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A NOTE ON A STRUCTURE THEOREM FOR PREHOMOGENEOUS VECTOR SPACES

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Abstract. In this note, we give a structure theorem for all prehomogeneous vector spaces defined over the complex number field $\mathbb{C}$. Also it means a necessary and sufficient condition for a triplet (G, ρ, V) defined over $\mathbb{C}$ to be a prehomogeneous vector space. For this purpose, we give a general structural correspondence between isotropy subgroups and fixed point sets when a group acts on a non-empty set.
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1. Introduction

Let G be a connected linear algebraic group, V a finite dimensional vector space ($\dim V \geq 1$), and $\rho : G \to GL(V)$ a rational linear representation of G on V, all defined over the complex number field $\mathbb{C}$. If V has a Zariski-dense G-orbit, the triplet (G, ρ, V) is called a *prehomogeneous vector space* (abbrev. PV). A point v in the Zariski-dense $G-$orbit is called a *generic point*, and the isotropy subgroup $G_v = \{g \in G | \ \rho(g) \cdot v = v\}$ at a generic point v is called a *generic isotropy subgroup*. When G is reductive, a PV (G, ρ, V) is called a *reductive PV*. Many types of reductive PVs are classified in [3], [4], [6], [7], [8], [9], [10], [11], etc. However, to classify all reductive PVs where each ρ is non-irreducible is terribly difficult. This is pointed out in the abstract of [7]. The main reason for difficulty under investigation is that we have to determine the prehomogeneity for a great many triplets composed of non-reductive groups. It is too complicated. On behalf of the classification of all reductive PVs, it is indispensable to investigate the structure for any triplet (G, ρ, V) where G is a connected linear algebraic group. Note that non-reductive linear algebraic groups and their rational linear representations are not classified.

In this note, we give a structure theorem for all PVs defined over $\mathbb{C}$. Also it means a necessary and sufficient condition for a triplet (G, ρ, V) defined over $\mathbb{C}$ to be a PV.

In section 2, as preliminaries, we give a structural correspondence between an arbitrary group and an arbitrary non-empty set on which a group acts (Theorem 2.15).

Although in this paper, the result is applied only to rational linear representations of algebraic groups, we give it in a general form, since the general result will be needed in a forthcoming paper.

In section 3, we give some structural facts for all triplets, a structure theorem for all PVs (Theorem 3.11), and some examples. In general, when a PV (G, ρ, V) satisfies $\dim G = \dim V$, then such a PV is called a *cuspidal PV* (or an *etàle PV*). In Proposition 3.18, we show that we can attach a cuspidal PV to any PV.

In section 4, as appendices, some facts are given.

Notation

As usual, $\mathbb{C}$ stands for the field of complex numbers. For a positive integer n, we denote by $\mathbb{C}^n$ the totality of column vectors with n components, also denote by $e_i^{(n)} = {}^t(0, \cdots, 1, \cdots, 0) \in \mathbb{C}^n$ $(i = 1, \cdots, n)$ the canonical base of $\mathbb{C}^n$. For positive integers m, n, we denote by $M(m, n)$ the totality of $m \times n$ matrices over $\mathbb{C}$. If $m = n$, we simply write $M(n)$ instead of $M(n, n)$. We denote by $GL(n)$ (resp. $SL(n)$) the general linear group $\{X \in M(n)|\ \det X \neq 0\}$ (resp. the special linear group $\{X \in M(n)|\ \det X = 1\}$), also denote by Λ_1 (resp. Λ_1^*) the standard representation of $GL(n)$ on $\mathbb{C}^n$ (resp. the dual representation of Λ_1). Let G be a group, ρ a representation of G, and H a subset of G. Then the restriction $\rho|_H$ is denoted by ρ for simplicity. Let G be a group, N a subgroup of G, and G/N the residue class of G by N. Then we denote by $\overline{g}$ an element $gN \in G/N$. For a triplet (G, ρ, V), we denote by I_G the identity element in G, denote by 0_V the zero element in V, and denote by I_V the identity transformation on V.

2. Preliminaries

We shall begin to consider a general situation. Let G be a group, X a non-empty set, $P(G)$ the power set of G, $P(X)$ the power set of X, $S(X)$ the permutation group of X, and $\rho : G \to S(X), g \mapsto \rho(g)$ a representation of G on X with $\rho(g) : X \to X$, $x \mapsto \rho(g) \cdot x$.

Definition 2.1. (1) For $H \in P(G)$, let $X^{\rho(H)} = \{x \in X|\ \rho(h) \cdot x = x \text{ for all } h \in H\}$. Then we call H a generator for $X^{\rho(H)}$ in $P(G)$. In particular, when H is the empty set ϕ, then $X^{\rho(\phi)}$ is defined as X.
(2) For $W \in P(X)$, let $G_W = \{g \in G|\ \rho(g) \cdot x = x \text{ for all } x \in W\}$, which is clearly a subgroup of G. Then we call W a generator for G_W in $P(X)$. In particular, when W is the empty set ϕ, then G_ϕ is defined as G.

Remark 2.2. When there is no confusion, for $\{g\} \in P(G)$ and for $\{x\} \in P(X)$, we write $X^{\rho(g)}$ instead of $X^{\rho(\{g\})}$, also G_x instead of $G_{\{x\}}$.

Proposition 2.3. *Let $g \in G$ and $x \in X$. Then the following assertions are equivalent. (1) $g \in G_x$. (2) $\rho(g) \cdot x = x$. (3) $x \in X^{\rho(g)}$.*

Proposition 2.4. *(1) For any $H \in P(G)$, we have $X^{\rho(H)} = \{x \in X|\ H \subset G_x\}$.*
(2) For any $W \in P(X)$, we have $G_W = \{g \in G|\ W \subset X^{\rho(g)}\}$.
(3) For any $H_i \in P(G)(i = 1, 2)$ satisfying $H_1 \subset H_2$, we have $X^{\rho(H_1)} \supset X^{\rho(H_2)}$.
(4) For any $W_i \in P(X)(i = 1, 2)$ satisfying $W_1 \subset W_2$, we have $G_{W_1} \supset G_{W_2}$.

Proof. For (1), if $H = \phi$, then it is clear by (1) of Definition 2.1. When $H \neq \phi$, by (1) of Definition 2.1, then $x \in X^{\rho(H)}$ if and only if $x \in X^{\rho(h)}$ for all $h \in H$. Hence by Proposition 2.3, it is equivalent to $h \in G_x$ for all $h \in H$, which means $H \subset G_x$.
For (2), if $W = \phi$, then it is clear by (2) of Definition 2.1. When $W \neq \phi$, by (2) of Definition 2.1, then $g \in G_W$ if and only if $g \in G_x$ for all $x \in W$. Hence by Proposition 2.3, it is equivalent to $x \in X^{\rho(g)}$ for all $x \in W$, which means $W \subset X^{\rho(g)}$.
For (3) (resp. for (4)), it is clear by (1) (resp. by (2)). □

Lemma 2.5. *(1) For any $H \in P(G)$, we have $H \subset G_{X^{\rho(H)}}$.*
(2) For any $H \in P(G)$, we have $X^{\rho(H)} = X^{\rho(G_{X^{\rho(H)}})}$.
(3) For $H \in P(G)$, the following statements are equivalent.
(i) $H = G_{X^{\rho(H)}}$.
(ii) For any $K \in P(G)$ satisfying $X^{\rho(H)} = X^{\rho(K)}$, we have $K \subset H$.

Proof. To prove (1), let h be any element in H. Then we have $X^{\rho(h)} \supset X^{\rho(H)}$ by (3) of Proposition 2.4. It is equivalent to $h \in G_{X^{\rho(H)}}$ by (2) of Proposition 2.4, hence we have $H \subset G_{X^{\rho(H)}}$.
For (2), we have $H \subset G_{X^{\rho(H)}}$ by (1). Then $X^{\rho(H)} \supset X^{\rho(G_{X^{\rho(H)}})}$ by (3) of Proposition 2.4. On the other hand, for any $x \in X^{\rho(H)}$, we have $G_x \supset G_{X^{\rho(H)}}$ by (4) of Proposition 2.4. It is equivalent to $x \in X^{\rho(G_{X^{\rho(H)}})}$ by (1) of Proposition 2.4.
Hence $X^{\rho(H)} \subset X^{\rho(G_{X^{\rho(H)}})}$, and we have our assertion.
For (3), first assume (i). For any $K \in P(G)$ satisfying $X^{\rho(H)} = X^{\rho(K)}$, we have $G_{X^{\rho(H)}} = G_{X^{\rho(K)}}$. Here $H = G_{X^{\rho(H)}}$ and $K \subset G_{X^{\rho(K)}}$ by (1), so we have $K \subset H$. Next assume (ii). Here put $K = G_{X^{\rho(H)}}$.
Then we obtain $X^{\rho(K)} = X^{\rho(G_{X^{\rho(H)}})} = X^{\rho(H)}$ by (2). So we have $K = G_{X^{\rho(H)}} \subset H$. Moreover $H \subset G_{X^{\rho(H)}} = K$ by (1). Hence we have $H = K = G_{X^{\rho(H)}}$. □

Remark 2.6. By (2) and (3) of Lemma 2.5, the group $G_{X^{\rho(H)}}$ is the maximal generator for $X^{\rho(H)}$ in $P(G)$, which is called the isotropy subgroup of G for $X^{\rho(H)}$.

Lemma 2.7. *(1) For any $W \in P(X)$, we have $W \subset X^{\rho(G_W)}$.*
(2) For any $W \in P(X)$, we have $G_W = G_{X^{\rho(G_W)}}$.
(3) For $W \in P(X)$, the following assertions are equivalent.
(i) $W = X^{\rho(G_W)}$.
(ii) For any $Z \in P(X)$ satisfying $G_W = G_Z$, we have $Z \subset W$.

Proof. To prove (1), let x be any element in W. Then we have $G_x \supset G_W$ by (4) of Proposition 2.4. It is equivalent to $x \in X^{\rho(G_W)}$ by (1) of Proposition 2.4, hence $W \subset X^{\rho(G_W)}$.
For (2), we have $W \subset X^{\rho(G_W)}$ by (1). Then $G_W \supset G_{X^{\rho(G_W)}}$ by (4) of Proposition 2.4. On the other hand, for any $g \in G_W$, we have $X^{\rho(g)} \supset X^{\rho(G_W)}$ by (3) of Proposition 2.4. It is equivalent to $g \in G_{X^{\rho(G_W)}}$ by (2) of Proposition 2.4. Hence $G_W \subset G_{X^{\rho(G_W)}}$. We have our assertion.
For (3), first assume (i). For any $Z \in P(X)$ satisfying $G_W = G_Z$, we have $X^{\rho(G_W)} = X^{\rho(G_Z)}$. Here $W = X^{\rho(G_W)}$ and $Z \subset X^{\rho(G_Z)}$ by (1), we have $Z \subset W$. Next assume (ii). Here put $Z = X^{\rho(G_W)}$. Then we obtain $G_Z = G_{X^{\rho(G_W)}} = G_W$ by (2). So

we have $Z = X^{\rho(G_W)} \subset W$. Moreover, by (1), $W \subset X^{\rho(G_W)} = Z$. Hence we have $W = X^{\rho(G_W)}$. □

Remark 2.8. By (2) and (3) of Lemma 2.7, the set $X^{\rho(G_W)}$ is the maximal generator for G_W in $P(X)$, which is usually called the fixed subset of X under G_W.

Proposition 2.9. *Let $H_i \in P(G)$ $(i = 1, 2)$ and $W_j \in P(X)$ $(j = 1, 2)$. Then the following assertions hold.*
(1) $X^{\rho(H_1)} \subset X^{\rho(H_2)}$ if and only if $G_{X^{\rho(H_1)}} \supset G_{X^{\rho(H_2)}}$.
(2) $X^{\rho(H_1)} = X^{\rho(H_2)}$ if and only if $G_{X^{\rho(H_1)}} = G_{X^{\rho(H_2)}}$.
(3) $G_{W_1} \subset G_{W_2}$ if and only if $X^{\rho(G_{W_1})} \supset X^{\rho(G_{W_2})}$.
(4) $G_{W_1} = G_{W_2}$ if and only if $X^{\rho(G_{W_1})} = X^{\rho(G_{W_2})}$.

Proof. Clearly, we have (1) (resp. (3)) by Proposition 2.4 and (2) of Lemma 2.5 (resp. by Proposition 2.4 and (2) of Lemma 2.7). Moreover we obtain (2) (resp. (4)) by (1) (resp. by (3)). □

Definition 2.10. (1) For $g \in G$, let $\sigma(g) : P(G) \to P(G)$ be the map defined as $H \mapsto \sigma(g) \cdot H = gHg^{-1}$. In particular, $\sigma(g) \cdot \phi$ is defined as ϕ.
(2) For $g \in G$, let $\tau(g) : P(X) \to P(X)$ be the map defined as

$$W \mapsto \tau(g) \cdot W = \rho(g)(W).$$

In particular, $\tau(g) \cdot \phi$ is defined as ϕ.

Here, let $S(P(G))$ be the permutation group of $P(G)$, and $S(P(X))$ the permutation group of $P(X)$. Then $\sigma(g)$ is an element in $S(P(G))$, also $\tau(g)$ is an element in $S(P(X))$.

Definition 2.11. (1) Let $\sigma : G \to S(P(G))$ be the map defined as $g \mapsto \sigma(g)$ by (1) of Definition 2.10. Obviously σ is a representation of G on $P(G)$.
(2) Let $\tau : G \to S(P(X))$ be the map defined as $g \mapsto \tau(g)$ by (2) of Definition 2.10. Also τ is a representation of G on $P(X)$.

Definition 2.12. (1) Let $\Phi : P(G) \to P(X)$ be the map defined as

$$H \mapsto \Phi(H) = X^{\rho(H)}.$$

(2) Let $\Psi : P(X) \to P(G)$ be the map defined as $W \mapsto \Psi(W) = G_W$.

Proposition 2.13. *Let σ, τ be as in Definition 2.11, and Ψ, Φ as in Definition 2.12. Then the following assertions hold, namely Ψ and Φ are $G-$compatible maps.*
(1) $\Phi(\sigma(g) \cdot H) = \tau(g) \cdot \Phi(H)$ (i.e., $X^{\rho(gHg^{-1})} = \rho(g)(X^{\rho(H)})$) for any $g \in G$ and for any $H \in P(G)$.
(2) $\Psi(\tau(g) \cdot W) = \sigma(g) \cdot \Psi(W)$ (i.e., $G_{\rho(g)(W)} = gG_Wg^{-1}$) for any $g \in G$ and for any $W \in P(X)$.

Proof. For (1), by (1) of Proposition 2.4, $x \in X^{\rho(gHg^{-1})}$ is equivalent to $gHg^{-1} \subset G_x$. It is transformed to $H \subset g^{-1}G_xg = G_{\rho(g^{-1})\cdot x}$. By (1) of Proposition 2.4 again, $H \subset G_{\rho(g^{-1})\cdot x}$ is equivalent to $\rho(g^{-1}) \cdot x \in X^{\rho(H)}$, which is transformed to $x \in \rho(g)(X^{\rho(H)})$.

For (2), by (2) of Proposition 2.4, $h \in G_{\rho(g)(W)}$ is equivalent to $\rho(g)(W) \subset X^{\rho(h)}$. It is equivalent to $W \subset \rho(g)^{-1}(X^{\rho(h)}) = \rho(g^{-1})(X^{\rho(h)})$. Here by (1), we have $\rho(g^{-1})(X^{\rho(h)}) = X^{\rho(g^{-1}hg)}$. Therefore, by (2) of Proposition 2.4 again, $W \subset X^{\rho(g^{-1}hg)}$ is equivalent to $g^{-1}hg \in G_W$, it is transformed to $h \in gG_Wg^{-1}$. □

Here the isotropy subgroup of G at an element $H \in P(G)$ by the representation σ is the normalizer of H in G, which is denoted by

$$N_G(H) = \{g \in G | \ \sigma(g) \cdot H = gHg^{-1} = H\}.$$

Also the isotropy subgroup of G at an element $W \in P(X)$ by the representation τ acts admissibly on W, and we denote it by

$$Inv_G(W) = \{g \in G | \ \tau(g) \cdot W = \rho(g)(W) = W\}.$$

Corollary 2.14. *(1) For any $H \in P(G)$, the set $X^{\rho(H)}$ is a $N_G(H)-$invariant subset of X by the representation τ (i.e., $N_G(H) \subset Inv_G(X^{\rho(H)})$).*
(2) For any $W \in P(X)$, the group G_W is an $Inv_G(W)-$invariant subgroup of G by the representation σ (i.e., $Inv_G(W) \subset N_G(G_W)$).
(3) Let $H = \Psi(W) = G_W$ and $W = \Phi(H) = X^{\rho(H)}$. Then $N_G(H) = Inv_G(W)$. In particular, $N_G(H) = G = Inv_G(W)$ means that H is a normal subgroup of G if and only if W is a $G-$invariant subset of X.

Theorem 2.15. *Let Φ, Ψ be as in Definition 2.12, and put*

$$\operatorname{Im}\Phi = \{\Phi(H) | \ H \in P(G)\} \text{ and } \operatorname{Im}\Psi = \{\Psi(W) | \ W \in P(X)\}.$$

Then there exists a $G-$compatible one-to-one correspondence between $\operatorname{Im}\Phi$ and $\operatorname{Im}\Psi$, which keeps relation of inclusion reversely in the sense of Proposition 2.9.

Proof. For any $H \in \operatorname{Im}\Psi$, there exists $W \in P(X)$ satisfying $H = G_W$. By (2) of Lemma 2.7, we have $\Psi \circ \Phi(H) = \Psi(X^{\rho(H)}) = G_{X^{\rho(H)}} = G_{X^{\rho(G_W)}} = G_W = H$. Similarly, for any $W \in \operatorname{Im}\Phi$, there exists $H \in P(G)$ satisfying $W = X^{\rho(H)}$. By (2) of Lemma 2.5, we have $\Phi \circ \Psi(W) = \Phi(G_W) = X^{\rho(G_W)} = X^{\rho(G_{X^{\rho(H)}})} = X^{\rho(H)} = W$. Hence $\Phi|_{\operatorname{Im}\Psi}$ and $\Psi|_{\operatorname{Im}\Phi}$ are bijections and they are inverse maps of each other. Clearly, by Proposition 2.13, $\Phi|_{\operatorname{Im}\Psi}$ and $\Psi|_{\operatorname{Im}\Phi}$ are $G-$compatible maps, also they keep relation of inclusion reversely in the sense of Proposition 2.9. Hence we have our assertion. □

Remark 2.16. In particular, $\operatorname{Im}\Psi$ is composed of all isotropy subgroups, and $\operatorname{Im}\Phi$ is composed of all fixed points sets when a group G acts on a non-empty set X by a representation ρ. Hence by Theorem 2.15, we have a one-to-one correspondence between the conjugacy classes of the isotropy subgroups and the isomorphism classes of the fixed point sets.

Corollary 2.17. *(1) For $H \in P(G)$, we have $H = G_{X^{\rho(H)}}$ if and only if $H \in \operatorname{Im}\Psi$. In particular, $G \in \operatorname{Im}\Psi$.*
(2) For $W \in P(X)$, we have $W = X^{\rho(G_W)}$ if and only if $W \in \operatorname{Im}\Phi$. In particular, $X \in \operatorname{Im}\Phi$.

Proof. For (1), if $H = G_{X^{\rho(H)}}$, then we have $H \in \operatorname{Im}\Psi$. On the other hand, if $H \in \operatorname{Im}\Psi$, then by Theorem 2.15, we have $\Psi \circ \Phi(H) = H$ (*i.e.*, $G_{X^{\rho(H)}} = H$). Especially, $G \subset G_{X^{\rho(G)}}(= G)$ by (1) of Lemma 2.5, hence $G \in \operatorname{Im}\Psi$.
For (2), if $W = X^{\rho(G_W)}$, then we have $W \in \operatorname{Im}\Phi$. On the other hand, if $W \in \operatorname{Im}\Phi$, then $\Phi \circ \Psi(W) = W$ (*i.e.*, $X^{\rho(G_W)} = W$) by Theorem 2.15.
Moreover $X \subset X^{\rho(G_X)}(= X)$ by (1) of Lemma 2.7, hence $X \in \operatorname{Im}\Phi$. □

Remark 2.18. By Remark 2.6 and (1) of Corollary 2.17, $\operatorname{Im}\Psi$ is composed of all maximal generators in $P(G)$. Similarly, by Remark 2.8 and (2) of Corollary 2.17, $\operatorname{Im}\Phi$ is composed of all maximal generators in $P(X)$.

3. A structure theorem for PVs

Firstly, we shall give some structural facts for all triplets. We consider everything over $\mathbb{C}$. Let G be a connected linear algebraic group, V a finite dimensional vector space ($\dim V \geq 1$), and $\rho : G \to GL(V)$ a rational linear representation of G on V. In addition, $P(G), P(V), S(P(G)), S(P(V)), \sigma, \tau, \Phi$, and Ψ are similarly defined by the assumptions and the definitions in section 2.

Proposition 3.1. *Let (G, ρ, V) be a triplet, $H \in P(G)$, and $W \in P(X)$. Then $V^{\rho(H)}$ is a subspace of V, and G_W is a closed subgroup of G. In particular, $V^{\rho(\phi)} = V$ and $G_\phi = G$ by Definition 2.1.*

Proof. Clearly, the set $V^{\rho(H)}$ is a subspace of V by (1) of Definition 2.1. For all $w \in W$, the map $\phi_w : \rho(G) \to V$ defined as $\rho(g) \mapsto \rho(g) \cdot w$ is a morphism. Since ρ and ϕ_w are continuous, the map $\phi_w \circ \rho$ is continuous. Hence $(\phi_w \circ \rho)^{-1}(\{w\}) = G_w$ is a closed subgroup of G. Hence by (2) of Definition 2.1, $G_W = \bigcap_{w \in W} G_w$ is a closed subgroup of G. □

Lemma 3.2. *Let (G, ρ, V) be a triplet. For $g \in G$ and $v \in V$, the following assertions hold.*
(1) $V^{\rho(G_v)} \supset V^{\rho(G_{\rho(g)\cdot v})}$ if and only if $V^{\rho(G_v)} = V^{\rho(G_{\rho(g)\cdot v})}$.
(2) $G_{\rho(g)\cdot v} \supset G_v$ if and only if $G_{\rho(g)\cdot v} = G_v$.

Proof. To prove (1), let $V^{\rho(G_v)} \supset V^{\rho(G_{\rho(g)\cdot v})}$. Then we have $V^{\rho(G_v)} \supset \rho(g)(V^{\rho(G_v)})$ by Proposition 2.13. Since $\rho(g) : V \to V$ is a linear automorphism on V, we have $\dim V^{\rho(G_v)} = \dim \rho(g)(V^{\rho(G_v)})$. Hence we have $V^{\rho(G_v)} = \rho(g)(V^{\rho(G_v)})$, which is equivalent to $V^{\rho(G_v)} = V^{\rho(G_{\rho(g)\cdot v})}$ by Proposition 2.13. On the other hand, let $V^{\rho(G_v)} = V^{\rho(G_{\rho(g)\cdot v})}$. Then $V^{\rho(G_v)} \supset V^{\rho(G_{\rho(g)\cdot v})}$ is obvious, so we have our assertion.
For (2), by (3) of Proposition 2.9, $G_{\rho(g)\cdot v} \supset G_v$ if and only if $V^{\rho(G_{\rho(g)\cdot v})} \subset V^{\rho(G_v)}$. Therefore, by (1), it is equivalent to $V^{\rho(G_{\rho(g)\cdot v})} = V^{\rho(G_v)}$. By (4) of Proposition 2.9, it is equivalent to $G_{\rho(g)\cdot v} = G_v$. We have our assertion. □

Proposition 3.3. *Let (G, ρ, V) be a triplet, $v \in V$, $N = N_G(G_v)$ the normalizer of G_v in G, and G/N the residue class group of G by N. Then the following assertions hold.*

(1) $N_G(G_v)$ *is a linear algebraic subgroup of* G.
(2) Let $g_i \in G$ $(i = 1, 2)$ *and* $w_i = \rho(g_i) \cdot v$. *Then* $G_{w_1} = G_{w_2}$ *if and only if* $\overline{g_1} = \overline{g_2}$.
(3) Let $\overline{g_i} \in G/N$ $(i = 1, 2)$. *Then* $\overline{g_1} = \overline{g_2}$ *if and only if* $\rho(g_1N) \cdot v = \rho(g_2N) \cdot v$. *Moreover,* $\rho(g_1N) \cdot v \neq \rho(g_2N) \cdot v$ *if and only if* $\rho(g_1N) \cdot v \cap \rho(g_2N) \cdot v = \phi$.
(4) For any $\overline{g} \in G/N$, *we have* $\rho(gN) \cdot v \subset \rho(g)(V^{\rho(G_v)})$.
(5) $\rho(G) \cdot v = \bigsqcup_{\overline{g} \in G/N} \rho(gN) \cdot v \subset \bigcup_{\overline{g} \in G/N} \rho(g)(V^{\rho(G_v)}) = \bigcup_{g \in G} \rho(g)(V^{\rho(G_v)})$
$= \bigcup_{g \in G} V^{\rho(G_{\rho(g)\cdot v})}$.
(6) For any $\overline{g} \in G/N$, *we have* $\rho(g)(V^{\rho(G_v)})$ *is a* $gNg^{-1}-$*invariant subspace of* V. *In particular, if* $g = I_G$, *then* $V^{\rho(G_v)}$ *is an* N*-invariant subspace of* V.
(7) For any $\overline{g} \in G/N$, *we have* $\rho(gN) \cdot v = \rho(G) \cdot v \cap \rho(g)(V^{\rho(G_v)})$.
(8) $G_v = N_v$.
(9) $\dim G - \dim \rho(G) \cdot v = \dim G_v = \dim N_v = \dim N - \dim \rho(N) \cdot v$.

Proof. For (1), it is a well-known fact (see [1, 1.7 in Section 1]).
For (2), by the assumption $w_i = \rho(g_i) \cdot v$ $(i = 1, 2)$ and (2) of Proposition 2.13, we have $G_{w_i} = G_{\rho(g_i)\cdot v} = g_i G_v g_i^{-1}$. Then $G_{w_1} = G_{w_2}$ is equivalent to $g_1 G_v g_1^{-1} = g_2 G_v g_2^{-1}$, which is transformed to $(g_2^{-1} g_1) G_v (g_2^{-1} g_1)^{-1} = G_v$. This means $g_2^{-1} g_1 \in N$, so we have our assertion.
For (3), we shall show the former assertion. Let $\overline{g_1} = \overline{g_2}$. Then we have $\rho(g_1N) \cdot v = \rho(g_2N) \cdot v$. On the other hand, if $\rho(g_1N) \cdot v = \rho(g_2N) \cdot v$, then there exist $n_1, n_2 \in N$ satisfying $\rho(g_1 n_1) \cdot v = \rho(g_2 n_2) \cdot v$. Hence we have $\rho({n_1}^{-1} {g_1}^{-1} g_2 n_2) \cdot v = v$, which is equivalent to $n_1^{-1} g_1^{-1} g_2 n_2 \in G_v$. It is transformed to $g_1^{-1} g_2 \in n_1 G_v n_2^{-1}$. Here $n_1, n_2 \in N$ and $G_v \subset N$. Hence $g_1^{-1} g_2 \in N$, which means $\overline{g_1} = \overline{g_2}$. For the latter assertion, let $\rho(g_1N) \cdot v \neq \rho(g_2N) \cdot v$. If $\rho(g_1N) \cdot v \cap \rho(g_2N) \cdot v \neq \phi$, then there exist $n_1, n_2 \in N$ satisfying $\rho(g_1 n_1) \cdot v = \rho(g_2 n_2) \cdot v$. By the same argument above, we have $\overline{g_1} = \overline{g_2}$, which is a contradiction. The remaining part of the latter assertion is obvious.
To prove (4), let w be an arbitrary element in $\rho(gN) \cdot v$. Then there exists $n \in N$ satisfying $w = \rho(gn) \cdot v$. Here $G_w = G_{\rho(gn)\cdot v} = (gn) G_v (gn)^{-1} = g G_v g^{-1}$. By (1) of Proposition 2.4 and (1) of Proposition 2.13, $g G_v g^{-1} \subset G_w$ if and only if $w \in V^{\rho(g G_v g^{-1})} = \rho(g)(V^{\rho(G_v)})$. So we have $\rho(gN) \cdot v \subset \rho(g)(V^{\rho(G_v)})$.
For (5), first, G is decomposed into the disjoint union $\bigsqcup_{\overline{g} \in G/N} gN$. Then by (3),

$$\rho(G) \cdot v = \rho\left(\bigsqcup_{\overline{g} \in G/N} gN \right) \cdot v = \bigsqcup_{\overline{g} \in G/N} \rho(gN) \cdot v.$$

By (4) and Proposition 2.13,

$$\bigsqcup_{\overline{g} \in G/N} \rho(gN) \cdot v \subset \bigcup_{\overline{g} \in G/N} \rho(g)(V^{\rho(G_v)}) = \bigcup_{g \in G} \rho(g)(V^{\rho(G_v)}) = \bigcup_{g \in G} V^{\rho(G_{\rho(g)\cdot v})}.$$

To prove (6), let n be an arbitrary element in N. By (1) of Proposition 2.13, we have

$$\rho(gng^{-1})(\rho(g)(V^{\rho(G_v)})) = \rho(gn)(V^{\rho(G_v)}) = V^{\rho((gn)G_v(gn)^{-1})}$$
$$= V^{\rho(gG_vg^{-1})} = \rho(g)(V^{\rho(G_v)}).$$

Hence $\rho(g)(V^{\rho(G_v)})$ is $gNg^{-1}-$invariant. Clearly, $\rho(g)(V^{\rho(G_v)})$ is a subspace of V. The latter assertion is obvious, so we have our result.
For (7), we have $\rho(gN)\cdot v \subset \rho(g)(V^{\rho(G_v)})$ by (4). Also clearly, $\rho(gN)\cdot v \subset \rho(G)\cdot v$. Therefore, we have $\rho(gN)\cdot v \subset \rho(G)\cdot v \cap \rho(g)(V^{\rho(G_v)})$. On the other hand, for any $w \in \rho(G)\cdot v \cap \rho(g)(V^{\rho(G_v)})$, there exists $h \in G$ satisfying $w = \rho(h)\cdot v \in \rho(g)(V^{\rho(G_v)})$, which is equivalent to $w = \rho(h)\cdot v \in V^{\rho(gG_vg^{-1})}$ by (1) of Proposition 2.13. Hence by (1) of Proposition 2.4, we have $gG_vg^{-1} \subset G_{\rho(h)\cdot v}$. It is transformed to $G_v \subset g^{-1}G_{\rho(h)\cdot v}g$, which is equivalent to $G_v \subset G_{\rho(g^{-1}h)\cdot v}$ by (2) of Proposition 2.13. Then by (2) of Lemma 3.2, we have $G_v = G_{\rho(g^{-1}h)\cdot v}$. Since $G_v = (g^{-1}h)G_v(g^{-1}h)^{-1}$ by (2) of Proposition 2.13, we obtain $g^{-1}h \in N$. Therefore $w = \rho(h)\cdot v \in \rho(gN)\cdot v$, we have our result.
For (8), since $N \subset G$, we have $N_v \subset G_v$. On the other hand, since $N = N_G(G_v) \supset G_v$, we have $N_v \supset G_v$. Hence $N_v = G_v$.
For (9), in general, $\dim \rho(G)\cdot v = \dim G/G_v = \dim G - \dim G_v$. Hence we have the first equality. Similarly, the third equality is clear by (1). Finally, by (8), we have the second equality. □

Definition 3.4. For a triplet (G, ρ, V) and for $W \in P(V)$, let $B_G(W)$ be the set $\bigcup_{g\in G} \rho(g)(W)$. We call $B_G(W)$ *the $G-$bundle for W*. In particular, if $W = \{w\}$, then $B_G(W)$ is the $G-$orbit $\rho(G)\cdot w$.

Lemma 3.5. *Let (G, ρ, V) be a triplet. For $v, w \in V$, the following assertions hold.*
(1) $B_G(V^{\rho(G_v)}) \subset B_G(V^{\rho(G_w)})$ if and only if there exists $g \in G$ satisfying

$$G_v \supset G_{\rho(g)\cdot w}.$$

(2) $B_G(V^{\rho(G_v)}) = B_G(V^{\rho(G_w)})$ if and only if there exists $g \in G$ satisfying

$$G_v = G_{\rho(g)\cdot w}.$$

Proof. For (1), first assume $B_G(V^{\rho(G_v)}) \subset B_G(V^{\rho(G_w)})$. Since

$$v \in V^{\rho(G_v)} \subset B_G(V^{\rho(G_v)}) \subset B_G(V^{\rho(G_w)}),$$

there exists $g \in G$ satisfying $v \in \rho(g)(V^{\rho(G_w)})$. By (1) of Proposition 2.13 and (1) of Proposition 2.4, $v \in \rho(g)(V^{\rho(G_w)}) = V^{\rho(gG_wg^{-1})}$ is equivalent to $gG_wg^{-1} \subset G_v$. It is also equivalent to $G_{\rho(g)\cdot w} \subset G_v$ by (2) of Proposition 2.13. On the other hand, if there exists $g \in G$ satisfying $G_{\rho(g)\cdot w} \subset G_v$, then by (3) of Proposition 2.9 and Proposition 2.13, we have $V^{\rho(G_v)} \subset V^{\rho(G_{\rho(g)\cdot v})} = \rho(g)(V^{\rho(G_w)})$. Hence

$$B_G(V^{\rho(G_v)}) \subset B_G(\rho(g)(V^{\rho(G_w)})) = B_G(V^{\rho(G_w)}),$$

we have our assertion.
For (2), first assume $B_G(V^{\rho(G_v)}) = B_G(V^{\rho(G_w)})$. By (1), there exist $g_1, g_2 \in G$

satisfying $G_v \supset G_{\rho(g_1)\cdot w}$ and $G_w \supset G_{\rho(g_2)\cdot v}$. Hence by (2) of Proposition 2.13, we have $G_v \supset G_{\rho(g_1)\cdot w} = g_1 G_w g_1^{-1} \supset g_1 G_{\rho(g_2)\cdot v} g_1^{-1} = G_{\rho(g_1 g_2)\cdot v}$. Here we transform $G_v \supset G_{\rho(g_1 g_2)\cdot v}$ to $G_{\rho((g_1 g_2)^{-1})\cdot v} \supset G_v$ by (2) of Proposition 2.13. Therefore, by (2) of Lemma 3.2, we have $G_{\rho((g_1 g_2)^{-1})\cdot v} = G_v$, which is also equivalent to $G_{\rho(g_1 g_2)\cdot v} = G_v$, and hence we have $G_v = G_{\rho(g_1)\cdot w}$. On the other hand, if there exists $g \in G$ satisfying $G_v = G_{\rho(g)\cdot w}$, then

$$B_G(V^{\rho(G_v)}) = B_G(V^{\rho(G_{\rho(g)\cdot w})}) = B_G(\rho(g)(V^{\rho(G_w)}) = B_G(V^{\rho(G_w)}).$$

Therefore, we have our assertion. □

Definition 3.6. Let (G, ρ, V) be a triplet and $v \in V$.
(1) We say that G_v satisfies the minimal condition if $G_v = G_w$ holds for any $w \in V$ satisfying $G_v \supset G_w$.
(2) We say that $V^{\rho(G_v)}$ satisfies the maximal condition if $V^{\rho(G_v)} = V^{\rho(G_w)}$ holds for any $w \in V$ satisfying $V^{\rho(G_v)} \subset V^{\rho(G_w)}$.
(3) We say that $B_G(V^{\rho(G_v)})$ satisfies the maximal condition if

$$B_G(V^{\rho(G_v)}) = B_G(V^{\rho(G_w)})$$

holds for any $w \in V$ satisfying $B_G(V^{\rho(G_v)}) \subset B_G(V^{\rho(G_w)})$.

Proposition 3.7. *Let (G, ρ, V) be a triplet and $v \in V$. Then the following statements are equivalent.*
(1) G_v satisfies the minimal condition.
(2) For any $g \in G$, $G_{\rho(g)\cdot v}$ satisfies the minimal condition.
(3) $V^{\rho(G_v)}$ satisfies the maximal condition.
(4) For any $g \in G$, $V^{\rho(G_{\rho(g)\cdot v})}$ satisfies the maximal condition.
(5) $B_G(V^{\rho(G_v)})$ satisfies the maximal condition.

Proof. Since $G_{\rho(g)\cdot v}(= gG_v g^{-1}) \supset G_w$ (resp. $G_{\rho(g)\cdot v} = G_w$) is equivalent to

$$G_v \supset G_{\rho(g^{-1})\cdot w}(= g^{-1}G_w g),$$

(resp. $G_v = G_{\rho(g^{-1})\cdot w}$), we obtain the equivalence of (1) and (2). By Proposition 2.9, we have the equivalence of (1) (resp. (2)) and (3) (resp. (4)). By Lemma 3.5, we have the equivalence of (1) and (5). □

Proposition 3.8. *Let (G, ρ, V) be a triplet and $V = \bigsqcup_{\lambda \in \Lambda} \rho(G) \cdot v_\lambda$ the $G-$orbital decomposition of V $(v_\lambda \in V)$. Then the following assertions hold.*
(1) Let $\tilde{\Lambda} = \{\lambda \in \Lambda \mid G_{v_\lambda}$ satisfies the minimal condition$\}$. Then we have

$$V = \bigcup_{\lambda \in \tilde{\Lambda}} B_G(V^{\rho(G_{v_\lambda})}).$$

(2) Let $\Lambda' \subset \Lambda$, $V = \bigcup_{\lambda \in \Lambda'} B_G(V^{\rho(G_{v_\lambda})})$, and no relation of inclusion between $B_G(V^{\rho(G_{v_\lambda})})$ and $B_G(V^{\rho(G_{v_\mu})})$ for any $\lambda \in \Lambda'$ and for any $\mu \in \Lambda'(\lambda \neq \mu)$. Then $\Lambda' \subset \tilde{\Lambda}$.

Proof. For (1), first, $(V =) \bigsqcup_{\lambda\in\Lambda} \rho(G)\cdot v_\lambda \supset \bigcup_{\lambda\in\tilde{\Lambda}} B_G(V^{\rho(G_{v_\lambda})})$ is clear. On the other hand, let v be an arbitrary element in V. Then there exist $\lambda_0 \in \Lambda$, $g_0 \in G$ satisfying $v = \rho(g_0)\cdot v_{\lambda_0} \in \rho(G)\cdot v_{\lambda_0}$. If G_v satisfies the minimal condition, then $G_{v_{\lambda_0}}$ satisfies the minimal condition by Proposition 3.7. Hence we have $\lambda_0 \in \tilde{\Lambda}$, and

$$v \in V^{\rho(G_v)} = \rho(g_0)(V^{\rho(G_{v_{\lambda_0}})}) \subset B_G(V^{\rho(G_{v_{\lambda_0}})}) \subset \bigcup_{\lambda\in\tilde{\Lambda}} B_G(V^{\rho(G_{v_\lambda})})$$

by Proposition 2.13. If G_v does not satisfy the minimal condition, then there exists $v_1 \in V$ satisfying $G_v \supsetneq G_{v_1}$. Similarly there exist $\lambda_1 \in \Lambda,\ g_1 \in G$ satisfying

$$v_1 = \rho(g_1)\cdot v_{\lambda_1} \in \rho(G)\cdot v_{\lambda_1}\ (\lambda_1 \neq \lambda_0).$$

By repeating this procedure, if there exists an infinite descending chain $G_v \supsetneq G_{v_1} \supsetneq \cdots$, then by (3) of Proposition 2.9, there exists an infinite ascending chain $V^{\rho(G_v)} \subsetneq V^{\rho(G_{v_1})} \subsetneq \cdots$. It is clearly a contradiction because V is a finite dimensional vector space. Hence there exists $v_k \in V$ stabilizing the infinite descending chain as $G_v \supsetneq G_{v_1} \supsetneq \cdots \supsetneq G_{v_k}$, which means that G_{v_k} satisfies the minimal condition. Hence there exist $\lambda_k \in \tilde{\Lambda},\ g_k \in G$ satisfying $v_k = \rho(g_k)\cdot v_{\lambda_k} \in \rho(G)\cdot v_{\lambda_k}$ by Proposition 3.7. Hence we have

$$v \in V^{\rho(G_v)} \subsetneq V^{\rho(G_{v_1})} \subsetneq \cdots \subsetneq V^{\rho(G_{v_k})} = \rho(g_k)(V^{\rho(G_{v_{\lambda_k}})}) \subset \bigcup_{\lambda\in\tilde{\Lambda}} B_G(V^{\rho(G_{v_\lambda})})$$

by (3) of Proposition 2.9 and Proposition 2.13.

To prove (2), let λ be an arbitrary element in Λ'. By Proposition 3.7, it is enough to show that $B_G(V^{\rho(G_{v_\lambda})})$ satisfies the maximal condition. For any $w \in V$ satisfying $B_G(V^{\rho(G_{v_\lambda})}) \subset B_G(V^{\rho(G_w)})$, there exist $\mu \in \Lambda',\ g \in G$ satisfying

$$w = \rho(g)\cdot v_\mu \in \rho(G)\cdot v_\mu$$

by the assumption in (2). Obviously, since $G_w = G_{\rho(g)\cdot v_\mu}$, we have

$$B_G(V^{\rho(G_{v_\lambda})}) \subset B_G(V^{\rho(G_w)}) = B_G(V^{\rho(G_{\rho(g)\cdot v_\mu})}) = B_G(V^{\rho(G_{v_\mu})})$$

by Proposition 2.13. Hence we have $\lambda = \mu$ by the assumption in (2), which means

$$B_G(V^{\rho(G_{v_\lambda})}) = B_G(V^{\rho(G_{v_\mu})}) = B_G(V^{\rho(G_w)}).$$

Hence we have $\lambda \in \tilde{\Lambda}$. □

Remark 3.9. Clearly by Proposition 3.8, we have $\Lambda' \subset \tilde{\Lambda} \subset \Lambda$. In general, there appear the two cases $\sharp\Lambda' < \infty$ or $\sharp\Lambda' = \infty$. For example of the former case, a triplet (G, ρ, V) with finitely many orbits (which is always a PV) has $\sharp\Lambda < \infty$, hence $\sharp\Lambda' < \infty$. Additionally, for example of the latter case, let

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \mid a, b \in \mathbb{C}, c \in \mathbb{C}^\times \right\},$$

$\rho = \Lambda_1$, and $V = \mathbb{C}^3$. Here we identify $v_{[x,y,z]}$ with ${}^t(x, y, z) (\in \mathbb{C}^3)$. Then by the direct calculation, we have $\Lambda = \{[0, y_0, z_0] |\ y_0, z_0 \in \mathbb{C}$ satisfying $y_0 \neq 0$ or $z_0 \neq 0,$

except $[0, ty_0, tz_0]$ with $t \in \mathbb{C} - \{1\}\} \sqcup \{[x_0, 0, 0] |\ x_0 \in \mathbb{C}^\times\} \sqcup \{[0, 0, 0]\}$. Here put $\Lambda' = \Lambda - (\{[x_0, 0, 0] |\ x_0 \in \mathbb{C}^\times\} \sqcup \{[0, 0, 0]\})$. For $[0, y_0, z_0] \in \Lambda'$, we have

$$G_{v_{[0,y_0,z_0]}} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G |\ ay_0 + bz_0 = 0 \right\}$$

and

$$V^{\rho(G_{v_{[0,y_0,z_0]}})} = \{{}^t(x\ y\ z) \in \mathbb{C}^3 |\ yz_0 - zy_0 = 0\} = B_G(V^{\rho(G_{v_{[0,y_0,z_0]}})})$$

of which dimension is 2. Clearly, $\mathbb{C}^3 = \bigcup_{[0,y_0,z_0] \in \Lambda'} B_G(V^{\rho(G_{v_{[0,y_0,z_0]}})})$, also there is no relation of inclusion between $B_G(V^{\rho(G_{v_{[0,y_0,z_0]}})})$ and $B_G(V^{\rho(G_{v_{[0,y_0{}',z_0{}']}})})$ for $[0, y_0, z_0], [0, y_0{}', z_0{}'] \in \Lambda'$ ($[0, y_0, z_0] \neq [0, y_0{}', z_0{}']$). Hence Λ' satisfies the assumptions in (2) of Proposition 3.8. So we have $\sharp\Lambda' = \infty$.

Proposition 3.10. *Let (G, ρ, V) be a PV with a generic point $v \in V$. Then the generic isotropy subgroup G_v (resp. $V^{\rho(G_v)}$ and $B_G(V^{\rho(G_v)})$) satisfies the minimal condition (resp. satisfy the maximal condition).*

Proof. By Proposition 3.7, it is enough to show that G_v satisfies the minimal condition. In general, for a PV (G, ρ, V), a point $v \in V$ satisfies $\dim G_v = \dim G - \dim V$ if and only if v is a generic point (see [11, Proposition 2 in section 2]). Let w be an arbitrary element in V satisfying $G_v \supset G_w$. Then by (1), $\dim G - \dim V = \dim G_v \geq \dim G_w = \dim G - \dim \overline{\rho(G) \cdot w} \geq \dim G - \dim V$. Hence we have $\dim G_v = \dim G_w$, which means $\overline{\rho(G) \cdot v} = \overline{\rho(G) \cdot w} = V$. Here $\rho(G) \cdot v$ and $\rho(G) \cdot w$ are open G−orbits in V, and V is an irreducible set on Zariski-topology, so we have $\rho(G) \cdot v \cap \rho(G) \cdot w \neq \phi$, which means $\rho(G) \cdot v = \rho(G) \cdot w$. Clearly there exists $g \in G$ satisfying $v = \rho(g) \cdot w$. By (2) of Lemma 3.2, $G_{\rho(g) \cdot w} \supset G_w$ if and only if $G_v = G_{\rho(g) \cdot w} = G_w$. □

Theorem 3.11. *Let (G, ρ, V) be a triplet. Then the following assertions are equivalent.*
(1) (G, ρ, V) is a PV with a generic point $v \in V$.
(2) There exists a point $v \in V$ satisfying (i) and (ii).
(i) $B_G(V^{\rho(G_v)})$ is Zariski-dense in V.
(ii) $(N_G(G_v), \rho, V^{\rho(G_v)})$ is a PV with a generic point $v \in V$.

Proof. First assume (1). For (2)-(i), by (5) of Proposition 3.3, we have $\rho(G) \cdot v \subset B_G(V^{\rho(G_v)})$. Then by (1), clearly $V = \overline{\rho(G) \cdot v} \subset \overline{B_G(V^{\rho(G_v)})} = V$. Thus we have (2)-(i). Here for simplicity, we put $N = N_G(G_v)$. For (2)-(ii), by (7) of Proposition 3.3, we have $\rho(N) \cdot v = \rho(G) \cdot v \cap V^{\rho(G_v)}$. Since $\rho(G) \cdot v$ is an open set in V by (1), hence $\rho(N) \cdot v$ is a non-empty open set in $V^{\rho(G_v)}$. Here $V^{\rho(G_v)}$ is an irreducible set, hence $\rho(N) \cdot v$ is Zariski-dense in $V^{\rho(G_v)}$. Next assume (2). For any $g \in G$, the map $\rho(g) : V \to V$ which is defined as $v \mapsto \rho(g) \cdot v$ is a morphism. By (2)-(ii) and the continuity of $\rho(g)$, we have

$$\rho(g)(V^{\rho(G_v)}) = \rho(g)(\overline{\rho(N) \cdot v}) \subset \overline{\rho(g)(\rho(N) \cdot v)} = \overline{\rho(gN) \cdot v}.$$

Since $N \subset G$, we have $\overline{\rho(gN)\cdot v} \subset \overline{\rho(G)\cdot v}$. Here by (2)-(i), we have

$$V = \overline{B_G(V^{\rho(G_v)})} = \overline{\bigcup_{g\in G} \rho(g)(V^{\rho(G_v)})} \subset \overline{\rho(G)\cdot v} = V.$$

We have our result. □

Remark 3.12. In general, for non PVs, the various cases appear with respect to (2) of Theorem 3.11. For example, let

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \mid a, b \in \mathbb{C}, c \in \mathbb{C}^\times \right\},$$

$\rho = \Lambda_1$, and $V = \mathbb{C}^3$ (these are in Remark 3.9). Then (G, ρ, V) is a non PV. For a point $v = {}^t(0, y_0, z_0) \in \mathbb{C}^3$ ($y_0 \neq 0$ or $z_0 \neq 0$), we have

$$G_v = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{C}, ay_0 + bz_0 = 0 \right\},\ N_G(G_v) = G,$$

and $\overline{\rho(N_G(G_v))\cdot v} = V^{\rho(G_v)} = B_G(V^{\rho(G_v)}) = \{{}^t(x, ty_0, tz_0) \in \mathbb{C}^3 \mid x, t \in \mathbb{C}\} \subsetneq \mathbb{C}^3$. Hence v satisfies (2)-(ii) of Theorem 3.11, however, does not satisfy (2)-(i) of Theorem 3.11. Additionally, for a point $w = {}^t(1, 0, 0) \in \mathbb{C}^3$, we have $G_w = G, N_G(G_w) = G$, and $\overline{\rho(N_G(G_w))\cdot w} = \{w\} \subsetneq V^{\rho(G_w)} = B_G(V^{\rho(G_w)}) = \{{}^t(x, 0, 0) \in \mathbb{C}^3 \mid x \in \mathbb{C}\} \subsetneq \mathbb{C}^3$. Therefore, w does not satisfy either (2)-(i) of Theorem 3.11 or (2)-(ii) of Theorem 3.11.

Corollary 3.13. *Let (G, ρ, V) be a PV with a generic point $v \in V$. Then we have* $\dim G - \dim V = \dim G_v = \dim(N_G(G_v))_v = \dim N_G(G_v) - \dim V^{\rho(G_v)}$.

Proof. By (9) of Proposition 3.3 and Theorem 3.11, it is clear. □

Example 3.14. Let $(G, \rho, V) = (GL(2), 3\Lambda_1, \mathbb{C}^4)$. Then it is an irreducible reductive PV with a generic point $v = {}^t(1, 0, 0, 1) \in V$ (see [5, Example 2.4]). Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2).$$

Then

$$3\Lambda_1\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^3 & a^2b & ab^2 & b^3 \\ 3a^2c & a^2d + 2abc & 2abd + b^2c & 3b^2d \\ 3ac^2 & 2acd + bc^2 & ad^2 + 2bcd & 3bd^2 \\ c^3 & c^2d & cd^2 & d^3 \end{pmatrix}$$

and the generic isotropy subgroup

$$G_v = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a^3 = d^3 = 1 \right\} \sqcup \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b^3 = c^3 = 1 \right\}.$$

By the direct calculations, we have $V^{\rho(G_v)} = \{{}^t(x, 0, 0, x) \in V \mid x \in \mathbb{C}\}$. Here put $H = \{\alpha I_2 \mid \alpha \in \mathbb{C}^\times\}$. Then we obtain $H \lhd G$ and

$$N_G(G_v) = H \sqcup \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} H \sqcup \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \end{pmatrix} H \sqcup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} H \sqcup \begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix} H \sqcup \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} H$$

where ω ($\neq 1$) is a cubic root of 1. Clearly, $(N_G(G_v), \rho, V^{\rho(G_v)})$ is a PV with a generic point v, and $(N_G(G_v))_v = G_v$. Since $\dim N_G(G_v) = 1$, $\dim V^{\rho(G_v)} = 1$, and $\dim G_v = 0$, we have the equations in Corollary 3.13.

Example 3.15. Let m, n be positive integers with $m > n \geq 2$, and

$$(G, \rho, V) = (GL(m), \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{n}, \overbrace{\mathbb{C}^m \oplus \cdots \oplus \mathbb{C}^m}^{n}).$$

Then it is a non-irreducible reductive PV with a generic point

$$v = (e_1^{(m)}, \cdots, e_n^{(m)}) \ (e_i^{(m)} \in \mathbb{C}^m, \ i = 1, \cdots, n)$$

(see [5, Proposition 7.1]). By the direct calculations, the generic isotropy subgroup

$$G_v = \{ \begin{pmatrix} I_n & B \\ 0 & C \end{pmatrix} \mid B \in M(n, m-n), \ C \in GL(m-n) \}.$$

Then we have

$$V^{\rho(G_v)} = \left\{ \left(\begin{pmatrix} X_1 \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} X_n \\ 0 \end{pmatrix} \right) \in V \mid X_i \in M(n, 1), \ i = 1, \cdots, n \right\},$$

and

$$N_G(G_v) = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in GL(n), \ B \in M(n, m-n), \ C \in GL(m-n) \right\}.$$

Hence $(N_G(G_v), \rho, V^{\rho(G_v)})$ is a PV with a generic point v, and $(N_G(G_v))_v = G_v$. Since $\dim N_G(G_v) = m^2 - mn + n^2, \dim V^{\rho(G_v)} = n^2$, and $\dim G_v = m^2 - mn$, we have the equations in Corollary 3.13.

Example 3.16. Let

$$G = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mid a \in \mathbb{C}, \ b \in \mathbb{C}^\times \right\}, \ \rho = \Lambda_1, \ \text{and } V = \mathbb{C}^2.$$

Then (G, ρ, V) is a non-reductive PV with a generic point $v = {}^t(0\ 1) \in V$, and the generic isotropy subgroup $G_v = \{I_2\}$. Hence we have $(G, \rho, V) = (N_G(G_v), \rho, V^{\rho(G_v)})$, which satisfies (1), (2), and (3) in Proposition 3.21.

Proposition 3.17. *For a triplet (G, ρ, V) and for $v \in V$, let $N_G(G_v)/G_v$ be the residue class group of $N_G(G_v)$ by G_v, and $\tilde{\rho} : N_G(G_v)/G_v \to GL(V^{\rho(G_v)})$ a map defined as $\overline{g} \mapsto \tilde{\rho}(\overline{g}) = \rho(g)|_{V^{\rho(G_v)}}$. Then $\tilde{\rho}$ is a representation of $N_G(G_v)/G_v$ on $V^{\rho(G_v)}$.*

Proof. Let $\overline{g} \in N_G(G_v)/G_v$ ($g \in N_G(G_v)$). For simplicity, put $N = N_G(G_v)$. Since $V^{\rho(G_v)}$ is an $N-$invariant subspace of V by (6) of Proposition 3.3, we have

$$\tilde{\rho}(\overline{g}) = \rho(g)|_{V^{\rho(G_v)}} \in GL(V^{\rho(G_v)})$$

by (1) of Proposition 2.13. Here let $\overline{g_1}, \ \overline{g_2} \in N_G(G_v)/G_v$. If $\overline{g_1} = \overline{g_2}$, then there exists $h \in G_v$ satisfying $g_1 = g_2 h$. For any $w \in V^{\rho(G_v)}$, we have

$$\tilde{\rho}(\overline{g_1}) \cdot w = \rho(g_1) \cdot w = \rho(g_2 h) \cdot w = \rho(g_2) \cdot w = \tilde{\rho}(\overline{g_2}) \cdot w.$$

Hence $\tilde{\rho}$ is well-defined.
Clearly, $\tilde{\rho}$ is a group homomorphism and we have our result. □

As well-known facts, an affine algebraic group is isomorphic to a linear algebraic group (see [1, Proposition 1.10]). Also, when G is an affine algebraic group, and H is a normal algebraic subgroup of G, then G/H is an affine algebraic group (see [1, Theorem 6.8]). Hence G/H is isomorphic to a linear algebraic group. For a triplet (G, ρ, V) and $v \in V$, the group $N_G(G_v)$ is a linear algebraic group by (1) of Proposition 3.3. Hence $N_G(G_v)/G_v$ is isomorphic to a linear algebraic group. For simplicity, in this paper, we identify $N_G(G_v)/G_v$ with a linear algebraic group, and $\tilde{\rho}$ with a rational linear representation of $N_G(G_v)/G_v$ on $V^{\rho(G_v)}$.

In general, when a PV(G, ρ, V) satisfies $\dim G = \dim V$, then such a PV is called a cuspidal PV.

Proposition 3.18. *Let (G, ρ, V) be a PV with a generic point $v \in V$. Then the triplet $(N_G(G_v)/G_v, \tilde{\rho}, V^{\rho(G_v)})$ is a cuspidal PV with a generic point $v \in V^{\rho(G_v)}$. Its generic isotropy subgroup is the unit group $\{\overline{I_G}\}$.*

Proof. For simplicity, put $N = N_G(G_v)$. By (2) of Theorem 3.11 and Corollary 3.13, we have

$$\overline{\tilde{\rho}(N/G_v) \cdot v} = \overline{\rho(N) \cdot v} = V^{\rho(G_v)},$$

and $\dim N/G_v = \dim N - \dim G_v = \dim N - \dim N_v = \dim V^{\rho(G_v)}$.
Hence $(N/G_v, \tilde{\rho}, V^{\rho(G_v)})$ is identified with a cuspidal PV. Moreover the generic isotropy subgroup

$$\begin{aligned}(N/G_v)_v &= \{\overline{g} \in N/G_v |\ \tilde{\rho}(\overline{g}) \cdot v = v\} = \{\overline{g} \in N/G_v |\ \rho(g) \cdot v = v\} \\ &= \{\overline{g} \in N/G_v |\ g \in G_v\} = \{\overline{I_G}\},\end{aligned}$$

we have our results. □

Remark 3.19. For any reductive PV with a non-zero relative invariant at least, there is a method of constructing a cuspidal PV ([2]). This method is called contraction, which is different from our method in Proposition 3.18. For example, let (G, ρ, V) be an irreducible reductive PV $(SL(2) \times GL(2), \Lambda_1 \otimes \Lambda_1^*, M(2))$ with a relative invariant $f(X) = \det X$ $(X \in M(2))$ and a generic point $I_2 \in M(2)$. Then by the direct calculations, contraction induces a cuspidal PV $(GL(1) \times GL(1), \Lambda_1^* \otimes 1 \oplus 1 \otimes \Lambda_1^*, \mathbb{C} \oplus \mathbb{C})$, while our method induces a cuspidal PV $(GL(1), \Lambda_1^*, \mathbb{C})$.

Lemma 3.20. *Let (G, ρ, V) be a triplet. Then the following assertions hold.*
(1) Let $g \in G$. Then $g \in \operatorname{Ker} \rho$ if and only if $V^{\rho(g)} = V$.
(2) $G_V = \operatorname{Ker} \rho$.
(3) If (G, ρ, V) is a PV with a generic point $v \in V$, then $G_{\rho(G) \cdot v} = \operatorname{Ker} \rho$.

Proof. To prove (1), let $g \in \operatorname{Ker} \rho$. Then we have $V^{\rho(g)} = V$. On the other hand, let $V^{\rho(g)} = V$. If $g \notin \operatorname{Ker} \rho$, then $\rho(g) - I_V : V \to V$ is a non-zero linear transformation on V. Clearly, we obtain $\operatorname{Im}(\rho(g) - I_V) \supsetneq \{0_V\}$ and $\dim \operatorname{Im}(\rho(g) - I_V) > 0$. Since $V^{\rho(g)} = \{w \in V |\ \rho(g) \cdot w = w\} = \{w \in V |\ (\rho(g) - I_V) \cdot w = 0\} = \operatorname{Ker}(\rho(g) - I_V)$, we have $\dim V^{\rho(g)} = \dim V - \dim \operatorname{Im}(\rho(g) - I_V) < \dim V$, which is a contradiction. We have our result.
To prove (2), let $g \in G_V$. Then we have $V \subset V^{\rho(g)}$ by (2) of Proposition 2.4. Hence $g \in \operatorname{Ker} \rho$ by (1). On the other hand, $G_V \supset \operatorname{Ker} \rho$ is clear.

For (3), since $\rho(G)\cdot v \subset V$, we have $G_{\rho(G)\cdot v} \supset G_V = \operatorname{Ker}\rho$ by (4) of Proposition 2.4 and (2). On the other hand, for any $h \in G_{\rho(G)\cdot v}$, we obtain $\rho(G)\cdot v \subset V^{\rho(h)}$ by (2) of Proposition 2.4. Since (G,ρ,V) is a PV, we have $V = \overline{\rho(G)\cdot v} \subset \overline{V^{\rho(h)}} = V^{\rho(h)}$. Hence $h \in \operatorname{Ker}\rho$ by (1). □

Proposition 3.21. *Let (G,ρ,V) be a PV with a generic point $v \in V$. Then the following assertions are equivalent.*
(1) $G = N_G(G_v)$ (i.e., $G \triangleright G_v$). (2) $V = V^{\rho(G_v)}$. (3) $G_v = \operatorname{Ker}\rho$.

Proof. Firstly, we shall show the equivalence of (1) and (2). Assume (1). Clearly, we have $\dim G = \dim N_G(G_v)$. By Corollary 3.13, we have $\dim V^{\rho(G_v)} = \dim V$, hence $V^{\rho(G_v)} = V$. On the other hand, assume (2). Since $\dim V = \dim V^{\rho(G_v)}$, we have $\dim G = \dim N_G(G_v)$ by Corollary 3.13. In general, a connected linear algebraic group is an irreducible set. Hence we have $G = \overline{N_G(G_v)}$. By (1) of Proposition 3.3, we have $G = N_G(G_v)$. Secondly, we shall show the equivalence of (2) and (3). Assume (2). Then we have $G_V = G_{V^{\rho(G_v)}} = G_v$ by (2) of Lemma 2.7. Hence by (2) of Lemma 3.20, we have $G_v = \operatorname{Ker}\rho$. Assume (3). Then (2) is clear.

□

4. Appendices

In this section, as appendices, we give some facts. The assumptions in this section are the same as those in section 3.

Proposition 4.1. *Let (G,ρ,V) be a triplet. Then the following assertions are equivalent.*
(1) ρ is irreducible.
(2) Let W be an arbitrary non-zero $G-$invariant subspace of V. Then $W \in \operatorname{Im}\Phi$ and $G_W = \operatorname{Ker}\rho$.

Proof. First assume (1), let W be an arbitrary non-zero $G-$invariant subspace of V. Then we obtain $W = V$. By (2) of Corollary 2.17, we have $W = V \in \operatorname{Im}\Phi$, and $G_W = G_V = \operatorname{Ker}\rho$ by (2) of Lemma 3.20. Next assume (2), let W be an arbitrary non-zero $G-$invariant subspace of V. Then $W \in \operatorname{Im}\Phi$ if and only if $W = V^{\rho(G_W)}$ by (2) of Corollary 2.17. Here $G_W = \operatorname{Ker}\rho$, hence $W = V^{\rho(G_W)} = V$, we have our result. □

Proposition 4.2. *Let (G,ρ,V) be a triplet and $\rho(G) \neq \{I_V\}$. If ρ is irreducible, then $V^{\rho(G)} = \{0_V\}$.*

Proof. For any $g \in G$, by (1) of Proposition 2.13, we have

$$\rho(g)(V^{\rho(G)}) = V^{\rho(gGg^{-1})} = V^{\rho(G)}.$$

Hence $V^{\rho(G)}$ is a $G-$invariant subspace of V. Since ρ is irreducible, there are the two cases $V^{\rho(G)} = \{0_V\}$ or $V^{\rho(G)} = V$. If the latter case appears , by (1) of Corollary 2.17 and (2) of Lemma 3.20, we have $G = G_{V^{\rho(G)}} = G_V = \operatorname{Ker}\rho$, which is a contradiction with $\rho(G) \neq \{I_V\}$. Hence $V^{\rho(G)} = \{0_V\}$. □

Corollary 4.3. *Let $(G,\rho,V) = (G,\rho_1 \oplus \cdots \oplus \rho_l, V_1 \oplus \cdots \oplus V_l)$ be a triplet where each ρ_i $(i = 1,\cdots,l)$ is an indecomposable representation of G on V_i and $\rho_i(G) \neq \{I_{V_i}\}$. If there exists ρ_i satisfying $V_i^{\rho_i(G)} \neq \{0_{V_i}\}$, then ρ_i is non-irreducible and G is non-reductive. However, the converse does not hold.*

Proof. Clearly, ρ_i is non-irreducible by Proposition 4.2. If G is reductive, then each ρ_i $(i = 1,\cdots,l)$ is an irreducible representation of G on V_i. By Proposition 4.2, we have $V_i^{\rho_i(G)} = \{0_{V_i}\}$, which is a contradiction. Hence we have the former assertion. For example of the latter assertion, let (G,ρ,V) be a triplet where

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a,c \in \mathbb{C}^\times, b \in \mathbb{C} \right\},\ \rho = \Lambda_1,\ \text{and } V = \mathbb{C}^2.$$

By the direct calculation, we have $V^{\rho(G)} = \{0_V\}$. □

Proposition 4.4. *Let (G,ρ,V) be a triplet. Then the following assertions hold.*
(1) For any $v \in V$, the group $G_{\rho(G)\cdot v}$ is a normal subgroup of G, and the set $V^{\rho(G_{\rho(G)\cdot v})}$ is a $G-$invariant subspace of V.
(2) Let v be any non-zero element in V. If ρ is irreducible, then $G_{\rho(G)\cdot v} = \operatorname{Ker}\rho$.

Proof. To prove (1), let g be any element in G. Then we have

$$gG_{\rho(G)\cdot v}g^{-1} = G_{\rho(g)\rho(G)\cdot v} = G_{\rho(G)\cdot v}$$

by (2) of Proposition 2.13. Hence $G_{\rho(G)\cdot v}$ is a normal subgroup of G.
Clearly, $V^{\rho(G_{\rho(G)\cdot v})}$ is a $G-$invariant subspace of V by Proposition 2.13.
For (2), by (1), $V^{\rho(G_{\rho(G)\cdot v})}$ is a $G-$invariant subspace of V. Since ρ is irreducible and $\{0_V\} \neq \rho(G)\cdot v \subset V^{\rho(G_{\rho(G)\cdot v})}$ by (1) of Lemma 2.7, we have $V^{\rho(G_{\rho(G)\cdot v})} = V$, which is equivalent to $V^{\rho(G_{\rho(G)\cdot v})} = V^{\rho(G_V)}$ by (2) of Corollary 2.17. Since $G_V \in \operatorname{Im}\Psi$ is the maximal generator for $V^{\rho(G_V)}$ in $P(G)$ and $G_V = \operatorname{Ker}\rho$ by (2) of Lemma 3.20, we have $G_{\rho(G)\cdot v} \subset G_V = \operatorname{Ker}\rho$. On the other hand, since $\rho(G)\cdot v \subset V$, we obtain $G_{\rho(G)\cdot v} \supset G_V = \operatorname{Ker}\rho$ by (4) of Proposition 2.4 and (2) of Lemma 3.20. So we have our results. □

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