# THE FIXED POINT PROPERTY FOR CLOSED NEIGHBORHOODS OF LINE SEGMENTS IN $L^{p}$ 

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#### Abstract

We prove that, in $L^{p}$-spaces with $p \in(1, \infty]$, closed neighborhoods of line segments are dismantlable and hence every monotone operator on these neighborhoods has a fixed point. We also give an example that, for $p=1$, closed neighborhoods of line segments need not be dismantlable. It is an open question whether every monotone self map of a closed neighborhood of a line segment in $L^{1}$ has a fixed point. Key Words and Phrases: Dismantlable ordered set, fixed point property, line segment, closed $L^{p}$-neighborhood. 2010 Mathematics Subject Classification: 06A07, 46B42, 47H07, 47H10.


## 1. Introduction

There are at least three types of fixed point results/theories that involve ordered structures. The general investigation of the fixed point property for ordered sets, that is, sets equipped with a reflexive, antisymmetric and transitive order relation $\leq$, started to come into its own with the publication of [8]. This investigation of the fixed point property is used as one of the driving forces to introduce structures in ordered sets in [13]. Although this theory has results for infinite ordered sets, its primary focus lies on finite ordered sets. In analysis, function spaces are endowed with a natural pointwise (almost everywhere) order. It is natural to use this order to prove fixed point theorems. This type of work is very well described in [5]. Finally, there is the use of order in metric fixed point theory. This work was recently summarized in [6] and much of it seems to ultimately rely on graphs, not order. Monotonicity continues to be useful in metric fixed point theory, see [3], [4], though it is usually paired with other conditions that are essential to guarantee existence of a fixed point.

To date, the connection between the fixed point property for ordered sets in general, which, by its very nature, relies exclusively on monotonicity and no additional conditions, and the use of order in analysis or metric fixed point theory appears to be quite minimal. In fact, except for the Abian-Brown Theorem, see [1, 2], papers
on the fixed point property in ordered sets rarely mention papers used in the application of order in analysis or metric fixed point theory and vice versa. Part of the separation is quite natural: A theory that primarily focuses on finite structures, such as the theory surrounding the fixed point property for ordered sets, is naturally separate from a theory in which the relevant underlying structures are infinite, such as analysis or metric fixed point theory. The notion of dismantlability, see Definition 3.7, which is fundamental in the fixed point theory for finite ordered sets, can be defined for infinite ordered sets as well. However, experience (see [7], [11]) shows that, although dismantlability is a reasonably simple notion, finding natural examples of infinite dismantlable ordered sets is quite nontrivial. A natural use of dismantlability in analysis is the alternative proof for Theorem 2.44 in [5] that is presented here in Lemma 6.2. Apparently, dismantlability has not been used in analysis beyond what could be considered an implicit use in Theorem 2.44 in [5].

In this paper, we present the first explicit application of dismantlability to analysis/metric fixed point theory that cannot be substituted by another approach as we prove that, for $p \in(1, \infty]$, closed $\rho$-neighborhoods of line segments in $L^{p}(\Omega)$ are dismantlable, see Theorems 8.2 and 8.4.

From the point-of view of analysis/metric fixed point theory, this means that any monotone operator on these neighborhoods must have a fixed point. That is, unlike for results that apply to more general convex, closed, bounded subsets of Banach spaces, such as, say, Theorems 3.3 and 3.6 in [3], no further conditions on the operator are required to guarantee the existence of a fixed point.

From the point of view of finite fixed point theory, Theorems 8.2 and 8.4 present, after [7], the second natural class of infinite dismantlable ordered sets. We also show that, for $p=1$, closed $\rho$-neighborhoods of line segments need not be dismantlable, see Example 9.3. Instead, a related class of ordered sets is dismantlable, see Theorem 10.4, which leads to the result that any monotone operator from a closed $\rho$-neighborhood of a line segment in $L^{1}(\Omega)$ to a closed $(\rho-\varepsilon)$-neighborhood of the line segment has a fixed point, see Corollary 10.5.

We conclude with a discussion of open questions that suggest that the application of methods from discrete fixed point theory to analysis/metric fixed point theory is only at its beginning. To keep the paper self-contained, all proofs are included. Readers who do not wish to re-read all exact definitions regarding ordered normed spaces can assume to work with $L^{p}$-spaces throughout.

## 2. Chain-Completeness

Let $E$ be an ordered normed space, that is, an ordered vector space with an order relation $\leq$ and a norm $\|\cdot\|$ such that $0 \leq x \leq y$ implies $0 \leq\|x\| \leq\|y\|$. For a review of the relevant notions, see [5]. The central focus of this paper is on neighborhoods of line segments, that is, on subsets of the following form.
Definition 2.1. Let $E$ be an ordered normed space, let $f, g \in E$ and let $\rho>0$. We define the closed $\rho$-neighborhood of the line segment from $f$ to $g$ to be

$$
L_{f, g, \rho}:=\{u \in E:(\exists t \in[0,1])\|u-[f+t(g-f)]\| \leq \rho\}
$$

When $f=0$, we will write $L_{g, \rho}$ for $L_{0, g, \rho}$.

Because the translation $T u:=u-f$ preserves the order and maps $L_{f, g, \rho}$ to $L_{g-f, \rho}=$ $L_{0, g-f, \rho}$, we will focus on line segments that start at the origin throughout this paper.

We first prove that $L_{g, \rho}$ is chain-complete as long as the underlying space has the property
(E0) Bounded and monotone sequences have (weak or) strong limits.
We recall the relevant definitions along the way.
Definition 2.2. Let $C$ be an ordered set. Then $C$ is called a chain or linearly ordered if and only if, for all $x, y \in C$, we have that $x \leq y$ or $y \leq x$.

For the definition of chain-completeness, we also need the idea of suprema and infima.
Definition 2.3. Let $P$ be an ordered set and let $A \subseteq P$. Then $u \in P$ is called an upper bound of $A$, also denoted by $u \geq A$, if and only if, for all $a \in A$, we have that $a \leq u$. Moreover, $s \in P$ is called the supremum or lowest upper bound of $A$ if and only if $s$ is an upper bound of $A$ and, for all upper bounds $u$ of $A$, we have $s \leq u$. The supremum of $A$ is also denoted $\bigvee A$.

Dually, $\ell \in P$ is called a lower bound of $A$, also denoted by $\ell \leq A$, if and only if, for all $a \in A$, we have that $a \geq \ell$. Moreover, $i \in P$ is called the infimum or greatest lower bound of $A$ if and only if $i$ is a lower bound of $A$ and, for all lower bounds $\ell$ of $A$, we have $i \geq \ell$. The infimum of $A$ is also denoted $\bigwedge A$.

We can now state the definition of chain-completeness.
Definition 2.4. Let $P$ be an ordered set. Then $P$ is called chain-complete if and only if each nonempty subchain $C \subseteq P$ has a supremum and an infimum.
Definition 2.5. Let $P$ be an ordered set and let $C \subseteq P$ be a chain. Then $K \subseteq C$ is called cofinal if and only if $\{p \in P: p \geq K\}=\{p \in P: p \geq C\}$.
Lemma 2.6. Let $E$ be an ordered normed space that satisfies (EO), let $u>0$ and let $C \subseteq E$ be a chain in $E$ such that, for all $c \in C$, we have $\|c\| \leq u$. Then $C$ has a supremum $d$ in $E$ and $d$ is the (weak or strong) limit of a countable cofinal subchain of $C$.
Proof. Because we can pick an element $c_{0} \in C$ and consider the chain $\left\{c-c_{0}: c \in\right.$ $\left.C, c \geq c_{0}\right\}$ instead of $C$, we can assume, without loss of generality, that $c \geq 0$ for all $c \in C$. There is nothing to prove if $C$ has a largest element, so we can assume that $C$ does not have a largest element.

Because $\|c\| \leq u$ for all $c \in C$, we have that $s:=\sup _{c \in C}\|c\|$ exists. For each $n \in \mathbb{N}$, choose a $c_{n} \in C$ such that $\left\|c_{n}\right\| \geq s-\frac{1}{n}$. Without loss of generality, we can assume that $c_{1} \leq c_{2} \leq c_{3} \leq \cdots$. Then, for each $c \in C$, there is an $n_{c} \in \mathbb{N}$ such that $\|c\|<\left\|c_{n_{c}}\right\|$ and hence $c<c_{n_{c}}$.

By (E0), the sequence $\left\{c_{n}\right\}_{n=1}^{\infty}$ has a (weak or strong) limit $d \in E$. Then $d \geq c$ for all $c \in C$.

Now let $v$ be any upper bound of $C$. Then, for all $n \in \mathbb{N}$, we have $v-c_{n} \geq 0$ and hence $v-d \geq 0$, that is $v \geq d$.
Lemma 2.7. Let $E$ be an ordered normed space that satisfies (E0), let $g \in E$ and let $\rho>0$. Then $L_{g, \rho}$ is chain-complete.
Proof. Let $C \subseteq L_{g, \rho}$ be a nonempty chain. For every $c \in C$, there is a $t_{c} \in[0,1]$ such that $\left\|c-t_{c} g\right\| \leq \rho$. Hence, for every $c \in C$, we have that $\|c\| \leq\left\|c-t_{c} g\right\|+\left\|t_{c} g\right\| \leq$
$\rho+\|g\|$. By Lemma 2.6, $C$ has a supremum in $E$ that is the (weak or strong) limit of a countable cofinal subchain of $C$. Because $L_{g, \rho}$ is closed, the supremum of $C$ is in $L_{g, \rho}$.

Existence of the infimum of $C$ in $L_{g, \rho}$ is proved dually.
Definition 2.8. Let $E$ be a Banach lattice. The norm of $E$ is called ordercontinuous iff every monotone order-bounded sequence converges.

So, in particular, in a Banach lattice with order-continuous norm, we have that $L_{g, \rho}$ is chain-complete.

The importance of chain-completeness lies in the well-known Abian-Brown Theorem.
Theorem 2.9. (The Abian-Brown Theorem, see [2].) Let P be a chain-complete ordered set and let $T: P \rightarrow P$ be order-preserving, that is, $x \leq y$ implies $T(x) \leq T(y)$. If there is a $p \in P$ with $p \leq T(p)$, then $T$ has a fixed point above $p$. Moreover, there is a unique smallest fixed point of $T$ above $p$.

## 3. Comparative Retractions and Dismantlability

Note that the Abian-Brown Theorem holds for ordered sets without any additional algebraic or analytical structure. In this section, we will focus entirely on ordered sets in general.
Definition 3.1. An ordered set $P$ has the fixed point property if and only if every order-preserving self map $T: P \rightarrow P$ has a fixed point $x=T(x)$.

Retractions are a standard tool when investigating fixed point properties for morphisms of any kind. Note that, in an ordered set $P$, any subset $Q \subseteq P$ is an ordered set with the induced order $\leq_{Q}:=\leq\left.\right|_{Q \times Q}$ obtained by restricting $\leq$ to $Q$.
Definition 3.2. Let $P$ be an ordered set. Then an order-preserving function $r: P \rightarrow$ $P$ is called $a$ retraction if and only if $r^{2}=r$. The ordered subset $r[P]$ is called $a$ retract of $P$.
Proposition 3.3. Let $P$ be an ordered set with the fixed point property and let $r: P \rightarrow P$ be a retraction. Then $r[P]$ has the fixed point property.
Proof. (The following standard argument applies to any fixed point property for morphisms and to retractions that are morphisms of the same type.) Let $P$ have the fixed point property and let $T: r[P] \rightarrow r[P]$ be order-preserving. Then $T \circ r$ : $P \rightarrow P$ is order-preserving, too, and hence it has a fixed point $x=T(r(x))$. Now $x \in T[r[P]] \subseteq r[P]$ implies $x=r(x)$. Therefore, $x=T(r(x))=T(x)$ is a fixed point of $T$. (Note that the argument only used that the function $r$ is idempotent and a morphism of the same type as $T$.)

In [8], Rival exhibited a situation in which the fixed point property for the retract implies the fixed point property for the surrounding ordered set. Theorem 3.6 below is the natural generalization of Rival's seminal result to chain-complete ordered sets. Definition 3.4. Let $P$ be an ordered set and let $p, q \in P$. We will write $p \sim q$ if and only if $p \leq q$ or $p \geq q$.
Definition 3.5. Let $P$ be an ordered set. A retraction $r: P \rightarrow P$ is called a comparative retraction if and only if, for all $p \in P$, we have $r(p) \sim p$. It is called an up-retraction if and only if, for all $p \in P$, we have $r(p) \geq p$, and, it is called a down-retraction if and only if, for all $p \in P$, we have $r(p) \leq p$.

Theorem 3.6. (Compare with Proposition 1 in [8].) Let $P$ be a chain-complete ordered set and let $r: P \rightarrow P$ be a comparative retraction. Then $P$ has the fixed point property if and only if $r[P]$ has the fixed point property.
Proof. The direction " $\Rightarrow$ " follows from Proposition 3.3.
For the converse " $\Leftarrow$ ", let $r[P]$ have the fixed point property and let $T: P \rightarrow P$ be order-preserving. Then $\left.r \circ T\right|_{r[P]}$ is an order-preserving self map of $r[P]$. Because $r[P]$ has the fixed point property, there is an $x \in r[P]$ such that $\left.r \circ T\right|_{r[P]}(x)=x$. If $T(x)=x$, then $x$ is a fixed point of $T$. In case $T(x) \neq x$, note that, because $r$ is a comparative retraction, we have that $x=\left.r \circ T\right|_{r[P]}(x)$ is comparable to $\left.T\right|_{r[P]}(x)=$ $T(x)$. By the Abian-Brown Theorem (see Theorem 2.9) or its dual, $T$ must have a fixed point.

Theorem 3.6 can be applied repeatedly until a certain subset is reached or until there are no more comparative retractions. This is the idea of dismantlability.
Definition 3.7. Let $P$ be a chain-complete ordered set. Then we say that $P$ is dismantlable to $Q \subseteq P$ if and only if there are subsets $P=P_{0} \supseteq P_{1} \supseteq \cdots \supseteq P_{n}=Q$ and comparative retractions $r_{i}: P_{i-1} \rightarrow P_{i}$ such that $r_{i}\left[P_{i-1}\right]=P_{i}$. If $Q$ is a singleton, we will simply call $P$ dismantlable.

Clearly, if a chain-complete ordered set is dismantlable to an ordered set with the fixed point property, then the original ordered set has the fixed point property, too.
Corollary 3.8. Let $P$ be a chain-complete ordered set that is dismantlable to an ordered set $Q$. Then $P$ has the fixed point property if and only if $Q$ has the fixed point property. In particular, dismantlable chain-complete ordered sets have the fixed point property.
Proof. This result follows from Theorem 3.6 if we can establish that retracts of chaincomplete ordered sets are chain-complete.

So let $P$ be a chain-complete ordered set, let $r: P \rightarrow r[P]$ be a retraction and let $C \subseteq r[P]$ be a nonempty chain. Then $C$ has a supremum $s$ in $P$. Because $r$ is order-preserving, $r(s)$ is an upper bound of $C=r[C]$. Let $u \in r[P]$ be an upper bound of $C$. Because $s$ is the supremum of $C$ in $P$, we infer $u \geq s$.

Because $r$ is order-preserving, $u=r(u) \geq r(s)$ follows. Hence $r(s)$ is the supremum of $C$ in $r[P]$.

Existence of the infimum is established dually. Hence $r[P]$ is chain-complete.
There may be more than one comparative retraction on a given ordered set, which leads to different ways to dismantle an ordered set. Hence, a given ordered set $P$ can usually be dismantled to different ordered subsets $Q_{i}$. When considering the fixed point property, the dismantling process is typically run until there is no nontrivial comparative retraction on the remaining ordered set. It can be shown that, if we use comparative retractions on chain-complete ordered sets and the process terminates after finitely many steps, this resulting ordered set is unique up to isomorphism (see [12]).

## 4. A Note on Computing Fixed Points

Aside from establishing the existence of a fixed point, in analysis, it is also important to compute the fixed point if it exists. This facet of fixed point theory is virtually nonexistent in the investigation of the fixed point property for finite ordered sets.

In chain-complete ordered sets $P$, papers as early as [1] provide a transfinite iterative scheme to compute a fixed point for $T: P \rightarrow P$ once we have a $u$ such that $T u \sim u$. For chain-complete subsets $P$ of ordered normed spaces, iterative schemes that take at most countably many steps are given in [5]. Thus, to compute a fixed point of an operator $T: P \rightarrow P$ on a dismantlable ordered set $P$, we could proceed as follows. For $i=1, \ldots, n$, define $T_{i}:=\left.r_{i} \circ r_{i-1} \circ \cdots \circ r_{1} \circ T\right|_{P_{i}}$ and let $T_{0}:=T$. Because $P_{n}$ is a singleton, $T_{n}$ has a fixed point. Now let $i \in\{1, \ldots, n\}$ be so that $T_{i}$ has a fixed point $u_{i}$. Because $r_{i}$ is a comparative retraction, $T_{i-1} u_{i} \sim u_{i}$. Now we can run an iterative scheme with $T_{i-1}$ and $u_{i}$ to obtain a fixed point $u_{i-1}$ of $T_{i-1}$. Thus, after the execution of $n$ iterative schemes as mentioned above, we have computed a fixed point of $T$.

As simple as this idea may be, its value is mainly theoretical. The iterative schemes to compute a fixed point of an operator $T$ from a $u \sim T u$ are infinite, which makes even their finite iteration transfinite. Thus, for applications of the results in this paper in which the computation of the fixed point is important, it would be interesting to determine if there is a way to shorten the scheme indicated here to a scheme that can compute a fixed point, or at least a close approximation, in finite time.

$$
\text { 5. DISMANTLABILITY OF } L_{g, \rho} \text { TO } r \leq g^{+} r_{\geq-g^{-}}\left[L_{g, \rho}\right]
$$

The first step in establishing whether $L_{g, \rho}$ is dismantlable is to dismantle $L_{g, \rho}$ to a nicer subset. Comparative retractions that map to the supremum or infimum with a fixed element will be our primary tool throughout.
Lemma 5.1. Let $E$ be an ordered normed space, let $A \subseteq E$ and let $h \in E$ be so that, for all $u \in A$, we have that $u \vee h$ exists and is in $A$. Then $r_{\geq h}(u):=u \vee h$ is a comparative retraction on $A$. The dual result holds for the dually defined function $r \leq h(u):=u \wedge h$.
Proof. Because $u \leq v$ implies that $u \vee h \leq v \vee h$, we conclude that $r_{\geq h}$ is orderpreserving. Because $(u \vee h) \vee h=u \vee h$, we conclude that $r_{\geq h}$ is idempotent. Because $u \vee h \geq u$, we conclude that $r_{\geq h}$ is a comparative retraction.

Recall that $g^{+}=g \vee 0$ and that $g^{-}=(-g) \vee 0$.
Lemma 5.2. Let $E$ be a Banach lattice that satisfies (E0), let $g \in E$ and let $\rho>0$. Then $L_{g, \rho}$ is dismantlable to $r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right] \subseteq L_{g, \rho}$.
Proof. Because $E$ satisfies (E0), by Lemma 2.7, $L_{g, \rho}$ is chain-complete. We will prove that, first, $L_{g, \rho}$ dismantles via $r_{\geq-g^{-}}$to $r_{\geq-g^{-}}\left[L_{g, \rho}\right]$ and then, that $r_{\geq-g^{-}}\left[L_{g, \rho}\right]$ dismantles via $r \leq g^{+}$to $r \leq g^{+} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$.

First we prove that $r_{\geq-g^{-}}\left[\bar{L}_{g, \rho}\right] \subseteq L_{g, \rho}$. Let $u \in L_{g, \rho}$ and let $t_{u} \in[0,1]$ be so that $\left\|u-t_{u} g\right\| \leq \rho$. Then

$$
t_{u} g \geq\left(t_{u} g\right) \wedge 0=\left(t_{u} g\right) \wedge\left(t_{u} 0\right)=t_{u}(g \wedge 0)=t_{u}\left(-g^{-}\right) \geq-g^{-}
$$

Hence

$$
\left\|u \vee\left(-g^{-}\right)-t_{u} g\right\|=\left\|u \vee\left(-g^{-}\right)-\left(t_{u} g\right) \vee\left(-g^{-}\right)\right\| \leq\left\|u-t_{u} g\right\| \leq \rho
$$

which shows that $r_{\geq-g^{-}}\left[L_{g, \rho}\right] \subseteq L_{g, \rho}$.

Next we prove that $r \leq g^{+}\left[r_{\geq-g^{-}}\left[L_{g, \rho}\right]\right] \subseteq L_{g, \rho}$. Let $u \in r_{\geq-g^{-}}\left[L_{g, \rho}\right]$ and let $t_{u} \in$ $[0,1]$ be so that $\left\|u-t_{u} g\right\| \leq \rho$. Then

$$
t_{u} g \quad \leq \quad\left(t_{u} g\right) \vee 0=\left(t_{u} g\right) \vee\left(t_{u} 0\right)=t_{u}(g \vee 0)=t_{u} g^{+} \leq g^{+}
$$

Hence

$$
\left\|u \wedge g^{+}-t_{u} g\right\|=\left\|u \wedge g^{+}-t_{u} g \wedge g^{+}\right\| \leq\left\|u-t_{u} g\right\| \leq \rho
$$

which shows that $r^{\leq g^{+}}\left[r_{\geq-g^{-}}\left[L_{g, \rho}\right]\right] \subseteq L_{g, \rho}$. Finally, every $u \in r_{\geq-g^{-}}\left[L_{g, \rho}\right]$ satisfies $u \geq-g^{-}$and we also have $g^{+}=g \vee 0 \geq g \wedge 0=-g^{-}$. Hence, for every $u \in r_{\geq-g^{-}}\left[L_{g, \rho}\right]$, we have $r \leq g^{+}(u)=u \wedge g^{+} \in r_{\geq-g^{-}}\left[L_{g, \rho}\right]$, which shows that $r \leq g^{+}\left[r_{\geq-g^{-}}\left[L_{g, \rho}\right]\right] \subseteq$ $r_{\geq-g^{-}}\left[L_{g, \rho}\right]$.

## 6. Existence of Order-Centers in $L_{g, \rho} \subseteq L^{p}(\Omega)$

The proof of Lemma 5.2 shows that, for every $u \in L_{g, \rho}$ the supremum $u \vee\left(-g^{-}\right)$ exists and is in $L_{g, \rho}$. If, in addition $-g^{-} \in L_{g, \rho}$, then $-g^{-}$is called a sup-center of $L_{g, \rho}$.
Definition 6.1. Let $P$ be an ordered set and let $c \in P$.
(1) The point $c \in P$ is called a sup-center of $P$ iff, for all $p \in P$, the supremum $c \vee p$ exists in $P$.
(2) The point $c \in P$ is called an inf-center of $P$ iff, for all $p \in P$, the infimum $c \wedge p$ exists in $P$.
(3) The point $c \in P$ is called an order-center of $P$ iff $c$ is a sup-center or an inf-center of $P$.
Order-centers have been used to establish fixed point theorems in [5]. In the language of the fixed point property, this can be seen as a consequence of the following result.
Lemma 6.2. (Compare with Theorem 2.44 in [5].) Let $E$ be an ordered set and let $A \subseteq E$ be chain-complete with a sup-center or an inf-center. Then $A$ is dismantlable. In particular, $A$ has the fixed point property.
Proof. Let $c \in A$ be an inf-center. Then, for all $u \in A$, we have that $u \wedge c \in A$ and hence $r_{\leq c}$ is a comparative retraction on $A$. Now all elements of $r_{\leq c}[A]$ are less than or equal to $c$ and $r(u):=c$ is a comparative retraction from $r_{\leq c}[A]$ to $\{c\}$.

Lemma 6.2 guarantees the fixed point property for chain-complete ordered sets with an order-center. Unfortunately, $L_{g, \rho}$ does not always have an order-center. In the following, we characterize when, in the space of $p$-integrable functions $L^{p}(\Omega)$ $(1 \leq p<\infty)$ over a measure space $(\Omega, \Sigma, \mu)$, a set $L_{g, \rho}$ has an order-center.
Proposition 6.3. Let $p \in[1, \infty), g \in L^{p}(\Omega)$ and let $\rho>0$. Then $L_{g, \rho}$ has a sup-center iff $-g^{-} \in L_{g, \rho}$. Moreover, when this is the case, $-g^{-}$is a sup-center of $L_{g, \rho}$.
Proof. First, consider the direction " $\Leftarrow$ :" The proof of Lemma 5.2 shows that $r_{\geq-g^{-}}\left[L_{g, \rho}\right] \subseteq L_{g, \rho}$, that is, for all $u \in L_{g, \rho}$, we have that $u \vee\left(-g^{-}\right) \in L_{g, \rho}$. Hence, if $-g^{-} \in L_{g, \rho}$, then $-g^{-}$is a sup-center for $L_{g, \rho}$. This also establishes the "moreover" part.

For the direction " $\Rightarrow$ ", let $c \in L_{g, \rho}$ be a sup-center of $L_{g, \rho}$.

First consider the element $a:=\left(1+\frac{\rho}{\left\|g^{+}\right\|_{p}}\right) g^{+}-g^{-}$. It is easy to see that $\|a-g\|_{p}=\rho$, which means that $a \in L_{g, \rho}$. Therefore $s^{+}:=c \vee a$ is in $L_{g, \rho}$ and satisfies $s^{+} \geq a$. Suppose, for a contradiction, that $s^{+}>a$. Then we would have the following for all $t \in[0,1]$. [As is customary, the indicator function of a set $A$ will be denoted $\mathbf{1}_{A}$.]

$$
\begin{aligned}
\left\|s^{+}-t g\right\|_{p} & =\left\|s^{+} \mathbf{1}_{\{x \in \Omega: g(x) \geq 0\}}+s^{+} \mathbf{1}_{\{x \in \Omega: g(x)<0\}}-t g^{+}+t g^{-}\right\|_{p} \\
& =\left\|\left[s^{+} \mathbf{1}_{\{x \in \Omega: g(x) \geq 0\}}-t g^{+}\right]+\left[s^{+} \mathbf{1}_{\{x \in \Omega: g(x)<0\}}+t g^{-}\right]\right\|_{p} \\
& \geq\left\|\left[s^{+} \mathbf{1}_{\{x \in \Omega: g(x) \geq 0\}}-t g^{+}\right]\right\|_{p} \\
& \geq\left\|\left(1+\frac{\rho}{\left\|g^{+}\right\|_{p}}\right) g^{+}-t g^{+}\right\|_{p} \\
& =\left(1-t+\frac{\rho}{\left\|g^{+}\right\|_{p}}\right)\left\|g^{+}\right\|_{p} \\
& =(1-t)\left\|g^{+}\right\|_{p}+\rho
\end{aligned}
$$

which is greater than $\rho$ for $t \in[0,1)$ and equal to $\rho$ for $t=1$. Moreover, for $t=1$, we have

$$
s^{+}-1 \cdot g>a-g=\left(1+\frac{\rho}{\left\|g^{+}\right\|_{p}}\right) g^{+}-g^{-}-g^{+}+g^{-}=\frac{\rho}{\left\|g^{+}\right\|_{p}} g^{+} \geq 0
$$

which implies that

$$
\left\|s^{+}-1 \cdot g\right\|_{p}>\left\|\frac{\rho}{\left\|g^{+}\right\|_{p}} g^{+}\right\|_{p}=\rho
$$

and we have arrived at a contradiction to $s^{+} \in L_{g, \rho}$. Therefore, $s^{+}=a$ and hence

$$
c \leq a=\left(1+\frac{\rho}{\left\|g^{+}\right\|_{p}}\right) g^{+}-g^{-} .
$$

Now consider the element $b:=\frac{\rho}{\left\|g^{-}\right\|_{p}} g^{-}$. Because $\|b\|=\rho$, we have that $b \in L_{g, \rho}$. Therefore, $s^{-}:=c \vee b$ is in $L_{g, \rho}$ and satisfies $s^{-} \geq b$. Suppose, for a contradiction, that $s^{-}>b$. Then we would have the following for all $t \in[0,1]$.

$$
\begin{aligned}
\left\|s^{-}-t g\right\|_{p} & =\left\|s^{-} \mathbf{1}_{\{x \in \Omega: g(x)<0\}}+s^{-} \mathbf{1}_{\{x \in \Omega: g(x) \geq 0\}}-t g^{+}+t g^{-}\right\|_{p} \\
& =\left\|\left[s^{-} \mathbf{1}_{\{x \in \Omega: g(x)<0\}}+t g^{-}\right]+\left[s^{-} \mathbf{1}_{\{x \in \Omega: g(x) \geq 0\}}-t g^{+}\right]\right\|_{p} \\
& \geq\left\|\frac{\rho}{\left\|g^{-}\right\|_{p}} g^{-}+t g^{-}\right\|_{p} \\
& =\left(\frac{\rho}{\left\|g^{-}\right\|_{p}}+t\right)\left\|g^{-}\right\|_{p}
\end{aligned}
$$

which is greater than $\rho$ for $t \in(0,1]$ and equal to $\rho$ for $t=0$. Moreover, for $t=0$, we have

$$
\left\|s^{-}-0 \cdot g\right\|_{p}=\left\|s^{-}\right\|_{p}>\left\|\frac{\rho}{\left\|g^{-}\right\|_{p}} g^{-}\right\|_{p}=\rho
$$

and we have arrived at a contradiction to $s^{-} \in L_{g, \rho}$. Therefore, $s^{-}=b$ and hence

$$
c \leq b=\frac{\rho}{\left\|g^{-}\right\|_{p}} g^{-}
$$

In summary,

$$
\begin{aligned}
c & \leq\left[\left(1+\frac{\rho}{\left\|g^{+}\right\|_{p}}\right) g^{+}+\left(-g^{-}\right)\right] \wedge\left(\frac{\rho}{\left\|g^{-}\right\|_{p}} g^{-}\right) \\
& \leq\left[\left(1+\frac{\rho}{\left\|g^{+}\right\|_{p}}\right) g^{+}\right] \wedge\left(\frac{\rho}{\left\|g^{-}\right\|_{p}} g^{-}\right)+\left(-g^{-}\right) \wedge\left(\frac{\rho}{\left\|g^{-}\right\|_{p}} g^{-}\right) \\
& =0+\left(-g^{-}\right)=-g^{-}
\end{aligned}
$$

Let $t \in[0,1]$ be so that $\|c-t g\|_{p} \leq \rho$. Then $t g+(-c) \geq t g+g^{-}>0$ and hence

$$
\left\|\left(-g^{-}\right)-t g\right\|_{p}=\left\|t g-\left(-g^{-}\right)\right\|_{p}=\left\|t g+g^{-}\right\|_{p} \leq\|t g+(-c)\|_{p}=\|c-t g\|_{p} \leq \rho
$$

which shows that $-g^{-} \in L_{g, \rho}$.
Proposition 6.4. Let $p \in[1, \infty), g \in L^{p}(\Omega), \rho>0$ and let

$$
m_{g}:=\min _{t \in[0,1]} t^{p}\left\|g^{+}\right\|_{p}^{p}+(1-t)^{p}\left\|g^{-}\right\|_{p}^{p}
$$

Then the following are equivalent.
(1) $g^{+} \in L_{g, \rho}$ and $-g^{-} \in L_{g, \rho}$.
(2) $L_{g, \rho}$ has a sup-center and an inf-center.
(3) $L_{g, \rho}$ has a sup-center or an inf-center.
(4) $\rho^{p} \geq m_{g}$.

Proof. First note the following. Because, for any $u$, the function $t \mapsto\|u-t g\|_{p}^{p}$ is continuous, this function assumes an absolute minimum on the compact interval $[0,1]$. Now

$$
\begin{aligned}
\min _{t \in[0,1]}\left\|\left(-g^{-}\right)-t g\right\|_{p}^{p} & =\min _{t \in[0,1]}\left\|t g+g^{-}\right\|_{p}^{p} \\
& =\min _{t \in[0,1]}\left\|t g^{+}+(1-t) g^{-}\right\|_{p}^{p} \\
& =\min _{t \in[0,1]} t^{p}\left\|g^{+}\right\|_{p}^{p}+(1-t)^{p}\left\|g^{-}\right\|_{p}^{p}=m_{g}
\end{aligned}
$$

and

$$
\begin{aligned}
\min _{t \in[0,1]}\left\|g^{+}-t g\right\|_{p}^{p} & =\min _{t \in[0,1]}\left\|(1-t) g^{+}+t g^{-}\right\|_{p}^{p} \\
& =\min _{t \in[0,1]}(1-t)^{p}\left\|g^{+}\right\|_{p}^{p}+t^{p}\left\|g^{-}\right\|_{p}^{p} \\
& =\min _{t_{*} \in[0,1]} t_{*}^{p}\left\|g^{+}\right\|_{p}^{p}+\left(1-t_{*}\right)^{p}\left\|g^{-}\right\|_{p}^{p}=m_{g}
\end{aligned}
$$

Therefore $g^{+} \in L_{g, \rho}$ iff $\rho^{p} \geq m_{g}$, and, $-g^{-} \in L_{g, \rho}$ iff $\rho^{p} \geq m_{g}$. This proves the equivalence of 1 and 4.

By Proposition 6.3 and its dual, 1 implies 2. Moreover, 2 trivially implies 3. Finally, if 3 is true, then, by Proposition 6.3 and its dual, we have that $-g^{-} \in L_{g, \rho}$ or $g^{+} \in L_{g, \rho}$, which implies, $\rho^{p} \geq m_{g}$. Therefore 3 implies 4 .

Proposition 6.4 characterizes when, in $L^{p}(\Omega)(1 \leq p<\infty)$, a set $L_{g, \rho}$ has an ordercenter. When $L_{g, \rho}$ does not have an order-center, Theorem 2.44 in [5] does not apply, so we need a different tool to establish the fixed point property. Dismantlability, which, as the proof of Lemma 6.2 shows, is a weaker condition than the existence of an order-center, will be this tool for $p \in(1, \infty]$. The author conjectures that retracts of ordered sets with a sup-center will have a sup-center, too, but was unable to prove it. (The problem is establishing the existence of the appropriate suprema.) For our purposes, establishing dismantlability will suffice.
Lemma 6.5. Let $P$ be an ordered set with a sup-center. Then every retract of $P$ is dismantlable.
Proof. Let $c \in P$ be a sup-center of $P$ and let $r: P \rightarrow P$ be a retraction. Define $f: r[P] \rightarrow r[P]$ by $f(x):=r(x \vee c)$. Clearly, $f$ is order-preserving and, for all $x \in r[P]$, we have that $f(x)=r(x \vee c) \geq r(x)=x$ and $f(x)=r(x \vee c) \geq r(c)=f(c)$.

Define $r_{1}: r[P] \rightarrow r[P]$ by, for every $x \in r[P]$, setting $r_{1}(x)$ to be the, by Theorem 2.9 guaranteed to exist, smallest fixed point of $f$ above $x$. By definition, for all $x \in r[P]$, we have $r_{1}(x) \geq x$. Because $x \leq y$ implies that the smallest fixed point of $f$ above $y$ is a fixed point of $f$ above $x, r_{1}$ is order-preserving. Because $r_{1}$ maps fixed points of $f$ to fixed points of $f, r_{1}$ is a retraction. Finally, because, for all $x \in r[P]$, we have $f(x) \geq f(c)$, we conclude that $r_{1}(x)=r_{1}(f(x)) \geq r_{1}(f(c))=r_{1}(c)$. In summary, the function $r_{1}$ is an up-retraction from $r[P]$ to $\left\{z \in r_{1}[r[P]]: z \geq r_{1}(c)\right\}$, which is dismantlable. Therefore, $r[P]$ is dismantlable.

## 7. Lemmas in Banach Lattices

The key idea for the proof is to dismantle $L_{g, \rho}$ successively to subsets of the form $r^{\leq(\varepsilon g)^{+}} r_{\geq-(\varepsilon g)^{-}}\left[L_{\varepsilon g, \rho}\right]$ and to, once $\varepsilon$ is small enough so that $\left\|-(\varepsilon g)^{-}\right\|_{p} \leq \rho$, apply Proposition 6.3 (in conjunction with Lemma 6.5). To not unnecessarily duplicate proofs for $p<\infty$ and for $p=\infty$, we prove as many lemmas as possible for Banach lattices. Note that Proposition 6.4 shows, in particular, that the only interesting case remaining is that of $g^{+} \neq 0$ and $g^{-} \neq 0$.
Lemma 7.1. Let $E$ be a Banach lattice that satisfies (E0), let $g \in E$ be so that $g^{+} \neq 0$ and $g^{-} \neq 0$, let $\rho>0$, and assume that there is a $t^{*} \in(0,1)$ such that, for all $u \in r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$, there is a $t_{u}^{*} \in\left[0, t^{*}\right]$ such that $\left\|f-t_{u}^{*} g\right\|_{p} \leq \rho$. Then $r_{\geq-t^{*} g^{-}}$ is a comparative retraction from $r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$ to $r^{\leq g^{+}} r_{\geq-t^{*} g^{-}}\left[L_{g, \rho}\right]$ and $r^{\leq t^{*} g^{+}}{ }^{\text {a }}$ is a comparative retraction from $r \leq g^{+} r_{\geq-t^{*} g^{-}}\left[L_{g, \rho}\right]$ to $r \leq t^{*} g^{+} r_{\geq-t^{*} g^{-}}\left[L_{g, \rho}\right]$. Moreover,

$$
r^{\leq t^{*} g^{+}} r_{\geq-t^{*} g^{-}}\left[L_{g, \rho}\right]=r^{\leq\left(t^{*} g\right)^{+}} r_{\geq-\left(t^{*} g\right)^{-}}\left[L_{t^{*} g, \rho}\right]
$$

Proof. Let $u \in r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$. We need to prove that $r_{\geq-t^{*} g^{-}}(u) \in L_{g, \rho}$. To this end, let $t_{u}^{*} \in\left[0, t^{*}\right]$ be so that $\left\|u-t_{u}^{*} g\right\|_{p} \leq \rho$. Then the following holds.

$$
\begin{aligned}
\left\|r_{\geq-t^{*} g^{-}}(u)-t_{u}^{*} g\right\| & =\left\|u \vee\left(-t^{*} g^{-}\right)-t_{u}^{*} g\right\| \\
& =\left\|u \vee\left(-t^{*} g^{-}\right)-t_{u}^{*} g \vee\left(-t^{*} g^{-}\right)\right\| \\
& \leq\left\|u-t_{u}^{*} g\right\| \leq \rho
\end{aligned}
$$

Therefore, $r_{\geq-t^{*} g^{-}}(u) \in L_{g, \rho}$ and it is greater than or equal to $-t^{*} g^{-}$and less than or equal to $g^{+}$. Hence $r_{\geq-t^{*} g^{-}}$is a comparative retraction from $r \leq g^{+} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$ to $r_{\geq-t^{*} g^{-}}\left[r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]\right]=r^{\leq g^{+}} r_{\geq-t^{*} g^{-}}\left[L_{g, \rho}\right] \subseteq L_{g, \rho}$. The proof that $r^{\leq t^{*} g^{+}}$is a comparative retraction from $r^{\leq g^{+}} r_{\geq-t^{*} g^{-}}\left[L_{g, \rho}\right]$ to $r^{\leq t^{*} g^{+}} r_{\geq-t^{*} g^{-}}\left[L_{g, \rho}\right]$ is similar.

For the "moreover" part, first note that $t^{*} g^{+}=\left(t^{*} g\right)^{\mp}$ and $-t^{*} g^{-}=-\left(t^{*} g\right)^{-}$. Now let $u \in r^{\leq t^{*} g^{+}} r_{\geq-t^{*} g^{-}}\left[L_{g, \rho}\right]$. By assumption, there is a $t_{u}^{*} \in\left[0, t^{*}\right]$ such that $\left\|u-t_{u}^{*} g\right\|_{p} \leq \rho$. Hence, with $t_{u}^{\prime}:=\frac{t_{u}^{*}}{t^{*}}$, we have $\rho \geq\left\|u-t_{u}^{*} g\right\|_{p}=\left\|u-t_{u}^{\prime}\left(t^{*} g\right)\right\|_{p}$. Thus $r \leq t^{*} g^{+} r_{\geq-t^{*} g^{-}}\left[L_{g, \rho}\right] \subseteq r^{\leq\left(t^{*} g\right)^{+}} r_{\geq-\left(t^{*} g\right)^{-}}\left[L_{t^{*} g, \rho}\right]$. The reverse inequality follows from the fact that $L_{t^{*} g, \rho} \subseteq L_{g, \rho}$.
Lemma 7.2. Let $E$ be a Banach lattice that satisfies (EO), let $g \in E$ be so that $g^{+} \neq 0$ and $g^{-} \neq 0$, let $\rho>0$, and assume that there are $t_{0}, t_{*} \in(0,1)$ such that, for all $t \in$ $\left[t_{0}, 1\right]$ and all $u \in r^{\leq(t g)^{+}} r_{\geq-(t g)^{-}}\left[L_{t g, \rho}\right]$, there is a $t_{u}^{*} \in\left[0, t_{*}\right]$ such that $\left\|u-t_{u}^{*}(t g)\right\|_{p} \leq$ $\rho$. Then $r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$ is dismantlable to $r \leq\left(t_{*} t_{0} g\right)^{+} r_{\geq-\left(t_{*} t_{0} g\right)^{-}}\left[L_{t_{*} t_{0} g, \rho}\right]$.
Proof. Let $t_{1}:=\max \left\{t_{*}, t_{0}\right\}$. By hypothesis, for $t=1 \in\left[t_{0}, 1\right]$ and all $u \in$ $r^{\leq(t g)^{+}} r_{\geq-(t g)^{-}}\left[L_{t g, \rho}\right]=r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$, there is a $t_{u}^{*} \in\left[0, t_{*}\right] \subseteq\left[0, t_{1}\right]$ such that $\left\|u-t_{u}^{*} g\right\|_{p} \leq \rho$. Therefore, by Lemma 7.1, $r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$ is dismantlable to $r^{\leq\left(t_{1} g\right)^{+}} r_{\geq-\left(t_{1} g\right)^{-}}\left[L_{t_{1} g, \rho}\right]$.

Inductively, if $t_{n}>t_{0}$, assume that $t_{n}=t_{*}^{n}$ and $r \leq g^{+} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$ is dismantlable to $r^{\leq\left(t_{n} g\right)^{+}} r_{\geq-\left(t_{n} g\right)^{-}}\left[L_{t_{n} g, \rho}\right]$. Let $t_{n+1}:=\max \left\{t_{*}^{n+1}, t_{0}\right\}$ and let $t_{n+1}^{*}:=\frac{t_{n+1}}{t_{n}} \geq t_{*}$. By hypothesis, for all $u \in r^{\leq\left(t_{n} g\right)^{+}} r_{\geq-\left(t_{n} g\right)^{-}}\left[L_{t_{n} g, \rho}\right]$, there is a $t_{u}^{*} \in\left[0, t_{*}\right] \subseteq\left[0, t_{n+1}^{*}\right]$ such that $\left\|u-t_{u}^{*}\left(t_{n} g\right)\right\|_{p} \leq \rho$. Therefore, by Lemma 7.1, $r \leq\left(t_{n} g\right)^{+} r_{\geq-\left(t_{n} g\right)^{-}}\left[L_{t_{n} g, \rho}\right]$ is dismantlable to $r^{\leq\left(t_{n+1}^{*} t_{n} g\right)^{+}} r_{\geq-\left(t_{n+1}^{*} t_{n} g\right)^{-}}\left[L_{t_{n+1}^{*} t_{n} g, \rho}\right]=r^{\leq\left(t_{n+1} g\right)^{+}} r_{\geq-\left(t_{n+1} g\right)^{-}}\left[L_{t_{n+1} g, \rho}\right]$. Moreover, if $t_{n+1}>t_{0}$, then $t_{n+1}=t_{*}^{n+1}$.

Because $t_{*}^{n} \rightarrow 0$ as $n \rightarrow \infty$, this induction stops at some $n \in \mathbb{N}$ with $t_{n}=t_{0}$. Hence $r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$ is dismantlable to $r^{\leq\left(t_{0} g\right)^{+}} r_{\geq-\left(t_{0} g\right)^{-}}\left[L_{t_{0} g, \rho}\right]$.

By hypothesis, for all $u \in r^{\leq\left(t_{0} g\right)^{+}} r_{\geq-\left(t_{0} g\right)^{-}}\left[L_{t_{0} g, \rho}\right]$, there is a $t_{u}^{*} \in\left[0, t_{*}\right]$ such that $\left\|u-t_{u}^{*}\left(t_{0} g\right)\right\|_{p} \leq \rho$. Therefore, by Lemma 7.1, $r^{\leq\left(t_{0} g\right)^{+}} r_{\geq-\left(t_{0} g\right)^{-}}\left[L_{t_{0} g, \rho}\right]$ is dismantlable to $r^{\leq\left(t_{*} t_{0} g\right)^{+}} r_{\geq-\left(t_{*} t_{0} g\right)^{-}}\left[L_{t_{*} t_{0} g, \rho}\right]$, which proves the claim.

## 8. Dismantlability of $L_{g, \rho}$ FOR $p>1$

We can now prove that the subset $L_{g, \rho}$ of $L^{p}(\Omega)$ is dismantlable for $p>1$. Interestingly enough, it need not be dismantlable when $p=1$, as we will see in Example 9.3. We first focus on $p \in(1, \infty)$.

Lemma 8.1. Let $p \in(1, \infty)$, let $g \in L^{p}(\Omega)$ be so that $g^{+} \neq 0$ and $g^{-} \neq 0$, let $\rho>0$, and let $k>\rho$ be so that

$$
\frac{2}{k^{p-1}} \frac{1}{\rho} \int_{\Omega}|g|^{p} d \mu<1
$$

Then, for every $u \in r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$, there is a $t_{u}^{*} \in\left[0,1-\frac{\rho}{k}\right]$ such that

$$
\left\|u-t_{u}^{*} g\right\|_{p} \leq \rho
$$

Proof. Let $u \in r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$, let

$$
N:=\{x \in \Omega: u(x)<0\}
$$

and let

$$
P:=\{x \in \Omega: u(x) \geq 0\}
$$

The function $t \mapsto\|u-t g\|_{p}^{p}$ is differentiable on $(-\infty, \infty)$. Consider its derivative.

$$
\begin{aligned}
& \frac{d}{d t}\|u-t g\|_{p}^{p} \\
&= \frac{d}{d t} \int_{\Omega}|u-t g|^{p} d \mu \\
&= \int_{\Omega} \frac{d}{d t}|u-t g|^{p} d \mu \\
&= \int_{\Omega} \operatorname{sgn}(u-t g) p|u-t g|^{p-1}(-g) d \mu \\
&= \int_{N} \operatorname{sgn}(u-t g) p|u-t g|^{p-1}|g| d \mu-\int_{P} \operatorname{sgn}(u-t g) p|u-t g|^{p-1}|g| d \mu \\
&= \int_{\{x \in N: u(x) \geq t g(x)\}} p|u-t g|^{p-1}|g| d \mu-\int_{\{x \in N: u(x)<t g(x)\}} p|u-t g|^{p-1}|g| d \mu \\
&-\int_{\{x \in P: u(x)>t g(x)\}} p|u-t g|^{p-1}|g| d \mu+\int_{\{x \in P: u(x) \leq t g(x)\}} p|u-t g|^{p-1}|g| d \mu \\
&= \int_{\{x \in \Omega:|u(x)| \leq t|g(x)|\}} p|u-t g|^{p-1}|g| d \mu-\int_{\{x \in \Omega:|u(x)|>t|g(x)|\}} p|u-t g|^{p-1}|g| d \mu
\end{aligned}
$$

Let $m:=\inf \{s \geq 0: \mu\{x \in \Omega:|u(x)| \leq s|g(x)|\}>0\}$. The function

$$
t \mapsto \int_{\{x \in \Omega:|u(x)| \leq t|g(x)|\}} p|u-t g|^{p-1}|g| d \mu
$$

is increasing on $\mathbb{R}$ and strictly increasing on $[m, \infty)$. Let

$$
M:=\sup \{s \leq 1: \mu\{x \in \Omega:|u(x)|>s|g(x)|\}>0\}
$$

The function

$$
t \mapsto \int_{\{x \in \Omega:|u(x)|>t|g(x)|\}} p|u-t g|^{p-1}|g| d \mu
$$

is decreasing on $\mathbb{R}$ and strictly decreasing on $(-\infty, M]$. Because $m \leq M$, the derivative $\frac{d}{d t}\|u-t g\|_{p}^{p}$ is strictly increasing. Because the derivative assumes negative as well as positive values, we conclude that $t \mapsto\|u-t g\|_{p}^{p}$ assumes a unique absolute minimum on the real numbers and, because the derivative is continuous, said absolute minimum is assumed when the following equality holds.

$$
\int_{\{x \in \Omega:|u(x)| \leq t|g(x)|\}} p|u-t g|^{p-1}|g| d \mu=\int_{\{x \in \Omega:|u(x)|>t|g(x)|\}} p|u-t g|^{p-1}|g| d \mu .
$$

Because both sides are nonnegative and because the left side is zero for $t=0$ and the right side is zero for $t=1$, the absolute minimum is assumed at a $t_{u} \in[0,1]$. Because $u \in L_{g, \rho}$, we have $\left\|u-t_{u} g\right\|_{p} \leq \rho$.

Now let $k$ be so that

$$
\frac{2}{k^{p-1}} \frac{1}{\rho} \int_{\Omega}|g|^{p} d \mu<1
$$

Then, for all $u \in L_{g, \rho}$ such that $t_{u}>1-\frac{\rho}{k}$, we have the following. [Note that, because $u \leq g^{+}$and $u \geq-g^{-}$, the functions $u$ and $g$ will never have opposite signs.]

$$
\begin{aligned}
& \int_{\left\{x \in \Omega:|u(x)| \leq\left(1-\frac{\rho}{k}\right)|g(x)|\right\}}\left|u-\left(1-\frac{\rho}{k}\right) g\right|^{p-1}|g| d \mu \\
& \quad<\int_{\left\{x \in \Omega:|u(x)|>\left(1-\frac{\rho}{k}\right)|g(x)|\right\}}\left|u-\left(1-\frac{\rho}{k}\right) g\right|^{p-1}|g| d \mu \\
& \quad=\int_{\left\{x \in \Omega:|u(x)|>\left(1-\frac{\rho}{k}\right)|g(x)|\right\}}| | u\left|-\left(1-\frac{\rho}{k}\right)\right| g| |^{p-1}|g| d \mu \\
& \quad \leq \int_{\left\{x \in \Omega:|u(x)|>\left(1-\frac{\rho}{k}\right)|g(x)|\right\}}| | g\left|-\left(1-\frac{\rho}{k}\right)\right| g| |^{p-1}|g| d \mu \\
& \quad=\left(\frac{\rho}{k}\right)^{p-1} \int_{\Omega}|g|^{p} d \mu
\end{aligned}
$$

Again, because $u$ and $g$ never have opposite signs and because $|u| \leq|g|$, we have that

$$
\left|u-\left(1-\frac{\rho}{k}\right) g\right| \leq|g|
$$

Therefore, we obtain the following.

$$
\begin{aligned}
\| u- & \left(1-\frac{\rho}{k}\right) g \|_{p}^{p} \\
= & \int_{\left\{x \in \Omega:|u(x)| \leq\left(1-\frac{\rho}{k}\right)|g(x)|\right\}}\left|u-\left(1-\frac{\rho}{k}\right) g\right|^{p} d \mu \\
& +\int_{\left\{x \in \Omega:|u(x)|>\left(1-\frac{\rho}{k}\right)|g(x)|\right\}}\left|u-\left(1-\frac{\rho}{k}\right) g\right|^{p} d \mu \\
\leq & \int_{\left\{x \in \Omega:|u(x)| \leq\left(1-\frac{\rho}{k}\right)|g(x)|\right\}}\left|u-\left(1-\frac{\rho}{k}\right) g\right|^{p-1}|g| d \mu \\
& +\int_{\left\{x \in \Omega:|u(x)|>\left(1-\frac{\rho}{k}\right)|g(x)|\right\}}\left|u-\left(1-\frac{\rho}{k}\right) g\right|^{p-1}|g| d \mu \\
\leq & \left(\frac{\rho}{k}\right)^{p-1} \int_{\Omega}|g|^{p} d \mu+\left(\frac{\rho}{k}\right)^{p-1} \int_{\Omega}|g|^{p} d \mu \\
= & \rho^{p} \frac{2}{k^{p-1}} \frac{1}{\rho} \int_{\Omega}|g|^{p} d \mu \\
< & \rho^{p}
\end{aligned}
$$

Thus, for all $u \in r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$, we have that there is a $t_{u}^{*} \leq 1-\frac{\rho}{k}$ such that $\left\|u-t_{u}^{*} g\right\|_{p} \leq \rho$.
Theorem 8.2. Let $p \in(1, \infty)$, let $g \in L^{p}(\Omega)$ be so that $g^{+} \neq 0$ and $g^{-} \neq 0$ and let $\rho>0$. Then $L_{g, \rho}$ is dismantlable.

Proof. Let $k>\rho$ be so that

$$
\frac{2}{k^{p-1}} \frac{1}{\rho} \int_{\Omega}|g|^{p} d \mu<1
$$

Then, for all $t \in[0,1]$, we have that

$$
\frac{2}{k^{p-1}} \frac{1}{\rho} \int_{\Omega}|t g|^{p} d \mu \leq \frac{2}{k^{p-1}} \frac{1}{\rho} \int_{\Omega}|g|^{p} d \mu<1
$$

Let $n \in \mathbb{N}$ be so that

$$
\left(1-\frac{\rho}{k}\right)^{n}<\frac{\rho}{\|g\|_{p}}
$$

By Lemma 8.1, for all $t \in[0,1]$, so, in particular, for all $t \in\left[\left(1-\frac{\rho}{k}\right)^{n-1}, 1\right]$, and for all $u \in r^{\leq(t g)^{+}} r_{\geq-(t g)^{-}}\left[L_{t g, \rho}\right]$, there is a $t_{u}^{*} \in\left[0,1-\frac{\rho}{k}\right]$ such that

$$
\left\|u-t_{u}^{*}(t g)\right\|_{p} \leq \rho
$$

Therefore, by Lemma 7.2 with

$$
t_{0}:=\left(1-\frac{\rho}{k}\right)^{n-1} \text { and } t_{*}:=\left(1-\frac{\rho}{k}\right)
$$

$L_{g, \rho}$ is dismantlable to $r^{\leq\left[\left(1-\frac{\rho}{k}\right)^{n} g\right]^{+}} r_{\geq-\left[\left(1-\frac{\rho}{k}\right)^{n} g\right]^{-}}\left[L_{\left(1-\frac{\rho}{k}\right)^{n} g, \rho}\right]$.
Because

$$
\left\|-\left[\left(1-\frac{\rho}{k}\right)^{n} g\right]^{-}\right\|_{p} \leq\left\|\left(1-\frac{\rho}{k}\right)^{n} g\right\|_{p}=\left(1-\frac{\rho}{k}\right)^{n}\|g\|_{p}<\frac{\rho}{\|g\|_{p}}\|g\|_{p}=\rho
$$

we have that

$$
-\left[\left(1-\frac{\rho}{k}\right)^{n} g\right]^{-} \in L_{\left(1-\frac{\rho}{k}\right)^{n} g, \rho}
$$

By the " $\Leftarrow$ "-direction of Proposition 6.3, $L_{\left(1-\frac{\rho}{k}\right)^{n} g, \rho}$ has a sup-center. Because

$$
r^{\leq\left[\left(1-\frac{\rho}{k}\right)^{n} g\right]^{+}} r_{\geq-\left[\left(1-\frac{\rho}{k}\right)^{n} g\right]^{-}\left[L_{\left(1-\frac{\rho}{k}\right)^{n} g, \rho}\right]}
$$

is a retract of $L_{\left(1-\frac{\rho}{k}\right)^{n} g, \rho}$, by Lemma 6.5, it is dismantlable.
Therefore, $L_{g, \rho}$ is dismantlable.
The idea for the proof for $L^{\infty}(\Omega)$ is the same, which induces the temptation to look for a more general result in Banach lattices. Example 9.3 below shows that, if such a result exists, it will be more subtle than a simple generalization of the arguments given here.
Lemma 8.3. Let $g \in L^{\infty}(\Omega)$ be so that $g^{+} \neq 0$ and $g^{-} \neq 0$, let $\rho>0$, and let $k>\rho$ be so that $\|g\|_{\frac{\infty}{k}}<1$. Then, for every $u \in r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$, there is a $t_{u}^{*} \in\left[0,1-\frac{\rho}{k}\right]$ such that $\left\|u-t_{u}^{*} g\right\|_{\infty} \leq \rho$.
Proof. Let $u \in r^{\leq g^{+}} r_{\geq-g^{-}}\left[L_{g, \rho}\right]$ and let $t_{u}$ be so that $\left\|u-t_{u} g\right\|_{\infty} \leq \rho$. If $t_{u} \in$ $\left[0,1-\frac{\rho}{k}\right]$, then there is nothing to prove. So consider the case that $t_{u} \in\left(1-\frac{\rho}{k}, 1\right]$.

In this case, we have the following. (Below, the supremum is, in each case, an essential supremum.)

$$
\begin{aligned}
& \left\|u-\left(1-\frac{\rho}{k}\right) g\right\|_{\infty}=\sup _{x \in \Omega}\left|u(x)-\left(1-\frac{\rho}{k}\right) g(x)\right| \\
& =\max \left\{\begin{array}{l}
\sup _{x \in \Omega, g(x)>0, u(x) \geq\left(1-\frac{\rho}{k}\right) g(x)}\left(u(x)-\left(1-\frac{\rho}{k}\right) g(x)\right), ~
\end{array}\right. \\
& \sup _{x \in \Omega, g(x)>0, u(x)<\left(1-\frac{\rho}{k}\right) g(x)}\left(\left(1-\frac{\rho}{k}\right) g(x)-u(x)\right), \\
& \sup _{x \in \Omega, g(x)<0, u(x) \geq\left(1-\frac{\rho}{k}\right) g(x)}\left(u(x)-\left(1-\frac{\rho}{k}\right) g(x)\right), \\
& \left.\sup _{x \in \Omega, g(x)<0, u(x)<\left(1-\frac{\rho}{k}\right) g(x)}\left(\left(1-\frac{\rho}{k}\right) g(x)-u(x)\right)\right\} \\
& \leq \max \left\{\begin{array}{l}
\sup _{x \in \Omega, g(x)>0, u(x) \geq\left(1-\frac{\rho}{k}\right) g(x)}\left(g(x)-\left(1-\frac{\rho}{k}\right) g(x)\right), ~
\end{array}\right. \\
& \sup \left(t_{u} g(x)-u(x)\right), \\
& \sup \left(u(x)-t_{u} g(x)\right), \\
& x \in \Omega, g(x)<0, u(x) \geq\left(1-\frac{\rho}{k}\right) g(x) \\
& \left.\sup _{x \in \Omega, g(x)<0, u(x)<\left(1-\frac{\rho}{k}\right) g(x)}\left(\left(1-\frac{\rho}{k}\right) g(x)-g(x)\right)\right\} \\
& \leq \max \left\{\rho, \frac{\rho}{k}\|g\|_{\infty}\right\} \leq \rho
\end{aligned}
$$

Hence, in case $t_{u} \in\left(1-\frac{\rho}{k}, 1\right]$, we can use $t_{u}^{*}:=1-\frac{\rho}{k}$, which proves the result.
Theorem 8.4. Let $g \in L^{\infty}(\Omega)$ be so that $g^{+} \neq 0$ and $g^{-} \neq 0$ and let $\rho>0$. Then $L_{g, \rho}$ is dismantlable.
Proof. Similar to the proof of Theorem 8.2. (It is easy to see that the " $\Leftarrow$ "-direction of Proposition 6.3, which is what was used in the proof of Theorem 8.2, also holds for $p=\infty$.)

## 9. NONDISMANTLABILITY FOR $p=1$

As noted earlier, as a subset of $L^{1}(\Omega)$, the set $L_{g, \rho}$ need not be dismantlable. The idea for the relevant example is to construct a set of maximal and minimal elements of $L_{g, \rho}$ such that every element of this set is fixed by every comparative retraction. We start with a characterization of maximal elements and then give the example.
Lemma 9.1. Let $\Omega:=[0,2), p:=1, \rho=\frac{1}{10}$ and $g:=\mathbf{1}_{[0,1)}-\mathbf{1}_{[1,2)}$. Let $u \in L_{g, \rho}$ be so that there is exactly one $t \in[0,1]$ such that $\|u-t g\|_{1} \leq \rho$. If $\left.u\right|_{[0,1)} \geq t$ and $\left.u\right|_{[1,2)} \geq-t$, then $u$ is maximal in $L_{g, \rho}$.

Proof. Because $s \mapsto\|u-s g\|_{1}$ is continuous, we have that $\|u-t g\|_{1}=\rho$.
Now let $h:[0,2) \rightarrow \mathbb{R}$ be a function such that $h>u$ in $L^{1}[0,2)$. Then $\left.h\right|_{[0,1)} \geq t$ and $\left.h\right|_{[1,2)} \geq-t$. Let $s \in[0,1]$. Then, for $s \geq t$, the following holds.

$$
\begin{aligned}
\|h-s g\|_{1} & =\int_{0}^{1}|h-s| d \lambda+\int_{1}^{2}|h-(-s)| d \lambda \\
& =\int_{0}^{1}|(h-t)+(t-s)| d \lambda+\int_{1}^{2}|h-(-t)| d \lambda+(s-t) \\
& \geq \int_{0}^{1}|h-t| d \lambda-(s-t)+\int_{1}^{2}|h-(-t)| d \lambda+(s-t) \\
& =\int_{0}^{1} h-t d \lambda+\int_{1}^{2} h-(-t) d \lambda \\
& >\int_{0}^{1} u-t d \lambda+\int_{1}^{2} u-(-t) d \lambda \\
& =\|u-t g\|_{1}=\rho
\end{aligned}
$$

Similarly, we prove that $\|h-s t\|_{1}>\rho$ for $s<t$.
Lemma 9.2. Let $\Omega:=[0,2), p:=1, \rho=\frac{1}{10}$ and $g:=\mathbf{1}_{[0,1)}-\mathbf{1}_{[1,2)}$. Let $u \in L_{g, \rho}$ and $t \in[0,1]$ be so that $\left.u\right|_{[0,1)} \geq t,\left.u\right|_{[1,2)} \geq-t$, $\lambda\{x \in[0,1): u(x)=t\}>0$, $\lambda\{x \in[1,2): u(x)=-t\}>0$ and $\|u-t g\|_{1}=\rho$. Then $u$ is maximal in $L_{g, \rho}$.
Proof. By Lemma 9.1, we only need to prove that, for $s \neq t$, we have $\|u-s g\|_{1}>\rho$. Let $s>t$. Then

$$
\begin{aligned}
\|u-s g\|_{1}= & \int_{0}^{1}|u-s| d \lambda+\int_{1}^{2}|u-(-s)| d \lambda \\
\geq & \int_{0}^{1}|u-t| d \lambda+(s-t) \lambda\{x \in[0,1): u(x)=t\} \\
& -(s-t) \lambda\{x \in[0,1): u(x)>t\}+\int_{1}^{2}|u-(-t)| d \lambda+(s-t) \\
\geq & \int_{0}^{1}|u-t| d \lambda+(s-t) \lambda\{x \in[0,1): u(x)=t\}+\int_{1}^{2}|u-(-t)| d \lambda \\
> & \|u-t g\|_{1}=\rho
\end{aligned}
$$

The proof that $\|u-s g\|_{1}>\rho$ for $s<t$ is similar.
Example 9.3. Let $\Omega:=[0,2), p:=1, \rho=\frac{1}{10}$ and $g:=\mathbf{1}_{[0,1)}-\mathbf{1}_{[1,2)}$ and consider $L_{g, \rho}$ in $L^{1}(\Omega)$. For any $t_{D}, t_{g} \in\left[\frac{2}{10}, 1\right], t_{G}, t_{d} \in\left[0, \frac{8}{10}\right]$ and any $A \subseteq[0,1)$ and $B \subseteq[1,2)$ such that $\lambda(A)=\lambda(B)=\frac{1}{2}$, we define

$$
\begin{aligned}
D_{t_{D}, B} & :=t_{D} \mathbf{1}_{[0,1)}+\left(-t_{D}\right) \mathbf{1}_{B}+\left(-t_{D}+\frac{2}{10}\right) \mathbf{1}_{[1,2) \backslash B} \\
G_{t_{G}, A} & :=t_{G} \mathbf{1}_{A}+\left(t_{G}+\frac{2}{10}\right) \mathbf{1}_{[0,1) \backslash A}+\left(-t_{G}\right) \mathbf{1}_{[1,2)} \\
d_{t_{d}, B} & :=t_{d} \mathbf{1}_{[0,1)}+\left(-t_{d}\right) \mathbf{1}_{B}+\left(-t_{d}-\frac{2}{10}\right) \mathbf{1}_{[1,2) \backslash B}
\end{aligned}
$$

$$
g_{t_{g}, A}:=t_{g} \mathbf{1}_{A}+\left(t_{g}-\frac{2}{10}\right) \mathbf{1}_{[0,1) \backslash A}+\left(-t_{g}\right) \mathbf{1}_{[1,2)} .
$$

By Lemma 9.2 and its dual, the $D_{t_{D}, B}$ and the $G_{t_{G}, A}$ are maximal and the $d_{t_{d}, B}$ and the $g_{t_{g}, A}$ are minimal.

Let $r$ be an up-retraction on $L_{g, \rho}$. Clearly, $r$ fixes the $D_{t_{D}, B}$ and the $G_{t_{G}, A}$. Let $t_{d} \in\left[0, \frac{8}{10}\right]$, let $B \subseteq[1,2)$ be so that $\lambda(B)=\frac{1}{2}$ and suppose for a contradiction that $r\left(d_{t_{d}, B}\right)>d_{t_{d}, B}$.

First consider the case

$$
\left.r\left(d_{t_{d}, B}\right)\right|_{[1,2)}>\left.d_{t_{d}, B}\right|_{[1,2)}=\left(-t_{d}\right) \mathbf{1}_{B}+\left.\left(-t_{d}-\frac{2}{10}\right) \mathbf{1}\right|_{[1,2) \backslash B}
$$

Because

$$
D_{t_{d}+\frac{2}{10},[1,2) \backslash B}=\left(t_{d}+\frac{2}{10}\right) \mathbf{1}_{[0,1)}+\left(-t_{d}\right) \mathbf{1}_{B}+\left(-t_{d}-\frac{2}{10}\right) \mathbf{1}_{[1,2) \backslash B} \geq d_{t_{d}, B}
$$

we obtain
$\left.r\left(D_{t_{d}+\frac{2}{10},[1,2) \backslash B}\right)\right|_{[1,2)} \geq\left. r\left(d_{t_{d}, B}\right)\right|_{[1,2)}>\left.d_{t_{d}, B}\right|_{[1,2)}=\left(-t_{d}\right) \mathbf{1}_{B}+\left(-t_{d}-\frac{2}{10}\right) \mathbf{1}_{[1,2) \backslash B}$
and then $r\left(D_{t_{d}+\frac{2}{10},[1,2) \backslash B}\right)>D_{t_{d}+\frac{2}{10},[1,2) \backslash B}$, which is not possible.
Now consider the case

$$
\left.r\left(d_{t_{d}, B}\right)\right|_{[0,1)}>\left.d_{t_{d}, B}\right|_{[0,1)}=t_{d} \mathbf{1}_{[0,1)}
$$

Then there are an $\varepsilon>0$ and a subset $C \subseteq[0,1)$ of positive measure such that $\lambda(C) \leq \frac{1}{2}$ and such that

$$
\left.r\left(d_{t_{d}, B}\right)\right|_{C}>\left.d_{t_{d}, B}\right|_{C}+\varepsilon \mathbf{1}_{C}=\left(t_{d}+\varepsilon\right) \mathbf{1}_{C}
$$

Let $A \subseteq[0,1)$ be a subset of $[0,1)$ such that $\lambda(A)=\frac{1}{2}$ and $C \subseteq A$. Because

$$
G_{t_{d}, A}=t_{d} \mathbf{1}_{A}+\left(t_{d}+\frac{2}{10}\right) \mathbf{1}_{[0,1) \backslash A}+\left(-t_{d}\right) \mathbf{1}_{[1,2)} \geq d_{t_{d}, B}
$$

we obtain

$$
\left.r\left(G_{t_{d}, A}\right)\right|_{C} \geq\left. r\left(d_{t_{d}, B}\right)\right|_{C}>\left.d_{t_{d}, B}\right|_{C}+\varepsilon \mathbf{1}_{C}=\left(t_{d}+\varepsilon\right) \mathbf{1}_{C}
$$

and then $r\left(G_{t_{d}, A}\right)>G_{t_{d}, A}$, which is not possible.
Therefore, we must have that $r\left(d_{t_{d}, B}\right)=d_{t_{d}, B}$. Similarly, we prove that $r$ must fix every $g_{t_{g}, A}$.

The proof that every down-retraction fixes every $D_{t_{D}, B}$ and every $G_{t_{G}, A}$ is similar, too. Thus, if $H$ is the set of all functions $D_{t_{D}, B}, G_{t_{G}, A}, d_{t_{d}, B}$ and $g_{t_{g}, A}$, then $H$ is fixed by every up- or down-retraction of $L_{g, \rho}$, which means that $H$ is fixed by every comparative retraction of $L_{g, \rho}$. Hence $L_{g, \rho}$ cannot be dismantlable.
Remark 9.4. By using isomorphisms, Example 9.3 can be transplanted onto any finite measure space that has disjoint subsets $A$ and $B$ of equal measure such that $A \cup B=\Omega$ and enough structure to allow the subsets needed in Example 9.3. However, in Example 9.3, we have that

$$
\int_{\{x \in \Omega: g(x)>0\}}|g(x)| d \lambda(x)=\int_{\{x \in \Omega: g(x)<0\}}|g(x)| d \lambda(x)
$$

The author has tried, but was unable to, construct examples that show that $L_{g, \rho}$, as a subset of $L^{1}(\Omega)$ is not dismantlable when

$$
\int_{\{x \in \Omega: g(x)>0\}}|g(x)| d \lambda(x) \neq \int_{\{x \in \Omega: g(x)<0\}}|g(x)| d \lambda(x) .
$$

The problem is that, when the integrals of the positive and negative parts of $g$ are not balanced, constructions as in Example 9.3 can be carried out, but to guarantee that none of the minimal elements are removed by an up-retraction, we need to use some maximal elements which cannot be guaranteed to not be removed by a downretraction and vice versa.
Remark 9.5. Let $\Omega:=[0,2), p:=1, \rho=\frac{1}{10}$ and $g:=\mathbf{1}_{[0,1)}-\mathbf{1}_{[1,2)}$ and consider $L_{g, \rho}$ in $L^{1}(\Omega)$ with notation as in Example 9.3. The function

$$
M:=\mathbf{1}_{\left[0, \frac{1}{2}\right)}+\left(1-\frac{2}{10}\right) \mathbf{1}_{\left[\frac{1}{2}, 1\right)}+(-1) \mathbf{1}_{\left[1, \frac{3}{2}\right)}+\left(-1+\frac{2}{10}\right) \mathbf{1}_{\left[\frac{3}{2}, 2\right)}
$$

satisfies $g_{1,\left[0, \frac{1}{2}\right)} \leq M \leq D_{1,\left[1, \frac{3}{2}\right)}$. However, because, for any $t \in[0,1]$, we have that

$$
\|M-t g\| \geq \frac{2}{10}
$$

we have that $M$ is not in $L_{g, \rho}$. Thus $L_{g, \rho}$, though convex in the sense of analysis (straight lines between two points are again in $L_{g, \rho}$ ), is not order-convex, that is, there are functions $u<v<w$ such that $u, w \in L_{g, \rho}$, but $v \notin L_{g, \rho}$.

Note that situations of this type can be constructed in any $L^{p}(\Omega)$ with $p \in(1, \infty)$ and for any $g$ such that $\left\|g^{+}\right\|_{p}>0$ and $\left\|g^{-}\right\|_{p}>0$. Thus, contrary to bounded balls, which are the most common setting for applying the fixed point property for ordered sets in analysis, in general, line segments $L_{g, \rho}$ will rarely be order-convex.

## 10. Dismantlable Subsets for $p \geq 1$

With $L_{g, \rho}$ as a subset of $L^{1}(\Omega)$ not necessarily dismantlable, we turn to a slightly modified subset that will be dismantlable in Banach lattices that satisfy (E0). The main problem in Example 9.3 is that, for certain functions $u$, there is no ability to adjust the value $t$ that minimizes $\|u-t g\|$. By multiplying the positive and negative parts by different values $p$ and $n$, we gain the flexibility needed to prove dismantlability. Limiting the choices so that $|p-n|$ is small, produces a set that is "close" to $L_{g, \rho}$ in the sense that every member of either set has a member of the other within a short distance.
Definition 10.1. Let $E$ be an ordered normed space, let $g \in E$ and let $\rho, \varepsilon>0$. We define

$$
L_{g, \rho, \varepsilon}:=\left\{u \in E:(\exists p, n \in[0,1],|p-n| \leq \varepsilon)\left\|u-\left[p g^{+}-n g^{-}\right]\right\| \leq \rho\right\}
$$

Proposition 10.2. $L_{g, \rho, \varepsilon}$ is closed and bounded and therefore chain-complete when E satisfies (E0).
Lemma 10.3. Let $E$ be a Banach lattice that satisfies (E0), let $g \in E$ and let $\rho, \varepsilon>0$. Then $L_{g, \rho, \varepsilon}$ is dismantlable to $r \leq g^{+} r_{\geq-g^{-}}\left[L_{g, \rho, \varepsilon}\right] \subseteq L_{g, \rho, \varepsilon}$.
Proof. Because $E$ satisfies (E0), we have that $L_{g, \rho, \varepsilon}$ is chain-complete.

First we prove that $r_{\geq-g^{-}}\left[L_{g, \rho, \varepsilon}\right] \subseteq L_{g, \rho, \varepsilon}$. Let $u \in L_{g, \rho, \varepsilon}$ and let $p_{u}, n_{u} \in[0,1]$ be so that $\left|p_{u}-n_{u}\right| \leq \varepsilon$ and $\left\|u-\left[p_{u} g^{+}-n_{u} g^{-}\right]\right\| \leq \rho$. Then

$$
p_{u} g^{+}-n_{u} g^{-} \geq-n_{u} g^{-} \geq-g^{-}
$$

Hence

$$
\begin{aligned}
\left\|u \vee\left(-g^{-}\right)-\left[p_{u} g^{+}-n_{u} g^{-}\right]\right\| & =\left\|u \vee\left(-g^{-}\right)-\left[p_{u} g^{+}-n_{u} g^{-}\right] \vee\left(-g^{-}\right)\right\| \\
& \leq\left\|u-\left[p_{u} g^{+}-n_{u} g^{-}\right]\right\| \leq \rho
\end{aligned}
$$

which shows that $r_{\geq-g^{-}}\left[L_{g, \rho, \varepsilon}\right] \subseteq L_{g, \rho, \varepsilon}$.
Next we prove that $r \leq g^{+}\left[r_{\geq-g^{-}}\left[L_{g, \rho, \varepsilon}\right]\right] \subseteq L_{g, \rho, \varepsilon}$. Let $u \in r_{\geq-g^{-}}\left[L_{g, \rho, \varepsilon}\right]$ and let $p_{u}, n_{u} \in[0,1]$ be so that $\left|p_{u}-n_{u}\right| \leq \varepsilon$ and $\left\|u-\left[p_{u} g^{+}-n_{u} g^{-}\right]\right\| \leq \rho$. Then

$$
p_{u} g^{+}-n_{u} g^{-} \leq p_{u} g^{+} \leq g^{+}
$$

Hence

$$
\begin{aligned}
\left\|u \wedge g^{+}-\left[p_{u} g^{+}-n_{u} g^{-}\right]\right\| & =\left\|u \wedge g^{+}-\left[p_{u} g^{+}-n_{u} g^{-}\right] \wedge g^{+}\right\| \\
& \leq\left\|u-\left[p_{u} g^{+}-n_{u} g^{-}\right]\right\| \leq \rho
\end{aligned}
$$

which shows that $r^{\leq g^{+}}\left[r_{\geq-g^{-}}\left[L_{g, \rho, \varepsilon}\right]\right] \subseteq L_{g, \rho, \varepsilon}$. Moreover, every $u \in r_{\geq-g^{-}}\left[L_{g, \rho, \varepsilon}\right]$ satisfies $u \geq-g^{-}$and we also have $g^{+}=g \vee 0 \geq g \wedge 0=-g^{-}$. Hence $r \leq g^{+}(u)=$ $u \wedge g^{+} \in r_{\geq-g^{-}}\left[L_{g, \rho, \varepsilon}\right]$, which shows that $r^{\leq g^{+}}\left[r_{\geq-g^{-}}\left[L_{g, \rho, \varepsilon}\right]\right] \subseteq r_{\geq-g^{-}}\left[L_{g, \rho, \varepsilon}\right]$.
Theorem 10.4. Let $E$ be a Banach lattice that satisfies (EO), let $g \in E$ and let $\rho, \varepsilon>0$. Then $L_{g, \rho, \varepsilon}$ is dismantlable.
Proof. Let $\delta \in(0, \varepsilon)$ be so that there is an $N \in \mathbb{N}$ such that $\delta N=1$. We will prove by induction that, for all $k \in\{0, \ldots, N\}, L_{g, \rho, \varepsilon}$ is dismantlable to $r \leq((1-k \delta) g)^{+} r_{\geq-((1-k \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right] \subseteq L_{g, \rho, \varepsilon}$. Because $r^{\leq 0^{+}} r_{\geq-0^{-}}\left[L_{g, \rho, \varepsilon}\right]=\{0\}$, this proves the result.

The base case, $k=0$, is proved in Lemma 10.3.
For the induction step, let $k \in\{0, \ldots, N-1\}$ be so that $L_{g, \rho, \varepsilon}$ is dismantlable to $r \leq((1-k \delta) g)^{+} r_{\geq-((1-k \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]$.

First we prove that

$$
\begin{gathered}
r_{\geq-((1-(k+1) \delta) g)^{-}}\left[r^{\leq((1-k \delta) g)^{+}} r_{\geq-((1-k \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]\right] \\
\subseteq r^{\leq((1-k \delta) g)^{+}} r_{\geq-((1-k \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]
\end{gathered}
$$

Let $u \in r^{\leq((1-k \delta) g)^{+}} r_{\geq-((1-k \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]$ and let $p_{u}, n_{u} \in[0,1]$ be so that $\left|p_{u}-n_{u}\right| \leq \varepsilon$ and $\left\|u-\left[p_{u} g^{+}-n_{u} g^{-}\right]\right\| \leq \rho$. Because $u \in r^{\leq((1-k \delta) g)^{+}} r_{\geq-((1-k \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]$, we can assume without loss of generality that $p_{u}, n_{u} \in[0,1-k \delta]$. Therefore,

$$
\left|p_{u}-\min \left\{n_{u}, 1-(k+1) \delta\right\}\right| \leq \varepsilon
$$

Now

$$
\begin{aligned}
\| u & \vee\left(-((1-(k+1) \delta) g)^{-}\right)-\left[p_{u} g^{+}-\min \left\{n_{u}, 1-(k+1) \delta\right\} g^{-}\right] \| \\
& =\left\|u \vee\left(-((1-(k+1) \delta) g)^{-}\right)-\left[p_{u} g^{+}-n_{u} g^{-}\right] \vee\left(-((1-(k+1) \delta) g)^{-}\right)\right\| \\
& \leq\left\|u-\left[p_{u} g^{+}-n_{u} g^{-}\right]\right\| \leq \rho
\end{aligned}
$$

which shows that

$$
r_{\geq-((1-(k+1) \delta) g)^{-}}\left[r^{\leq((1-k \delta) g)^{+}} r_{\geq-((1-k \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]\right]
$$

which is equal to $r \leq((1-k \delta) g)^{+} r_{\geq-((1-(k+1) \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]$, is contained in $L_{g, \rho}$ and hence in $r \leq((1-k \delta) g)^{+} r_{\geq-((1-k \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]$.

Now we prove that

$$
\begin{gathered}
r^{\leq((1-(k+1) \delta) g)^{+}}\left[r^{\leq((1-k \delta) g)^{+}} r_{\geq-((1-(k+1) \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]\right] \\
\subseteq r^{\leq((1-k \delta) g)^{+}} r_{\geq-((1-(k+1) \delta) g)^{-}\left[L_{g, \rho, \varepsilon}\right]} .
\end{gathered}
$$

Let $u \in r^{\leq((1-k \delta) g)^{+}} r_{\geq-((1-(k+1) \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]$ and let $p_{u}, n_{u} \in[0,1]$ be so that $\left|p_{u}-n_{u}\right| \leq \varepsilon$ and $\left\|u-\left[p_{u} g^{+}-n_{u} g^{-}\right]\right\| \leq \rho$.
Because $u \in r^{\leq((1-k \delta) g)^{+}} r_{\geq-((1-(k+1) \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]$, we can assume without loss of generality that $p_{u} \in\left[0,1-k \delta \bar{\delta}\right.$ and $n_{u} \in[0,1-(k+1) \delta]$. Therefore,

$$
\left|\min \left\{p_{u}, 1-(k+1) \delta\right\}-n_{u}\right| \leq \varepsilon
$$

Now

$$
\begin{aligned}
\| u & \wedge((1-(k+1) \delta) g)^{+}-\left[\min \left\{p_{u}, 1-(k+1) \delta\right\} g^{+}-n_{u} g^{-}\right] \| \\
& =\left\|u \wedge((1-(k+1) \delta) g)^{+}-\left[p_{u} g^{+}-n_{u} g^{-}\right] \wedge((1-(k+1) \delta) g)^{+}\right\| \\
& \leq\left\|u-\left[p_{u} g^{+}-n_{u} g^{-}\right]\right\| \leq \rho
\end{aligned}
$$

which shows that the set

$$
r^{\leq((1-(k+1) \delta) g)^{+}}\left[r^{\leq((1-k \delta) g)^{+}} r_{\geq-((1-(k+1) \delta) g)^{-}\left[L_{g, \rho, \varepsilon}\right]}\right],
$$

which is equal to $r^{\leq((1-(k+1) \delta) g)^{+}} r_{\geq-((1-(k+1) \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]$ is contained in the set $r \leq((1-k \delta) g)^{+} r_{\geq-((1-(k+1) \delta) g)^{-}}\left[L_{g, \rho, \varepsilon}\right]$. Hence $L_{g, \rho, \varepsilon}$ is dismantlable to the ordered set $r \leq((1-(k+1) \delta) g)^{+} r_{\geq-((1-(k+1) \delta) g)^{-}\left[L_{g, \rho, \varepsilon}\right] .}$
Corollary 10.5. Let $E$ be a Banach lattice that satisfies (EO), let $g \in E$ and let $\rho>\rho^{\prime}>0$. Then every order-preserving map from $L_{g, \rho}$ to $L_{g, \rho^{\prime}}$ has a fixed point.
Proof. Let $\varepsilon:=\frac{1}{\|g\|+1}\left(\rho-\rho^{\prime}\right)$. Then, trivially, $L_{g, \rho^{\prime}} \subseteq L_{g, \rho^{\prime}, \varepsilon}$ Now let $u \in L_{g, \rho^{\prime}, \varepsilon}$. Then there are $p, n \in[0,1]$ such that $|p-n| \leq \varepsilon$ and such that $\left\|u-\left[p g^{+}-n g^{-}\right]\right\| \leq \rho^{\prime}$. Thus

$$
\begin{aligned}
\|u-p g\| & =\left\|u-\left[p g^{+}-p g^{-}\right]-\left(n g^{-}-n g^{-}\right)\right\| \\
& =\left\|u-\left[p g^{+}-n g^{-}\right]-\left(n g^{-}-p g^{-}\right)\right\| \\
& \leq\left\|u-\left[p g^{+}-n g^{-}\right]\right\|+|n-p|\left\|g^{-}\right\| \\
& \leq \rho^{\prime}+\frac{1}{\|g\|+1}\left(\rho-\rho^{\prime}\right)\left\|g^{-}\right\| \leq \rho
\end{aligned}
$$

Hence $L_{g, \rho^{\prime}, \varepsilon} \subseteq L_{g, \rho}$.
Now let $T: L_{g, \rho} \rightarrow L_{g, \rho^{\prime}}$ be order-preserving. Then $\left.T\right|_{L_{g, \rho^{\prime}, \varepsilon}}$ is an order-preserving self map of $L_{g, \rho^{\prime}, \varepsilon}$, which has a fixed point. Hence $T$ has a fixed point.

Ordered sets $L_{g, \rho, \varepsilon}$ were the author's initial attempt to find an analogue of a fence, that is, an ordered set $f_{0}<f_{1}>f_{2}<\cdots>f_{2 n}$ with no further comparabilities,
in analysis. Fences are very common in finite ordered sets and fences have the fixed point property. The next most common finite ordered sets are crowns, that is, ordered sets $c_{0}<c_{1}>c_{2}<\cdots>c_{2 n}=c_{0}$ with no further comparabilities. On a crown, the function $c_{k} \mapsto c_{k+2} \bmod 2 n$ is a fixed point free order-preserving self map. Analogues of crowns are easily found in analysis.
Example 10.6. Represent $S^{1}$ as $[0,2 \pi)$ with arithmetic modulo $2 \pi$. Consider the set

$$
T:=\left\{u \in L^{p}[0,2 \pi):(\exists s \in[0,2 \pi))\left\|u-\mathbf{1}_{[s, s+1)}\right\|_{p} \leq \frac{1}{2}\right\} .
$$

Then $T$ is closed, bounded and $H(u)[\cdot]:=u(\cdot-1)$ is a fixed point free order-preserving self map: Clearly, $H$ is order-preserving. Now suppose, for a contradiction, that $u \in T$ is so that $H(u)=u$. Then, because $u \in T$, there is an $s_{u}$ such that

$$
\left\|u-\mathbf{1}_{\left[s_{u}, s_{u}+1\right)}\right\|_{p} \leq \frac{1}{2} .
$$

Hence

$$
\begin{aligned}
\left\|\left.u\right|_{\left[s_{u}, s_{u}+1\right)}\right\|_{p} & \geq\left\|\mathbf{1}_{\left[s_{u}, s_{u}+1\right)}\right\|_{p}-\left\|\left.u\right|_{\left[s_{u}, s_{u}+1\right)}-\mathbf{1}_{\left[s_{u}, s_{u}+1\right)}\right\|_{p} \\
& \geq\left\|\mathbf{1}_{\left[s_{u}, s_{u}+1\right)}\right\|_{p}-\left\|u-\mathbf{1}_{\left[s_{u}, s_{u}+1\right)}\right\|_{p} \geq \frac{1}{2} .
\end{aligned}
$$

Thus, because $H(u)=u,\left\|\left.u\right|_{\left[\left(s_{u}+1\right),\left(s_{u}+1\right)+1\right)}\right\|_{p} \geq \frac{1}{2}$ and $\left\|\left.u\right|_{\left[\left(s_{u}+2\right),\left(s_{u}+2\right)+1\right)}\right\|_{p} \geq \frac{1}{2}$ and $\left\|\left.u\right|_{\left[\left(s_{u}+3\right),\left(s_{u}+3\right)+1\right)}\right\|_{p} \geq \frac{1}{2}$. But then, for all $s \in[0,2 \pi)$, there are $k_{1}, k_{2} \in$ $\left\{s_{u}, s_{u}+1, s_{u}+2, s_{u}+3\right\}$ such that $\left\|u-\mathbf{1}_{[s, s+1)}\right\|_{p} \geq\left\|\left.u\right|_{\left[k_{1}, k_{1}+1\right)}+\left.u\right|_{\left[k_{2}, k_{2}+1\right)}\right\|_{p}>\frac{1}{2}$, contradicting that $u \in T$.

## 11. Conclusion and Open Questions

In Theorems 8.2 and 8.4, we have constructed classes of nontrivial infinite dismantlable ordered sets $L_{g, \rho}$ that are a natural generalization of infinite dimensional balls in $L^{p}(\Omega)$ for $p>1$. By Theorem 2.44 in [5], it is well-known that infinite dimensional balls have the fixed point property, because they have an order-center. Proposition 6.4 shows that, for small enough $\rho$, the ordered sets $L_{g, \rho}$ do not have an order-center. Therefore, so far, dismantlability is the only viable method to establish the fixed point property for these ordered sets $L_{g, \rho}$. The present results appear to be the first application of dismantlability to ordered sets that arise in analysis that goes beyond the well-established use of order-centers. As is often the case in analysis, $L^{1}(\Omega)$ behaves differently from $L^{p}(\Omega)$ with $p>1$ : Example 9.3 shows that sets $L_{g, \rho}$ need not be dismantlable in $L^{1}(\Omega)$. Theorem 10.4 shows that the sets $L_{g, \rho}$ can be "approximated" with ordered sets that do have the fixed point property. This is a new phenomenon in the fixed point theory for ordered sets, because, in this theory, there usually is no surrounding universe as there is in analysis. Using this approximation to prove the fixed point property would require the establishment of further connections between analytical and order-theoretical properties. Corollary 10.5 shows that many order preserving self maps of $L_{g, \rho}$ in $L^{1}(\Omega)$ have fixed points, so it stands to reason that $L_{g, \rho}$ in $L^{1}(\Omega)$ should have the fixed point property. A proof currently eludes the author, but a resolution either way would be interesting: Further pursuit of this
question could lead to infinite ordered sets that have the fixed point property, but not for a finitary reason, such as dismantlability. This would be very interesting, as the author is not aware of any such examples. To provide more structure for the many interesting follow-up questions, we conclude with some of these questions in list form.
(1) Does $L_{g, \rho}$ as a subset of $L^{1}(\Omega)$ have the fixed point property?
(2) If $L_{g, \rho}$ as a subset of $L^{1}(\Omega)$ has the fixed point property, would these ordered sets be candidates for examples of ordered sets with the fixed point property whose product does not have the fixed point property? Roddy proved in [9] that the product of two finite ordered sets with the fixed point property must have the fixed point property, too. A full extension to infinite ordered sets, if possible at all, appears to be challenging, see [10]. (It is known, see Exercises $12-15$ in [13], that the product of a dismantlable ordered set with an ordered set that has the fixed point property again has the fixed point property, so $L_{g, \rho}$ as a subset of $L^{p}(\Omega)$ with $p>1$ does not provide new insights here.)
(3) Are there nontrivial examples of sets $L_{g, \rho}$ in spaces $L^{1}(\Omega)$ that are dismantlable?
(4) Can the arguments given here be generalized to prove the fixed point property for neighborhoods of polygonal paths? In $L^{p}(\Omega)$ with $p>1$, for neighborhoods of polygonal paths of the form $L_{0, g, \rho} \cup L_{g, g+h, \rho^{\prime}}$ such that $\mu(\{x \in \Omega: g(x) \neq 0, h(x) \neq 0\})=0$, the arguments here can be generalized to show dismantlability: The disjointness of the supports allows us to first dismantle $L_{g, g+h, \rho^{\prime}}$ and to then dismantle $L_{0, g, \rho}$. This idea can be extended to neighborhoods of polygonal paths with more line segments, as long as the functions that define the direction of each line segment have pairwise disjoint supports. Note that the resulting domains are not convex.

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