal{D}(y,v) = 0$, i.e., $x = u$ and $y = v$. Therefore,

$$\mathcal{D}_+((x,y),(u,v)) = 0 \Rightarrow (x,y) = (u,v)$$

for all $(x,y),(u,v) \in X^2$.

(D'') Clearly, $\mathcal{D}_+((x,y),(u,v)) = \mathcal{D}_+((u,v),(x,y))$ for all $(x,y),(u,v) \in X^2$.

(D''') Let $(x_n, y_n) \to (x,y)$ as $n \to \infty$. Then

$$\begin{aligned} \mathcal{D}_+((x,y),(u,v)) &= \mathcal{D}(x,u) + \mathcal{D}(y,v) \\ &\le \limsup\{c_1\mathcal{D}(x_n,u) + c_2\mathcal{D}(y_n,v)\} \\ &\le c_0 \limsup \mathcal{D}_+((x_n,y_n),(u,v)) \end{aligned}$$

where $c_0 = \max\{c_1, c_2\}$.

Thus $\mathcal{D}_+$ satisfies all the axioms of JS-metric. Hence $(X^2, \mathcal{D}_+)$ is a $\mathcal{D}_+$-JS-metric space. Proceeding in this way, we can define a distance function on any n-tuple set X^n for $n \ge 2$.

Example 2.1. Let $X = \mathbb{R}$ and $\mathcal{D}$ be a distance function on X defined by

$$\mathcal{D}(x,y) = \begin{cases} 3, & (x,y) = (0,1) \textit{ or } (1,0); \\ |x-y|, & \textit{otherwise.} \end{cases}$$

Our first aim is to show that $(X, \mathcal{D})$ is a JS-metric space. Conditions (D1) and (D2) are trivially hold. Now we check the condition (D3). Let $x, y \in X$ such that $C(\mathcal{D}, X, x) \neq \phi$. Then following two possibilities may occur:

Case I. Let $x = 0$ and $y = 1$. Then $\mathcal{D}(x,y) = 3$ and $\mathcal{D}(x_n,y) = |x_n - 1|$ and

$$\mathcal{D}(x,y) = 3 \le c \limsup |x_n - y| \le c$$

which holds for all $c \ge 3$. Again, if $x = 1$ and $y = 0$, then we have $\mathcal{D}(1,0) = 3$ and

$$\mathcal{D}(x,y) = 3 \le c \limsup \mathcal{D}(x_n,y) = c \limsup |x_n - 0| = c$$

which also holds for all $c \ge 3$.

Case II. Suppose $(x,y) \neq (0,1),(1,0)$. Then for any other (x,y) with $C(\mathcal{D}, X, x) \neq \phi$,

$$\mathcal{D}(x,y) = |x-y| \le c \limsup \mathcal{D}(x_n,y) = c \limsup |x_n - y|$$

which holds for any $c \geq 1$. Therefore, all the axioms of JS-metric spaces hold. Hence, $(X, \mathcal{D})$ is a JS-metric space and this implies that $(X^2, \mathcal{D}_+)$ is also a JS-metric space under the metric $\mathcal{D}_+$ on X^2 defined by

$$\mathcal{D}_+((x,y),(u,v)) = \mathcal{D}(x,u) + \mathcal{D}(y,v).$$

Next, we define another function $\mathcal{D}_m : X^2 \to \mathbb{R}^+$ by

$$\mathcal{D}_m((x,y),(u,v)) = \max\{\mathcal{D}(x,u), \mathcal{D}(y,v)\}.$$

Then, it can be checked that $\mathcal{D}_m$ also satisfies the axioms of distance function in JS-metric spaces. Hence, $(X^2, \mathcal{D}_m)$ is also a $\mathcal{D}_m$-JS-metric space. In a similar fashion, one can define n-tuple $\mathcal{D}_m$-JS-metric space for any $n \geq 2$. In order to state our main results, the following propositions will be necessary.

Proposition 2.2. *Let $(z_n) = (x_n, y_n)$ be a sequence in $(X^2, \mathcal{D}_+)$. Suppose (z_n) $\mathcal{D}_+$-converges to $x^* = (x,y)$ and $u^* = (u,v)$. Then $x^* = u^*$.*

Proof. By the condition (D'''), we have

$$\begin{aligned} \mathcal{D}_+((x,y),(u,v)) &\leq c \limsup \mathcal{D}_+((x_n,y_n),(u,v)) \\ &\leq c \limsup\{\mathcal{D}(x_n,u) + \mathcal{D}(y_n,v)\} = 0 \\ \Rightarrow (x,y) = (u,v). \end{aligned}$$

Proposition 2.3. *Let (x_n) be a convergent sequence in $(X, \mathcal{D})$, converging to $x \in X$. Then $\mathcal{D}(x,x) = 0$.*

Proof. By the hypothesis of JS-metric spaces, we can find some $c > 0$ such that

$$\mathcal{D}(x,x) \leq c \limsup_{n\to\infty} \mathcal{D}(x, x_n) = 0.$$

Similarly, we can deduce the following result.

Proposition 2.4. *Let (z_n) be a convergent sequence in $(X^2, \mathcal{D}_+)$, converging to $(x,y) \in X$, where $z_n = (x_n, y_n)$. Then $\mathcal{D}_+((x,y),(x,y)) = 0$.*

If $(X, \mathcal{D})$ is a complete JS-metric space then one can easily check that $(X^2, \mathcal{D}_+)$ and $(X^2, \mathcal{D}_m)$ are complete, too. Let us consider $(x,y) \in X^2$. We define

$$\delta_F(\mathcal{D},(x,y)) = \sup\{\mathcal{D}(F^i(x,y), F^j(x,y)) : i,j \in \mathbb{N}\}$$

and

$$\delta_F(\mathcal{D},(y,x)) = \sup\{\mathcal{D}(F^i(y,x), F^j(y,x)) : i,j \in \mathbb{N}\}.$$

Throughout this article, we assume the partial order '$\leq$' on X^2 as follows:

$$(u,v) \leq (x,y) \Leftrightarrow u \leq x, v \geq y$$

for all $x,y,u,v \in X$ and we consider $(X^2, \mathcal{D}_+)$ as partially ordered complete $\mathcal{D}_+$-JS-metric space.

Before stating the coupled fixed point results, we would like to draw the reader's attention to an important thing regarding this structure. The authors of [13] have already proved that the existence of a fixed point of a contractive mapping satisfying certain conditions is guaranteed only when we choose $k \in [0,1) \cap [0, \frac{1}{c})$, where c is the least value for which condition $(D3)$ is satisfied in Definition 1.1 (see, Theorem 3.2 in [13]). If the least value $c = 0$, then it leads to a trivial case. Similarly, to establish the coupled fixed point results, we choose $k \in [0,1) \cap [0, \frac{1}{c_0})$ in the following result, where c_0 denotes the least value for which condition (D''') is satisfied in $\mathcal{D}_+$-JS-metric spaces.

2.1. **Coupled fixed point results.** In this section, we extend the results of Berinde [1] which generalize the results of Bhaskar and Lakshmikantham [2]. The contraction condition 1.2 in the setting of $(X^2, \mathcal{D}_+)$ is presented by

$$\mathcal{D}(F(x,y), F(u,v)) + \mathcal{D}(F(y,x), F(v,u)) \leq k[\mathcal{D}(x,u) + \mathcal{D}(y,v)] \tag{2.1}$$

for all $x \geq u; y \leq v$ and $k \in [0,1) \cap [0, \frac{1}{c_0})$. We define an operator $T_F : X^2 \to X^2$ by

$$T_F(x,y) = (F(x,y), F(y,x)) \tag{2.2}$$

for all $(x,y) \in X^2$. Then we can write the contraction condition 2.1 as follows:

$$\mathcal{D}_+(T_F(X), T_F(U)) \leq k\mathcal{D}_+(X,U) \tag{2.3}$$

where $X = (x,y), U = (u,v) \in X^2$ with $x \geq u; y \leq v$ and $k \in [0,1) \cap [0, \frac{1}{c_0})$.

Remark 2.1.1. From the above presentation, it is clear that the coupled fixed point theorem for F reduces to usual Banach fixed point theorem for the operator T_F because F has a coupled fixed point iff T_F has a fixed point.

By the notation $\delta(\mathcal{D}_+, T_F, z_0)$, we define

$$\delta(\mathcal{D}_+, T_F, z_0) = \sup\{\mathcal{D}_+(T_F^i(z_0), T_F^j(z_0)); i, j \in \mathbb{N}\}.$$

The following results are the extended version of the results given in Berinde [1].

Theorem 2.1.2. *Let $F : X^2 :\to X$ be a mapping with mixed monotone property on a partially ordered complete $\mathcal{D}_+$-JS-metric space $(X^2, \mathcal{D}_+)$. Suppose for all $x \geq u; y \leq v$, F satisfies the contraction condition 2.1. If there exists $z_0 = (x_0, y_0) \in X^2$ with the following conditions:*

(1) $x_0 \leq F(x_0, y_0)$ *and* $y_0 \geq F(y_0, x_0)$ *or*
(2) $x_0 \geq F(x_0, y_0)$ *and* $y_0 \leq F(y_0, x_0)$,
(3) $\delta_F(\mathcal{D}, (x_0, y_0)) < \infty$ *and* $\delta_F(\mathcal{D}, (y_0, x_0)) < \infty$,

then there exists a coupled fixed point $\widetilde{z} = (\widetilde{x}, \widetilde{y}) \in X^2$ of F, i.e., $\widetilde{x} = F(\widetilde{x}, \widetilde{y})$; $\widetilde{y} = F(\widetilde{y}, \widetilde{x})$.

Proof. By the hypothesis of the theorem, let us assume, there exists $z_0 = (x_0, y_0) \in X^2$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. We denote $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$ and we also denote

$$F^2(x_0, y_0) = F(F(x_0, y_0), F(y_0, x_0)) = F(x_1, y_1) = x_2;$$

$$F^2(y_0, x_0) = F(F(y_0, x_0), F(x_0, y_0)) = F(y_1, x_1) = y_2.$$

Processing in this way, by the mixed monotone property of F, we get

$$F^n(x_0, y_0) = F(F^{n-1}(x_0, y_0), F^{n-1}(y_0, x_0)) = x_n;$$

$$F^n(y_0, x_0) = F(F^{n-1}(y_0, x_0), F^{n-1}(x_0, y_0)) = y_n.$$

In view of Remark 2.1.1, to prove the existence of a coupled fixed point of F, it is sufficient to establish the existence of a fixed point of the operator T_F given by Equation 4. In order to show this we consider

$$z_1 = (x_1, y_1) = (F(x_0, y_0), F(y_0, x_0)) = T_F(x_0, y_0) = T_F(z_0)$$

and

$$z_2 = (x_2, y_2) = (F^2(x_0, y_0), F^2(y_0, x_0)) = (F(x_1, y_1), F(y_1, x_1)) = T_F(z_1) = T_F^2(z_0).$$

In a similar way, we obtain

$$z_n = (x_n, y_n) = (F^n(x_0, y_0), F^n(y_0, x_0)) = \cdots = T_F^n(z_0)$$

for all $n \in \mathbb{N}$. Hence, (z_n) is a Picard sequence with initial approximation z_0.

Again, due to mixed monotone property of F, it is easy to show that for all $n \geq 0$, $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$. This implies that $z_n \leq z_{n+1}$, i.e., (z_n) is a non-decreasing sequence.

Our next intention is to prove that (z_n) is a Cauchy sequence. Since F satisfies the contraction condition 2.1, for all $n \geq 0$ and $i \leq j$, we get

$$\begin{aligned}
&\mathcal{D}(F^{n+i}(x_0, y_0), F^{n+j}(x_0, y_0)) + \mathcal{D}(F^{n+i}(y_0, x_0), F^{n+j}(y_0, x_0)) \\
&\leq k[\mathcal{D}(F^{n+i-1}(x_0, y_0), F^{n+j-1}(x_0, y_0)) + \mathcal{D}(F^{n+i-1}(y_0, x_0), F^{n+j-1}(y_0, x_0))] \\
&\Rightarrow \mathcal{D}_+(T_F^{n+i}(z_0), T_F^{n+j}(z_0)) \leq k\mathcal{D}_+(T_F^{n-1+i}(z_0), T_F^{n-1+j}(z_0)) \ [by\ 2.3] \\
&\Rightarrow \delta(\mathcal{D}_+, T_F, T_F^n(z_0)) \leq k\delta(\mathcal{D}_+, T_F, T_F^{n-1}(z_0)).
\end{aligned}$$

This is true for all $n \in \mathbb{N}$. Hence for all $i \leq j$, we obtain

$$\begin{aligned}
\mathcal{D}_+(T_F^{n+i}(z_0), T_F^{n+j}(z_0)) &\leq k\delta(\mathcal{D}_+, T_F, T_F^{n-1}(z_0)) \\
&\leq k^2\delta(\mathcal{D}_+, T_F, T_F^{n-2}(z_0)) \\
&\vdots \\
&\leq k^n\delta(\mathcal{D}_+, T_F, z_0). \qquad (2.4)
\end{aligned}$$

Again, we know that

$$\begin{aligned}
\delta(\mathcal{D}_+, T_F, z_0) &= \sup\{\mathcal{D}_+(T_F^i(z_0), T_F^j(z_0)) : i, j \in \mathbb{N}\} \\
&= \sup\{\mathcal{D}(F^i(x_0, y_0), F^j(x_0, y_0)) + \mathcal{D}(F^i(y_0, x_0), F^j(y_0, x_0))\} \\
&= \delta_F(\mathcal{D}, (x_0, y_0)) + \delta_F(\mathcal{D}, (y_0, x_0)).
\end{aligned}$$

As $\delta_F(\mathcal{D}, (x_0, y_0)) < \infty$ and $\delta_F(\mathcal{D}, (y_0, x_0)) < \infty$, so we must have

$$\delta(\mathcal{D}_+, T_F, z_0) < \infty.$$

Employing this in (2.4), for all $m \in \mathbb{N}$, we obtain

$$\begin{aligned}
\mathcal{D}_+(z_n, z_{n+m}) &= \mathcal{D}_+(T_F^n(z_0), T_F^{n+m}(z_0)) \\
&\leq \delta(\mathcal{D}_+, T_F, T_F^n(z_0)) \\
&\leq k^n\delta(\mathcal{D}_+, T_F, z_0) \\
&\to 0\ as\ n \to \infty.
\end{aligned}$$

This implies that (z_n) is a Cauchy sequence. As $(X^2, \mathcal{D}_+)$ is complete, so the sequence (z_n) converges to $\widetilde{z}$ for some $\widetilde{z} = (\widetilde{x}, \widetilde{y}) \in X^2$.

Next, we prove that $\widetilde{z} = (\widetilde{x}, \widetilde{y})$ is a coupled fixed point of F, i.e., a fixed point of T_F. Now,

$$\begin{aligned}&\mathcal{D}_+(z_{n+1}, T_F(\widetilde{z})) = \mathcal{D}_+(T_F(z_n), T_F(\widetilde{z})) \le k\mathcal{D}_+(z_n, \widetilde{z})\\ &\Rightarrow \mathcal{D}_+(z_{n+1}, T_F(\widetilde{z})) = 0 \ as\ n \to \infty\\ &\Rightarrow z_n \to T_F(\widetilde{z}) \ as\ n \to \infty. \end{aligned} \tag{2.5}$$

Since limit of a convergent sequence in this structure is unique, so we must have $\widetilde{z} = T_F(\widetilde{z})$, i.e., $\widetilde{z}$ is a fixed point of T_F. In view of Remark 2.1.1, we can conclude that $\widetilde{z} = (\widetilde{x}, \widetilde{y})$ is a coupled fixed point of F, that is, $\widetilde{x} = F(\widetilde{x}, \widetilde{y})$ and $\widetilde{y} = F(\widetilde{y}, \widetilde{x})$.

Next, we present some additional conditions for the uniqueness of a coupled fixed point of F.

Theorem 2.1.3. *Let $\widetilde{z} = (\widetilde{x}, \widetilde{y})$ and $w = (u, v)$ be two comparable coupled fixed points of F with $\mathcal{D}_+(w, \widetilde{z}) < \infty$. Then $w = \widetilde{z}$.*

Proof. We have

$$\begin{aligned}&\mathcal{D}_+(w, \widetilde{z}) = \mathcal{D}_+(T_F(w), T_F(\widetilde{z})) \le k\mathcal{D}_+(w, \widetilde{z})\\ &\Rightarrow \mathcal{D}_+(w, \widetilde{z}) = 0\\ &\Rightarrow w = \widetilde{z}\\ &\Rightarrow (u, v) = (\widetilde{x}, \widetilde{y}).\end{aligned}$$

Hence the proof follows.

Theorem 2.1.4. *Let $w = (u, v)$ and $\widetilde{z} = (\widetilde{x}, \widetilde{y})$ be two incomparable coupled fixed points of F. Suppose there exists an upper bound or lower bound $z^* = (x^*, y^*) \in X^2$ of w and $\widetilde{z}$ with $\mathcal{D}_+(w, z^*) < \infty$ and $\mathcal{D}_+(\widetilde{z}, z^*) < \infty$. Then $w = \widetilde{z}$.*

Proof. Clearly, for every $n \in \mathbb{N}$, $T_F^n(z^*)$ is comparable to $w = T_F^n(w)$ as well as to $\widetilde{z} = T_F^n(\widetilde{z})$. By the contraction principle 2.3, we obtain

$$\mathcal{D}_+(T_F(w), T_F(z^*)) \le k\mathcal{D}_+(w, z^*),$$

and

$$\mathcal{D}_+(T_F^2(w), T_F^2(z^*)) \le k\mathcal{D}_+(T_F(w), T_F(z^*)) \le k^2\mathcal{D}_+(w, z^*).$$

Proceeding in this way, one can obtain,

$$\mathcal{D}_+(T_F^n(w), T_F^n(z^*)) \le k^n\mathcal{D}_+(w, z^*). \tag{2.6}$$

By using the axioms of $\mathcal{D}_+$-JS-metric spaces and the above inequality, we have

$$\mathcal{D}_+(w, T_F^n(z^*)) \le c \limsup \mathcal{D}_+(T_F^n(w), T_F^n(z^*)) \le k^n c\mathcal{D}_+(w, z^*).$$

Since, $\mathcal{D}_+(w, z^*) < \infty$ and $0 \le k < 1$, $\mathcal{D}_+(w, T_F^n(z^*)) \to 0$, whenever $n \to \infty$. This implies that the sequence $(T_F^n(z^*))$ converges to w.

Analogously, it can be proved that the sequence $(T_F^n(z^*))$ also converges to $\widetilde{z}$. In view of Proposition 2.2, we must have $\widetilde{z} = w$, that is, $(\widetilde{x}, \widetilde{y}) = (u, v)$.

Next, we are interested in finding additional conditions for the equality of the components of a coupled fixed point. In order to show this we consider the following conditions:

(C1) Let $(\widetilde{x}, \widetilde{y})$ be a coupled fixed point of F such that $\widetilde{x}$ and $\widetilde{y}$ are comparable in X with $\mathcal{D}(\widetilde{x}, \widetilde{y}) < \infty$.

(C2) Suppose every pair of elements $x, y \in X$ has either an upper bound or a lower bound $w \in X$ with $\mathcal{D}(x,w) < \infty, \mathcal{D}(y,w) < \infty$, $\mathcal{D}(x,y) < \infty$ and $\mathcal{D}(w,w) < \infty$.

(C3) Let x_0, y_0 be comparable in X with $\mathcal{D}(x_0, y_0) < \infty$.

Theorem 2.1.5. *By adding any of the above mentioned conditions with the hypotheses of Theorem 2.1.2, one can derive the equality of the components of coupled fixed point.*

Proof. We prove this theorem in the following steps:

Step I. Suppose the condition (C1) is satisfied along with the hypotheses of Theorem 2.1.2. We consider $X = (\widetilde{x}, \widetilde{y})$ and $U = (\widetilde{y}, \widetilde{x})$. Using the contraction principle in Theorem 2.1.2, we get,

$$\begin{aligned}
&\Rightarrow \mathcal{D}(F(\widetilde{x},\widetilde{y}), F(\widetilde{y},\widetilde{x})) + \mathcal{D}(F(\widetilde{y},\widetilde{x}), F(\widetilde{x},\widetilde{y})) \leq k(\mathcal{D}(\widetilde{x},\widetilde{y}) + \mathcal{D}(\widetilde{y},\widetilde{x}))\\
&\Rightarrow \mathcal{D}(F(\widetilde{x},\widetilde{y}), F(\widetilde{y},\widetilde{x})) \leq k\mathcal{D}(\widetilde{x},\widetilde{y})\\
&\Rightarrow \mathcal{D}(\widetilde{x},\widetilde{y}) \leq k\mathcal{D}(\widetilde{x},\widetilde{y})\\
&\Rightarrow \mathcal{D}(\widetilde{x},\widetilde{y}) = 0, \ i.e., \ \widetilde{x} = \widetilde{y}.
\end{aligned}$$

Step II. Suppose the condition (C2) is satisfied along with the hypotheses of Theorem 2.1.2 and $(\widetilde{x}, \widetilde{y})$ is coupled fixed point of F such that $\widetilde{x}$, $\widetilde{y}$ are not comparable. Let $\widetilde{w} \in X$ be an upper bound of $\widetilde{x}$ and $\widetilde{y}$ with $\mathcal{D}(\widetilde{x},\widetilde{w}) < \infty$, $\mathcal{D}(\widetilde{y},\widetilde{w}) < \infty$, $\mathcal{D}(\widetilde{x},\widetilde{y}) < \infty$ and $\mathcal{D}(\widetilde{w},\widetilde{w}) < \infty$. Then, $\widetilde{x} \leq \widetilde{w}$ and $\widetilde{y} \leq \widetilde{w}$. With respect to partial order in (X^2, D_+), we must have that

$$(\widetilde{x},\widetilde{y}) \geq (\widetilde{x},\widetilde{w}); (\widetilde{x},\widetilde{w}) \leq (\widetilde{w},\widetilde{x}); (\widetilde{w},\widetilde{x}) \geq (\widetilde{y},\widetilde{x}).$$

Let us consider $X = (\widetilde{x},\widetilde{y})$ and $U = (\widetilde{x},\widetilde{w})$. Since X, U are comparable, so from contraction conditions 2.1 and 2.3, we get

$$\begin{aligned}
&\mathcal{D}(F(\widetilde{x},\widetilde{y}), F(\widetilde{x},\widetilde{w})) + \mathcal{D}(F(\widetilde{y},\widetilde{x}), F(\widetilde{w},\widetilde{x})) \leq k[\mathcal{D}(\widetilde{x},\widetilde{x}) + \mathcal{D}(\widetilde{y},\widetilde{w})]\\
&\Rightarrow \mathcal{D}_+(T_F(X), T_F(U)) \leq k[\mathcal{D}(\widetilde{x},\widetilde{x}) + \mathcal{D}(\widetilde{y},\widetilde{w})].
\end{aligned}$$

In view of Proposition 2.3, we have $\mathcal{D}(\widetilde{x},\widetilde{x}) = 0$ and hence we have

$$\mathcal{D}_+(T_F(X), T_F(U)) \leq k\mathcal{D}(\widetilde{y},\widetilde{w}). \tag{2.7}$$

Now, since $X = (\widetilde{x},\widetilde{y})$ is a fixed point of T_F, so $T_F^n(X) = X$ for all $n \in \mathbb{N}$. Hence, Equation 2.7 is reduced to

$$\begin{aligned}
&\mathcal{D}_+(T_F^n(X), T_F^n(U)) \leq k^n \mathcal{D}_+(X,U)\\
&\Rightarrow \mathcal{D}_+(X, T_F^n(U)) \leq k^n \mathcal{D}(\widetilde{y},\widetilde{w})\\
&\Rightarrow \mathcal{D}_+(X, T_F^n(U)) = 0
\end{aligned} \tag{2.8}$$

as $n \to \infty$ and $\mathcal{D}(\widetilde{y},\widetilde{w}) < \infty$. This implies that the sequence $(T_F^n(U))$ converges to X.

Next, we consider that $Y = (\widetilde{y},\widetilde{x})$ and $V = (\widetilde{y},\widetilde{w})$. Then obviously Y, V are comparable and hence we get

$$\begin{aligned}
\mathcal{D}_+(T_F(Y), T_F(V)) &\leq k\mathcal{D}_+(Y,V)\\
&\leq k[\mathcal{D}(\widetilde{y},\widetilde{y}) + \mathcal{D}(\widetilde{x},\widetilde{w})]\\
&\leq k\mathcal{D}(\widetilde{x},\widetilde{w}).
\end{aligned}$$

In a similar fashion, we have

$$\begin{aligned}&\mathcal{D}_+(T_F^n(Y),T_F^n(V))\le k^n\mathcal{D}_+(Y,V)\\&\Rightarrow \mathcal{D}_+(Y,T_F^n(V))\le k^n\mathcal{D}(\widetilde{x},\widetilde{w})\\&\Rightarrow \mathcal{D}_+(Y,T_F^n(V))=0\end{aligned}\tag{2.9}$$

as $n\to\infty$ and $\mathcal{D}(\widetilde{x},\widetilde{w})<\infty$. This implies that the sequence $(T_F^n(V))$ converges to Y. Again, since U and V are comparable, then using the contraction principle 2.3, we yield

$$\begin{aligned}&\mathcal{D}_+(T_F^n(U),T_F^n(V))\le k^n\mathcal{D}_+(U,V)\\&\Rightarrow \mathcal{D}_+(T_F^n(U),T_F^n(V))\le k^n\{\mathcal{D}(\widetilde{x},\widetilde{y})+\mathcal{D}(\widetilde{w},\widetilde{w})\}\\&\Rightarrow \mathcal{D}_+(T_F^n(U),T_F^n(V))=0\end{aligned}\tag{2.10}$$

as $n\to\infty$ and $\mathcal{D}(\widetilde{x},\widetilde{y})<\infty$, $\mathcal{D}(\widetilde{w},\widetilde{w})<\infty$. By the axioms of $\mathcal{D}_+$-JS-metric space along with (2.8), (2.9) and (2.10), there exists $c>0$ such that

$$\begin{aligned}&\mathcal{D}_+(X,Y)\le c\limsup\mathcal{D}_+(T_F^n(U),T_F^n(V))=0\\&\Rightarrow X=Y\\&\Rightarrow(\widetilde{x},\widetilde{y})=(\widetilde{y},\widetilde{x})\\&\Rightarrow \widetilde{x}=\widetilde{y}.\end{aligned}$$

Alternatively, one can find the equality of components of a couple fixed point by taking $\widetilde{w}\in X$ as a lower bound of $\widetilde{x}$ and $\widetilde{y}$ with $\mathcal{D}(\widetilde{x},\widetilde{w})<\infty$, $\mathcal{D}(\widetilde{y},\widetilde{w})<\infty$, $\mathcal{D}(\widetilde{x},\widetilde{y})<\infty$ and $\mathcal{D}(\widetilde{w},\widetilde{w})<\infty$.

Step III. Suppose the condition (C3) is satisfied along with the hypotheses of Theorem 2.1.2. Due to mixed monotone property of F, for each $n\ge 1$, $x_n=F(x_{n-1},y_{n-1})$ and $y_n=F(y_{n-1},x_{n-1})$ are also comparable and $x_n\to\widetilde{x}$ and $y_n\to\widetilde{y}$ as $n\to\infty$. By the axioms of JS-metric spaces, we obtain

$$\mathcal{D}(\widetilde{x},\widetilde{y})\le c\limsup\mathcal{D}(x_n,y_n).\tag{2.11}$$

Again, by taking $X=(x_n,y_n)$ and $U=(y_n,x_n)$ in the contraction condition of Theorem 2.1.2, for all $n\ge 0$, we get

$$\begin{aligned}&\mathcal{D}(F(x_n,y_n),F(y_n,x_n))\le k\mathcal{D}(x_n,y_n)\\&\Rightarrow \mathcal{D}(x_{n+1},y_{n+1})\le k\mathcal{D}(x_n,y_n).\end{aligned}\tag{2.12}$$

Using Inequalities 2.11 and 2.12, we must have

$$\mathcal{D}(\widetilde{x},\widetilde{y})\le c\limsup\mathcal{D}(x_n,y_n)\le\limsup k^n c\mathcal{D}(x_0,y_0)=0$$

as $n\to\infty$. This implies that $\mathcal{D}(\widetilde{x},\widetilde{y})=0$. Hence, we obtain $\widetilde{x}=\widetilde{y}$.

As Berinde's theorems are extended versions of the main results of Bhaskar and Lakshmikantham [2], we can deduce immediate consequences of the above results which are the sharpened versions of the main results given in Bhaskar and Lakshmikantham [2].

The improved version of Theorem 2.1 in Bhaskar and Lakshmikantham [2] is given as follows:

Corollary 2.1.6. *Let $F : X^2 \to X$ be a mapping with mixed monotone property on X. Assume that there exists $k \in [0,1) \cap [0, \frac{1}{c_0})$ such that*

$$\mathcal{D}(F(x,y), F(u,v)) \leq \frac{k}{2}\mathcal{D}_+((x,y),(u,v))$$

for $x \geq u; y \leq v$. If there exist $x_0, y_0 \in X$ such that

(A) $x_0 \leq F(x_0, y_0); y_0 \geq F(y_0, x_0)$;
(B) $\delta_F(\mathcal{D}, (x_0, y_0)) < \infty$ *and* $\delta_F(\mathcal{D}, (y_0, x_0)) < \infty$,

then there exist $x, y \in X$ such that $x = F(x,y); y = F(y,x)$.

Remark 2.1.7. The authors of [2] considered two alternative hypotheses to establish the existence of coupled fixed points. These are: either the function F is continuous or if (x_n) and (y_n) are non-decreasing and non-increasing sequences respectively with $x_n \to x$ and $y_n \to y$, then $x_n \leq x$; $y_n \geq y$ for all $n \in \mathbb{N}$. But the Corollary 2.1.6 ensures the existence of coupled fixed points without assuming any of the above mentioned hypotheses.

Remark 2.1.8. As every b-metric space is a JS-metric space with $c = s \geq 1$ in Definition 1.1, so one can obtain the coupled fixed point results in partially ordered b-metric space from our obtained results. In particular, one can deduce the coupled fixed point result due to Bota et al. [3] (Theorem 2.2 in [3]) from Corollary 2.1.6 directly.

Remark 2.1.9. The uniqueness of a coupled fixed point and the equality of the component of a coupled fixed point of F in Corollary 2.1.6 are guaranteed by the Theorem 2.1.3, 2.1.4 and 2.1.5 respectively.

If we replace the distance function '$\mathcal{D}_+$' on X^2 by $\mathcal{D}_m$', then we can also prove the existence of coupled fixed point. In this direction, we present the following theorem.

Theorem 2.1.10. *Let $F : X^2 \to X$ be a mapping with mixed monotone property on X. Assume that there exists $k \in [0,1)$ such that*

$$\mathcal{D}_m((F(x,y), F(y,x)), (F(u,v), F(v,u)) \leq k\mathcal{D}_m((x,y),(u,v))$$

for $x \geq u; y \leq v$. If there exist $x_0, y_0 \in X$ with

(A) $x_0 \leq F(x_0, y_0); y_0 \geq F(y_0, x_0)$;
(B) $\delta_F(\mathcal{D}, (x_0, y_0)) < \infty$ *and* $\delta_F(\mathcal{D}, (y_0, x_0)) < \infty$,

then F has a coupled fixed point $(x, y) \in X^2$, that is, $x = F(x,y)$ and $y = F(y,x)$.

Proof. Proof is almost similar to the proof of Theorem 2.1.2. Hence, we skip the proof.

Finally, we furnish some examples which substantiate our obtained results.

Example 2.1.11. Let $X = [0,1]$ and we define the distance function as

$$\mathcal{D}(x,y) = |x| + |y|.$$

At first, we prove that $(X, \mathcal{D})$ is a JS-metric space. In order to prove this, we check the axioms of JS-metric spaces.

(1) $\mathcal{D}(x,y) = 0 \Rightarrow |x| + |y| = 0 \Rightarrow |x| = |y| = 0$, i.e., $x = y = 0$.
(2) Clearly, $\mathcal{D}(x,y) = \mathcal{D}(y,x)$.

(3) Let (x_n) be a sequence converging to some x in X. Then for any $y \in X$, we have $\mathcal{D}(x,y) = |x| + |y|$. Again, $\mathcal{D}(x_n, y) = |x_n| + |y|$ and

$$\limsup \mathcal{D}(x_n, y) = \limsup(|x_n| + |y|) = |x| + |y|.$$

So, we can always find some $c \geq 1$ such that $\mathcal{D}(x,y) \leq c \limsup \mathcal{D}(x_n, y)$.

All the axioms are satisfied. Hence, $(X, \mathcal{D})$ is a JS-metric space. Now, we consider the metric space $(X^2, \mathcal{D}_+)$, where,

$$\mathcal{D}_+((x,y),(u,v)) = \mathcal{D}(x,u) + \mathcal{D}(y,v).$$

It is clear that $(X^2, \mathcal{D}_+)$ is a $\mathcal{D}_+$-JS metric space. Next, we consider a function $F : X^2 \to X$ defined by

$$F(x,y) = \begin{cases} 0, & \text{whenever } x = 0; \\ \dfrac{(2x - y^2)}{5}, & \text{otherwise.} \end{cases}$$

Then we check the axioms of Theorem (2.1.2)

(1) Let $x_1 \leq x_2$. Then for all $y \in X$, we have $2x_1 - y^3 \leq 2x_2 - y^3$ which implies that $F(x_1, y) \leq F(x_2, y)$, i.e., F is monotonic non-decreasing in its 1st component. Again, for all $x \in X$ with $x \neq 0$, whenever $y_1 \leq y_2$, we get $2x - y_1^3 \geq 2x - y_2^3$ which shows that $F(x, y_1) \geq F(x, y_2)$ and for $x = 0$, $F(x, y_1) = F(x, y_2) = 0$ for all $y_1, y_2 \in X$. So for all $x \in X$, $F(x, y_1) \geq F(x, y_2)$ with $y_1 \leq y_2$, i.e., F is monotonic non-increasing function in its 2nd component. Thus F has mixed monotone property.

(2) Let $(x_0, y_0) = (1, 0)$. Then,

$$x_1 = F(x_0, y_0) = F(1, 0) = \frac{2}{5} \leq x_0 = 1$$

and

$$y_1 = F(y_0, x_0) = F(0, 1) = 0 \geq y_0 = 0.$$

It is clear that for all $n \in \mathbb{N}$, $(F^n(x_0, y_0))$ is a decreasing sequence converging to 0 and $(F^n(y_0, x_0))$ is a constant sequence. Hence, $\delta_F(\mathcal{D}, (x_0, y_0)) < \infty$ and $\delta_F(\mathcal{D}, (y_0, x_0)) < \infty$.

(3) Now we prove that F satisfies the contraction condition 2.1.
Let $(x,y), (u,v) \in X^2$ with $x \geq u$ and $y \leq v$.

(a) Suppose, $x = u = 0$, then for all $y, v \in X$, $\mathcal{D}(F(x,y), F(u,v)) = 0$. Hence,

$$\begin{aligned} \mathcal{D}(F(x,y), F(u,v)) + \mathcal{D}(F(y,x), F(v,u)) &= \mathcal{D}(F(y,x), F(v,u)) \\ &\leq \frac{|2y| + |2v|}{5} \\ &\leq \frac{2}{5} D_+((x,y),(u,v)). \end{aligned}$$

(b) For $y = v = 0$,

$$\mathcal{D}(F(x,y), F(u,v)) + \mathcal{D}(F(y,x), F(v,u)) \leq \frac{2}{5} \mathcal{D}_+((x,y),(u,v)).$$

(c) Analogously, for $u = 0, x \neq 0$ and $y = 0, v \neq 0$,

$$\mathcal{D}(F(x,y), F(u,v)) + \mathcal{D}(F(y,x), F(v,u)) \leq \frac{2}{5}\mathcal{D}_+((x,y),(u,v)).$$

(d) Let $(x,y),(u,v) \in X^2$ with $x \geq u$ and $y \leq v$ and $u \neq 0, y \neq 0$. Then

$$\begin{aligned}\mathcal{D}(F(x,y), F(u,v)) + \mathcal{D}(F(y,x), F(v,u)) &= \frac{|2x - y^3|}{5} + \frac{|2u - v^3|}{5} \\ &\quad + \frac{|2y - x^3|}{5} + \frac{|2v - u^3|}{5} \\ &\leq \frac{3}{5}(|x| + |y| + |u| + |v|) \\ &\leq \frac{3}{5}\mathcal{D}_+((x,y),(u,v)).\end{aligned}$$

From the above illustrations, it is clear that F satisfies the contraction condition 2.1.

Thus F satisfies all the conditions of Theorem 2.1.2. Hence, F has a coupled fixed point. Note that $(0,0)$ is the unique coupled fixed point of F.

Next, we construct another example in support of Corollary 2.1.6.

Example 2.1.12. Let us consider $X = \mathbb{R} \cup \{\infty, -\infty\}$ and we define the distance function $\mathcal{D}$ on X as $\mathcal{D}(x,y) = |x| + |y|$ for all $x, y \in X$. From previous example, it is clear that $(X, \mathcal{D})$ is a JS-metric space and so $(X^2, \mathcal{D}_+)$ is a $\mathcal{D}_+$-JS metric space. Let us define a function $F : X^2 \to X$ by

$$F(x,y) = \frac{x - y}{3}, \ \forall x, y \in X.$$

Then,

(i) Let $x_1 \leq x_2$. Then for all $y \in X$, we have $x_1 - y \leq x_2 - y$ which implies that $F(x_1, y) \leq F(x_2, y)$, i.e., F is monotonic non-decreasing in its 1st component. Again, for all $x \in X$, whenever $y_1 \leq y_2$, we get $x - y_1 \geq x - y_2$ which shows that $F(x, y_1) \geq F(x, y_2)$. So F is monotonic non-increasing function in its 2nd component. Thus F has mixed monotone property.

(ii) Let $(x,y),(u,v) \in X^2$. Then,

$$\begin{aligned}\mathcal{D}(F(x,y), F(u,v)) &= \frac{|x - y|}{3} + \frac{|u - v|}{3} \\ &\leq \frac{1}{3}(|x| + |u|) + \frac{1}{3}(|y| + |v|) \\ &\leq \frac{2}{3}\frac{\mathcal{D}_+((x,y),(u,v))}{2}.\end{aligned}$$

This shows that F satisfies the contraction condition.

(iii) Let us set $x_0 = -3$ and $y_0 = 2$. Then,

$$x_1 = F(x_0, y_0) = F(-3, 2) = \frac{-5}{3} > x_0 = -3$$

and

$$y_1 = F(y_0, x_0) = F(2, -3) = \frac{5}{3} < y_0 = 2.$$

Again, it is easy to show that for all $i, j \in \mathbb{N}$, $\delta_F(\mathcal{D}, (x_0, y_0)) < \infty$ and $\delta_F(\mathcal{D}, (y_0, x_0)) < \infty$.

Thus all the conditions of the Corollary 2.1.6 are satisfied. Therefore F has a coupled fixed point. Here, $(0, 0)$ is a coupled fixed point of F. Notice that this is not unique since $(\infty, -\infty)$ is also a coupled fixed point of F.

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