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NONLINEAR CONTRACTION AND FUZZY COMPACT OPERATOR IN FUZZY BANACH ALGEBRAS

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Abstract. In this paper, at first, we consider the concept of fuzzy Banach algebras and fuzzy compact operators. Then we apply a fixed point theorem to solve the operator equation $AxBx = x$ in the fuzzy Banach algebras under a nonlinear contraction.
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1. Preliminaries

Definition 1.1. ([22]) A *triangular norm* (shortly, *t-norm*) is a binary operation on the unit interval $[0,1]$, i.e., a function $* : [0,1] \times [0,1] \to [0,1]$ such that, for all $a, b, c \in [0,1]$, the following four axioms are satisfied:

(T1) $a * b = b * a$ (commutativity);
(T2) $a * (b * c) = (a * b) * c$ (associativity);
(T3) $a * 1 = a$ (boundary condition);
(T4) $a * b \leq a * c$ whenever $b \leq c$ (monotonicity).

We say the t-norm $*$ has Σ property and write $* \in \Sigma$ whenever, for any $\lambda \in (0,1)$, there exists $\gamma \in (0,1)$ (which does not depend on n) such that

$$\underbrace{(1-\gamma) * \cdots * (1-\gamma)}_{n} > 1 - \lambda \tag{1.1}$$

for each $n \geq 1$.

The theory of fuzzy space has much progressed as developing the theory of randomness. Some mathematicians have defined fuzzy norms on a vector space from various points of view [3, 9, 14]. Following Cheng and Mordeson [7], Nădăban and Dzitac [18] and Saadati and Vaezpour [21] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [15] and investigated some properties of fuzzy normed spaces [3, 18, 20].

Definition 1.2. Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq N(x,s) * N(y,t)$;
(N_5) $\lim_{t\to\infty} N(x,t) = 1$;
(N_6) for $x \neq 0$, $N(x,\cdot)$ is left continuous on $\mathbb{R}$.

The pair $(X, N, *)$ is called a *fuzzy normed vector space*.

Definition 1.3. [4] (1) Let $(X, N, *)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} x_n = x$.

(2) Let $(X, N, *)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*. We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [4]).

Definition 1.4. [16] A *fuzzy normed algebra* $(X, N, *, \diamond)$ is a random normed space $(X, N, *)$ with algebraic structure such that

(N_7) $N(xy, ts) \geq N(x,t) \diamond N(y,s)$ for all $x, y \in X$ and all $t, s > 0$. in which $\diamond$ is a continuous t-norm.

Every normed algebra $(X, \|\cdot\|)$ defines a fuzzy normed algebra $(X, N, \min, \cdot)$, where

$$N(x,t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ if and only if

$$\|xy\| \leq \|x\|\|y\| + s\|y\| + t\|x\| \ (x, y \in X;\ t, s > 0).$$

This space is called the induced fuzzy normed algebra.
Note that the last example is hold with $(X, N, \cdot, \cdot)$ (for details and more examples see [6, 16]).

Definition 1.5. Let $(X, N, *)$ be a fuzzy normed space. We define the open ball $B_x(r,t)$ and the closed ball $B_x[r,t]$ with center $x \in X$ and radius $0 < r < 1$ for any $t > 0$ as follows:

$$B_x(r,t) = \{y \in X \ : \ N(x-y,t) > 1 - r\},$$
$$B_x[r,t] = \{y \in X \ : \ N(x-y,t) \geq 1 - r\}.$$

Theorem 1.6. ([1]) *Let $(X, N, *)$ be a fuzzy normed space. Every open ball $B_x(r, t)$ is an open set.*

Different kinds of topologies can be introduced in a fuzzy normed space [1, 22]. The (r, t)-*topology* is introduced by a family of neighborhoods

$$\{B_x(r, t)\}_{x\in X,\ t>0,\ r\in(0,1)}.$$

In fact, every fuzzy norm N on X generates a topology ((r, t)-topology) on X which has as a base the family of open sets of the form

$$\{B_x(r, t)\}_{x\in X,\ t>0,\ r\in(0,1)}.$$

Remark 1.7. Since $\left\{B_x\left(\frac{1}{n}, \frac{1}{n}\right) : n = 1, 2, 3, \cdots\right\}$ is a local base at x, the (r, t)-topology is first countable.

Theorem 1.8. ([1]) *Every fuzzy normed space $(X, N, *)$ is a Hausdorff space.*

Definition 1.9. Let $(X, N, *)$ be a fuzzy normed space. A subset A of X is said to be fuzzy *bounded* if there exist $t > 0$ and $r \in (0, 1)$ such that $N(x - y, t) > 1 - r$ for all $x, y \in A$.

Note that, A fuzzy normed space $(X, N, *)$ is called compact if $(X, N, *)$ is a compact topological space.

Definition 1.10. The fuzzy normed space $(X, N, *)$ is said to be fuzzy compact (simply F-compact) if every sequence $\{p_m\}_m$ in X has a convergent subsequence $\{p_{m_k}\}$. A subset A of a fuzzy normed space $(X, N, *)$ is said to be F-compact if every sequence $\{p_m\}$ in A has a subsequence $\{p_{m_k}\}$ convergent to a vector $p \in A$.

By [10] a set is compact topological if and only if it is F-compact.

Theorem 1.11. ([1]) *Every F-compact subset A of a fuzzy normed space $(X, N, *)$ is closed and fuzzy bounded.*

Theorem 1.12. ([22]) *If $(X, N, *)$ is a fuzzy normed space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} N(x_n, t) = N(x, t)$ almost everywhere.*

Theorem 1.13. ([1]) *Let $(X, N, *)$ be a fuzzy normed space such that every Cauchy sequence in X has a convergent subsequence. Then $(X, N, *)$ is complete.*

Lemma 1.14. ([1]) *If $(X, N, *)$ is a fuzzy normed space, then*
(1) *The function $(x, y) \longrightarrow x + y$ is continuous.*
(2) *The function $(\alpha, x) \longrightarrow \alpha x$ is continuous.*

Note that, in [2] the authors proved that every fuzzy normed space is topological vector space (see also Theorem 2 of [13] and [23]).

Theorem 1.15. (The Schauder fixed point theorem) *Let K be a convex subset of a topological vector space X and A is a continuous mapping of K into itself so that $A(K)$ is contained in a F-compact subset of K, then A has a fixed point.*

Proof. The proof is depended to topological vector space properties, therefore we omit it. □

Lemma 1.16. *Let $(X, N, *)$ be a fuzzy normed space, in which $* \in \Sigma$. If we define $E_{\lambda,N} : X^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,N}(x) = \inf\{t > 0 : N(x,t) > 1 - \lambda\}$$

for each $\lambda \in]0,1[$ and $x \in X$, then we have the following:

(1) *For any $\kappa \in]0,1[$, there exists $\lambda \in]0,1[$ such that*

$$E_{\kappa,N}(x_1 - x_k) \leq E_{\lambda,N}(x_1 - x_2) + E_{\lambda,N}(x_2 - x_3) + \cdots + E_{\lambda,N}(x_{k-1} - x_k)$$

for any $x_1, ..., x_k \in X$;

(2) *For any sequence $\{x_n\}$ in X, we have $\mu_{x_n - x}(t) \longrightarrow 1$ if and only if*

$$E_{\lambda,N}(x_n - x) \to 0.$$

Also the sequence $\{x_n\}$ is Cauchy w.r.t. f if and only if it is Cauchy with $E_{\lambda,N}$.

Proof. The proof is the same as in Lemma 1.6 of [19]. □

Note that, λ in Lemma 1.16 (1) does not depend on k (see [19]).

Definition 1.17. A linear operator $\Lambda : (X, N, *) \longrightarrow (Y, N', \diamond)$ is said to be *fuzzy bounded* if there exists a constant $h \in \mathbb{R} - \{0\}$ such that

$$N'(\Lambda x, t) \geq N(hx, t) \tag{1.2}$$

for all $x \in X$ and $t > 0$.

Theorem 1.18. ([1]) *Every linear operator $\Lambda : (X, N, *) \longrightarrow (Y, N, \diamond)$ is fuzzy bounded if and only if it is continuous.*

Definition 1.19. (F-Compact linear operator). Let $(X, N, *)$ and $(Y, N, *)$ be fuzzy normed spaces. An operator $\Lambda : X \longrightarrow Y$ is called a F-compact linear operator if Λ is linear and if for every fuzzy bounded subset M of X, the closure $\overline{\Lambda(M)}$ is F-compact.

Definition 1.20. Let $(X, N, *)$ be fuzzy normed space and $A \subset X$. We say A is *totally bounded* if for each $0 < r < 1$ and $t > 0$ there exists a finite subset S of X such that

$$A \subseteq \bigcup_{x \in S} B_x(r, t).$$

Lemma 1.21. *Let $(X, N, *)$ be fuzzy normed space and $A \subset X$. Then*

(a) *If $\overline{A}$ is F-compact, A is totally bounded.*

(b) *If A is totally bounded and X is complete, $\overline{A}$ is F-compact.*

Proof. (a) We assume that $\overline{A}$ is compact and show that, any fixed $0 < r_0 < 1$ and $t_0 > 0$ being given, there exists a finite subset S of X such that

$$A \subseteq \bigcup_{x \in S} B_x(r_0, t).$$

If $A = \emptyset$, then $S = \emptyset$. If $A \neq \emptyset$ we pick any $x_1 \in A$. If $N(x_1 - z, t_0) > 1 - r_0$ for all $z \in A$, then $\{x_1\} = S$. Otherwise, let $x_2 \in A$ be such that $N(x_1 - x_2, t_0) \leq 1 - r_0$. If for all $z \in A$,

$$N(x_j - z, t_0) > 1 - r_0 \text{ for } j = 1 \text{ or } 2. \tag{1.3}$$

Then $\{x_1, x_2\} = S$. Otherwise let $z = x_3 \in A$ be a point not satisfying (1.3). If for all $z \in A$,

$$N(x_j - z, t_0) > 1 - r_0 \text{ for } j = 1, 2 \text{ or } 3,$$

then $\{x_1, x_2, x_3\} = S$. Otherwise we continue by selecting an $x_4 \in A$, etc. We assert the existence of a positive integer n such that the set $\{x_1, x_2, ..., x_n\} = S$ obtained after n such steps. In fact, if there were no such n, our construction would yield a sequence $\{x_j\}$ satisfying

$$N(x_j - x_k, t_0) \leq 1 - r_0 \text{ for } j \neq k.$$

Obviously, $\{x_j\}$ could not have a subsequence which is Cauchy. Hence $\{x_j\}$ could not have a subsequence which converges in X. But this contradicts the compactness of $\overline{A}$ because $\{x_j\}$ lies in A, by the construction. Hence there must be a finite set S for A. Since $0 < r_0 < 1$ and t_0 were arbitrary, we conclude that A is totally bounded.
(b) see [10]. □

Further, the operator Λ is called completely continuous if it is continuous and totally bounded.

2. Fixed point theorem

In this section, we assume that $(X, N, *)$ is fuzzy Banach space. Let Φ be the set of all non-decreasing functions

$$\phi : [0, \infty) \longrightarrow [0, \infty).$$

Here, $\phi^n(t)$ denotes the n-th iterative function of $\phi(t)$. Our results extend and improve some results of [8, 5].

Lemma 2.1. ([11, 12]) *If* $\phi \in \Phi$ *and satisfy* $\displaystyle\sum_{j=1}^{\infty} \phi^j(t) < \infty$ *for* $t > 0$ *then* $\phi(t) < t$ *for* $t > 0$.

The concept of fuzzy Lipschitzian mapping was introduced in [17]. A mapping $A : X \longrightarrow X$ is called fuzzy $\mathcal{D}$-Lipschitzian if

$$N(A(x) - A(y), \phi(t)) \geq N(x - y, t) \tag{2.1}$$

for $x, y \in X$ and $t > 0$, where $\phi \in \Phi$.

We call the function ϕ a $\mathcal{D}$-function of A on X.

Lemma 2.2. *Let* $(X, N, *)$ *be fuzzy normed space,* $t > 0$ *and* $0 < a < 1$. *Then there exists* $k \geq 1$ *such that*

$$N(x, t) * a = N\left(x, \frac{t}{k}\right),$$

for $x \in X$.

Proof. Let $x \in X$, $t > 0$ and $0 < a < 1$ be fixed. Then $N(x, t) * a \in (0, 1)$. By the intermediate value theorem there exists an $\ell > 0$ such that $N(x, t) * a = N(x, \ell)$. Put $\ell = \frac{t}{k}$ in which $k = k(a)$. □

Theorem 2.3. *Let $(X, N, *)$ be a complete fuzzy normed space, in which $* \in \Sigma$, and let $A : X \longrightarrow X$ such that*

$$N(Ax_1 - Ax_2, \phi(t)) \geq N(x_1 - x_2, t) * a,$$

for all $x_1, x_2 \in X$, $t > 0$, $\mathcal{D}$-function ϕ and some $a \in (0, 1)$ such that $\sum_{j=1}^{\infty} k^j(a)\phi^j(t) < \infty$. Then A has a unique fixed point.

Proof. Choose $x_0 \in X$ and let $t > 0$. Choose $x_1 \in X$ with $Ax_0 = x_1$. In general choose x_{n+1} such that $Ax_n = x_{n+1}$. Now,

$$\begin{aligned}\mu_{Ax_n - Ax_{n+1}}(\phi^{n+1}(t)) &\geq N(x_n - x_{n+1}, \phi^n(t)) * a = N(Ax_{n-1} - Ax_n, \phi^n(t)) * a \\ &= N(Ax_{n-1} - Ax_n, \frac{\phi^n(t)}{k}) \geq \ldots \geq N(x_0 - x_1, \frac{t}{k^{n+1}}).\end{aligned}$$

Note for each $\lambda \in (0, 1)$ that (see Lemma 1.9. of [19])

$$\begin{aligned}E_{\lambda,N}(Ax_n - Ax_{n+1}) &= \inf\{\phi^{n+1}(t) > 0 : N(Ax_n - Ax_{n+1}, \phi^{n+1}(t)) > 1 - \lambda\} \\ &\leq \inf\{\phi^{n+1}(t) > 0 : N(x_0 - x_1, \frac{t}{k^{n+1}}) > 1 - \lambda\} \\ &\leq \phi^{n+1}\left(\inf\left\{t > 0 : N\left(x_0 - x_1, \frac{t}{k^{n+1}}\right) > 1 - \lambda\right\}\right) \\ &= k^{n+1}\phi^{n+1}(E_{\lambda,N}(x_0 - x_1)),\end{aligned}$$

in which $k \geq 1$. Now, by the proof of Lemma 1.9 of [19], the sequence $\{Ax_n\}$ is Cauchy and so is convergent since X is complete. By Theorem 2.3 of [19], A has a unique fixed point. □

Theorem 2.4. *Let S be a closed, convex and fuzzy bounded subset of a Banach algebra $(X, N, *, *)$ in which $* \in \Sigma$ and let $A : X \longrightarrow X$, $B : S \longrightarrow X$ be two operators such that*

(a) *A is $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$-function ϕ,*

(b) *B is completely continuous, and*

(c) *$x = AxBy \Longrightarrow x \in S$, for all $y \in S$.*

Then the operator equation

$$AxBx = x \tag{2.2}$$

has a solution, whenever $\sum_{j=1}^{\infty} k^j(a)\phi^j(\ell) < \infty$, for $\ell > 0$ in which

$$a = N(B(S), 1) * N(ATy, 1),$$

$T : S \to X$ is a mapping defined by

$$Ty = z,$$

where $z \in X$ is the unique solution of the equation

$$z = AzBy, \ y \in S.$$

Also $N(B(S), 1) = \inf_{y \in S} N(By, 1)$.

Proof. Let $y \in S$. Now, define a mapping $A_y : X \to X$ by

$$A_y(x) = AxBy, \ x \in X.$$

Since, for $x_i \in X$ $(i = 1, 2)$ and $t > 0$,

$$\begin{aligned} N(A_y x_1 - A_y x_2, \phi(t)) &= N(Ax_1 B_y - Ax_2 By, \phi(t)) \\ &\geq N(Ax_1 - Ax_2, \phi(t)) * N(By, 1) \\ &\geq N(x_1 - x_2, t) * N(B(S), 1) \end{aligned}$$

Then, A_y is a nonlinear contraction on X with a $\mathcal{D}$-function ϕ and $N(B(S), 1) \in (0, 1)$. Now an application of a fixed point Theorem 2.3 yields that there is a unique point $x^* \in X$ such that

$$A_y(x^*) = x$$

or, equivalently,

$$x^* = Ax^*By.$$

Since hypothesis (c) holds, we have that $x^* \in S$. Now we show that T is continuous. Let $\{y_n\}$ be a sequence in S converging to a point y. Since S is closed, $y \in S$. Now,

$$\begin{gathered} N(Ty_n - Ty, \phi(t) + \epsilon) = N(ATy_n By_n - ATyBy, \phi(t) + \epsilon) \\ \geq N(ATy_n By_n - ATyBy_n, \phi(t)) * N(ATyBy_n - ATyBy, \epsilon) \\ \geq N(ATy_n - ATy, \phi(t)) * N(By_n, 1) * N(ATy, 1) * N(By_n - By, \epsilon) \\ \geq N(Ty_n - Ty, t) * N(By_n, 1) * N(ATy, 1) * N(By_n - By, \epsilon), \end{gathered}$$

for $t > 0$ and $\epsilon \in (0, 1)$. For great $n \in \mathbb{N}$ and by continuity of B we have

$$N(Ty_n - Ty, \phi(t) + \epsilon) \geq N(Ty_n - Ty, t) * N(By_n, 1) * N(ATy, 1).$$

Take infimum on $\epsilon \in (0, 1)$ we get

$$\begin{aligned} N(Ty_n - Ty, \phi(t)) &\geq N(Ty_n - Ty, t) * N(B(S), 1) * N(ATy, 1) \\ &\geq N(Ty_n - Ty, t) * a \end{aligned}$$

in which $a = N(B(S), 1) * N(ATy, 1) \in (0, 1)$. Now, by proof of Theorem 2.3, $N(Ty_n - Ty, \phi(t)$ tend to 1 for each $t > 0$, whenever $n \to \infty$ and consequently T is continuous on S. Next we show that T is a compact operator on S.

Let $\eta > 0$ and $t > 0$ be given. For any $z \in S$ we have

$$\begin{aligned} N\left(Az, 2\phi\left(\frac{\sqrt{t}}{3}\right)\right) &\geq N\left(Aa, \phi\left(\frac{\sqrt{t}}{3}\right)\right) * N\left(Az - Aa, \phi\left(\frac{\sqrt{t}}{3}\right)\right) \\ &\geq N\left(Aa, \phi\left(\frac{\sqrt{t}}{3}\right)\right) * N\left(z - a, \frac{\sqrt{t}}{3}\right) \\ &\geq c \end{aligned}$$

where

$$c = N\left(Aa, \phi\left(\frac{\sqrt{t}}{3}\right)\right) * N\left(S, \frac{\sqrt{t}}{3}\right)$$

for some fixed $a \in S$. Then we can find $r = r_{\eta,t}$ such that

$$N\left(S, \frac{t}{3}\right) * N(B(S), 1) * c * (1 - r) > 1 - \eta.$$

Since B is completely continuous, $B(S)$ is totally bounded. Then there is a set $Y=\{y_1,...,y_n\}$ in S such that

$$B(S)\subset\bigcup_{j=1}^{n}B_{w_j}(r,\sqrt{t}),$$

where $w_j=B(y_j)$. Therefore for any $y\in S$ we have a $y_k\in Y$ such that

$$N(By-By_k,\sqrt{t})>1-r.$$

Also we have

$$\begin{aligned}N(Ny_n-Ny,t)&\geq N\left(Ty_n-Ty,\phi\left(\frac{t}{3}\right)+2\sqrt{t}\phi\left(\frac{\sqrt{t}}{3}\right)\right)\\&\geq N\left(AzBy-Az_kBy_k,\phi\left(\frac{t}{3}\right)+2\sqrt{t}\phi\left(\frac{\sqrt{t}}{3}\right)\right)\\&\geq N\left(AzBy-Az_kBy,\phi\left(\frac{t}{3}\right)\right)*N\left(Az_kBy-Az_kBy_k,2\sqrt{t}\phi\left(\frac{\sqrt{t}}{3}\right)\right)\\&\geq N\left(Az-Az_k,\phi\left(\frac{t}{3}\right)\right)*N(By,1)*N\left(Az_k,2\phi\left(\frac{\sqrt{t}}{3}\right)\right)*N(By_n-By,\sqrt{t})\\&\geq N\left(z-z_k,\frac{t}{3}\right)*N(B(S),1)*c*(1-r)\\&\geq N\left(S,\frac{t}{3}\right)*N(B(S),1)*c*(1-r)\geq 1-\eta.\end{aligned}$$

This is true for every $y\in S$ and hence

$$T(S)\subset B_{z_i}(\eta,t),$$

where $z_i=T(y_i)$. As a result $T(S)$ is totally bounded. Since T is continuous, it is a compact operator on S. Now an application of Schauder's fixed point yields that T has a fixed point in S. Then, by the definition of T

$$x=Tx=A(Tx)Bx=AxBx,$$

and so the operator equation $x=AxBx$ has a solution in S. □

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