Fixed Point Theory, 20(2019), No. 1, 255-270
DOI: 10.24193/fpt-ro.2019.1.17
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

THE SET OF SOLUTIONS TO A NONLINEAR INTEGRODIFFERENTIAL EQUATION IN N VARIABLES

LE THI PHUONG NGOC*, HUYNH THI HOANG DUNG**,*** AND NGUYEN THANH LONG***

*University of Khanh Hoa, 01 Nguyen Chanh Str., Nha Trang City, Vietnam
E-mail: ngoc1966@gmail.com

**Department of Basic Sciences, University of Architecture Ho Chi Minh City,
196 Pasteur Str., Dist. 3, Ho Chi Minh City, Vietnam
E-mail: dunghth1980@gmail.com

***Department of Mathematics and Computer Science, University of Science
Vietnam National University Ho Chi Minh City, 227 Nguyen Van Cu Str., Dist. 5
Ho Chi Minh City, Vietnam
E-mail: longnt2@gmail.com

Abstract. In this paper, we study the existence and the compactness of the set of solutions for a nonlinear integrodifferential equation in N variables. The main tools are the fixed point theorem of Krasnosel'skii with the definition of a suitable Banach space and a sufficient condition for relative compactness of subsets in this space. An illustrative example is given.

Key Words and Phrases: Nonlinear integrodifferential equation, the fixed point theorem of Krasnosel'skii.
2010 Mathematics Subject Classification: 45G10, 47H10, 47N20, 65J15.

1. Introduction

In this paper, we consider the following nonlinear integrodifferential equation in N variables

$$u(x) = g(x) + \int_{B_x} H(x, y, u(y), D_1u(y))dy + \int_{\Omega} K(x, y, u(y), D_1u(y))dy, \qquad (1.1)$$

where $x = (x_1, ..., x_N) \in \Omega = [0, 1]^N$ and $g : \Omega \to \mathbb{R}$, $H : \Delta\times\mathbb{R}^2 \to \mathbb{R}$, $K : \Omega\times\Omega\times\mathbb{R}^2 \to \mathbb{R}$ are given functions, with $\Delta = \{(x, y) \in \Omega \times \Omega : y \in B_x\}$, $B_x = [0, x_1] \times ... \times [0, x_N]$. Denote by $D_1u = \frac{\partial u}{\partial x_1}$, the partial derivative of a function $u(x)$ defined on Ω, with respect to the first variable.

The problem of existence of solutions with certain properties for nonlinear integral and integrodifferential equations has attracted the interest of scientists not only because of their major role in the fields of functional analysis but also because of their important role in numerous applications, for example, mechanics, physics, population dynamics, economics and other fields of science, see Corduneanu [7], Deimling [8], E.

Zeidler [18]. In general, existence results of integral and integrodifferential equations in one variable or N variables, have been obtained via the fundamental methods in which the fixed point theorems are often applied, see [2] - [18] and the references given therein. It is also known that existence theory for nonlinear integral equations, strongly related with the evolution on fixed point theory, has been boosted ahead after the remarkable work of Krasnosel'skii which signaled a new era in the research of the subject, see [17] and the references given therein. Two main results of fixed point theory are Schauder's Theorem and Banach's Theorem (also called contraction mapping principle), Krasnosel'skii combined them into the following result.

Theorem 1.1. (see [4], [8]). *Let M be a nonempty bounded closed convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that $U : M \to X$ is a contraction and $C : M \to X$ is a completely continuous operator such that $U(x) + C(y) \in M$, $\forall x, y \in M$. Then $U + C$ has a fixed point in M.*

In [5], using a fixed point theorem of Krasnosel'skii, Avramescu and Vladimirescu have proved the existence of asymptotically stable solutions to the following integral equation

$$u(t) = q(t) + \int_0^t K(t, s, u(s))ds + \int_0^\infty G(t, s, u(s))ds, \ t \in \mathbb{R}_+, \tag{1.2}$$

where the real functions are supposed to be continuous satisfying suitable conditions.

In [14], based on the well known Banach fixed point theorem coupled with Bielecki type norm and a certain integral inequality with explicit estimate, B. G. Pachpatte proved uniqueness and other properties of solutions of the following Fredholm type integrodifferential equation

$$x(t) = g(t) + \int_a^b f(t, s, x(s), x'(s), ..., x^{(n-1)}(s))ds, \ t \in [a, b], \tag{1.3}$$

where x, g, f are real valued functions and $n \geq 2$ is an integer. With the same methods, B. G. Pachpatte studied the existence, uniqueness and some basic properties of solutions of the Fredholm type integral equation in two variables, see [15],

$$u(x, y) = f(x, y) + \int_0^a \int_0^b g\left(x, y, s, t, u(s, t), D_1u(s, t), D_2u(s, t)\right) dtds, \tag{1.4}$$

yet of certain Volterra integral and integrodifferential equations in two variables, see [16].

In [6], El-Borai et al. have proved the existence of a unique solution of a nonlinear integral equation of type Volterra-Hammerstein in n-dimensions of the form

$$\mu\phi(x, t) = f(x, t) + \lambda \int_0^t \int_\Omega F(t, \tau)K(x, y)\gamma\left(\tau, y, \phi(y, \tau)\right) dyd\tau, \tag{1.5}$$

where $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$; μ, λ are constants. After that, in [2], M. A. Abdou et al. investigated the following mixed nonlinear integral equation of the second kind in n-dimensions

$$\mu\phi(x,t) = \lambda \int_{\Omega} k(x,y)\gamma\left(t,y,\phi(y,t)\right) dy + \lambda \int_0^t \int_{\Omega} G(t,\tau)k(x,y)\gamma\left(\tau,y,\phi(y,\tau)\right) dyd\tau$$
$$+ \lambda \int_0^t F(t,\tau)\phi(x,\tau)d\tau + f(x,t), \tag{1.6}$$

where $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$. Also using the Banach fixed point theorem, the existence of a unique solution of these equations was proved.

M. A. Abdou et al. also considered the existence of integrable solution of nonlinear integral equation, of type Hammerstein-Volterra of the second kind, by using the technique of measure of weak noncompactness and Schauder fixed point theorem, see [1].

In [10], M. Lauran established sufficient conditions for the existence of solutions of the integral equation of Volterra type by using the concepts of nonexpansive operators, contraction principles and the Schaefer's fixed point theorem.

In [3], A. Aghajani et al. proved some results on the existence, uniqueness and estimation of the solutions of Fredholm type integro-differential equations in two variables, by using Perov's fixed point theorem.

Recently, in [11] - [13], using tools of functional analysis and a fixed point theorem of Krasnosel'skii type, solvability and asymptotically stable of nonlinear functional integral equations in one variable or two variables, or N variables have been investigated.

Motivated by the problems in the above mentioned works, in this paper, we consider (1.1). This paper consists of four sections. The main result is given in Section 3, where we state and prove Theorem 3.1 about the existence and the compactness of the solutions set for (1.1). The proof of this theorem requires a preliminary study of an appropriate Banach space $(X_1, \ \|\cdot\|_{X_1})$ defined below, and a sufficient condition for relative compactness of subsets of X_1 presented in Section 2. We note more that the Banach space X_1 with the property $C^1(\Omega;\mathbb{R}) \subset X_1 \subset C(\Omega;\mathbb{R})$ has not been used before. Finally, in Section 4, we give an illustrative example.

2. Preliminaries

Put $\Omega = [0,1]^N$ and let $B_x = [0,x_1] \times ... \times [0,x_N]$, $x = (x_1,...,x_N) \in \mathbb{R}^N$,

$$\Delta = \{(x,y) \in \Omega \times \Omega : y \in B_x\}.$$

We also denote by $x = (x_1, ..., x_N) = (x_1, x')$, with $x' = (x_2, ..., x_N)$.

First, we construct an appropriate Banach space for (1.1) as follows.

By $X = C(\Omega;\mathbb{R})$, we denote the space of all continuous functions from Ω into $\mathbb{R}$ equipped with the standard norm:

$$\|u\|_X = \sup_{x\in\Omega} |u(x)|, \ u \in X. \tag{2.1}$$

Put

$$X_1 = \{u \in X = C(\Omega;\mathbb{R}) : D_1 u \in X\}, \tag{2.2}$$

and

$$\|u\|_{X_1} = \|u\|_X + \|D_1 u\|_X, \ \ u \in X_1. \tag{2.3}$$

It is clear that $C^1(\Omega;\mathbb{R}) \subset X_1 \subset X$ and that they do not coincide. Indeed, there exists $u(x) = u(x_1, ..., x_N) = \left|x_1 - \frac{1}{2}\right| + \left|x_2 - \frac{1}{2}\right| + e^{x_1+...+x_N} \in X$, but $u \notin X_1$. And there exists $v(x) = v(x_1, ..., x_N) = x_1^2 \left|x_2 - \frac{1}{2}\right| + e^{x_1+...+x_N} \in X_1$, but $v \notin C^1(\Omega;\mathbb{R})$.

We have the following lemma.

Lemma 2.1. *X_1 is a Banach space with the norm defined by* (2.3).

Proof. Let $\{u_m\} \subset X_1$ be a Cauchy sequence in X_1, then

$$\|u_m - u_p\|_{X_1} = \|u_m - u_p\|_X + \|D_1 u_m - D_1 u_p\|_X \to 0, \text{ as } m,\, p \to \infty, \tag{2.4}$$

it implies that $\{u_m\}$ and $\{D_1 u_m\}$ are also the Cauchy sequences in X. Because X is a complete space, $\{u_m\}$ converges to u and $\{D_1 u_m\}$ converges to v in X, i.e.,

$$\|u_m - u\|_X \to 0,\ \|D_1 u_m - v\|_X \to 0, \text{ as } m \to \infty.$$

Clearly, it is enough to prove $\|u_m - u\|_{X_1} \to 0$, as $m \to \infty$, in the case $D_1 u = v$. This is true, by the fact that

$$u_m(x_1, x') - u_m(0, x') = \int_0^{x_1} D_1 u_m(s, x')ds,\ \ \forall x = (x_1, x') \in \Omega. \tag{2.5}$$

As $m \to \infty$, $\|u_m - u\|_X \to 0$ and $\|D_1 u_m - v\|_X \to 0$, so

$$u_m(x_1, x') - u_m(0, x') \to u(x_1, x') - u(0, x'),\ \ \forall (x_1, x') \in \Omega, \tag{2.6}$$

and

$$\int_0^{x_1} D_1 u_m(s, x')ds \to \int_0^{x_1} v(s, x')ds,\ \ \forall (x_1, x') \in \Omega. \tag{2.7}$$

Note that (2.7) holds, since

$$\begin{aligned} &\left| \int_0^{x_1} D_1 u_m(s, x')ds - \int_0^{x_1} v(s, x')ds \right| \\ &\le \int_0^{x_1} |D_1 u_m(s, x') - v(s, x')|\, ds \le \|D_1 u_m - v\|_X \to 0. \end{aligned} \tag{2.8}$$

It implies from (2.5)-(2.7) that

$$u(x_1, x') - u(0, x') = \int_0^{x_1} v(s, x')ds,\ \forall (x_1, x') \in \Omega. \tag{2.9}$$

Therefore, $D_1 u = v \in X$. Hence, $u \in X_1$ and $u_m \to u$ in X_1. Lemma 2.1 is complete.

Next, we have the following lemma establishing a necessary and sufficient condition for relative compactness of subsets in X_1.

Lemma 2.2. *Let $F \subset X_1$. Then F is relatively compact in X_1 if and only if the following conditions are satisfied*

$$\begin{aligned} &\text{(i) } \exists M > 0 : \|u\|_{X_1} \le M,\ \forall u \in F; \\ &\text{(ii) } \forall \varepsilon > 0, \exists \delta > 0 : \forall x,\, \bar{x} \in \Omega,\ |x - \bar{x}| < \delta \\ &\qquad \Longrightarrow \sup_{u \in F} \left(|u(x) - u(\bar{x})| + |D_1 u(x) - D_1 u(\bar{x})|\right) < \varepsilon. \end{aligned} \tag{2.10}$$

Proof. (a) Let F be relatively compact in X_1. Then F is bounded, (2.10)(i) follows. It remains to show that (2.10)(ii) holds.

For every $\varepsilon > 0$, considering a collection of open balls in X_1, with center at $u \in F$ and radius $\frac{\varepsilon}{3}$, as follows

$$B_{X_1}\left(u, \frac{\varepsilon}{3}\right) = \left\{\bar{u} \in X_1 : \|u - \bar{u}\|_{X_1} < \frac{\varepsilon}{3}\right\}, \ u \in F.$$

It is not difficult to verify that $\overline{F} \subset \bigcup_{u \in F} B_{X_1}\left(u, \frac{\varepsilon}{3}\right)$, where $\overline{F}$ is the closure of F. Since $\overline{F}$ compact in X_1, the open cover $\bigcup_{u \in F} B_{X_1}\left(u, \frac{\varepsilon}{3}\right)$ of $\overline{F}$ contains a finite subcover, it means that there are $u_1, u_2, ..., u_s \in F$ such that

$$\overline{F} \subset \bigcup_{i=1}^{s} B_{X_1}\left(u_i, \frac{\varepsilon}{3}\right).$$

The functions u_i, $D_1 u_i$, $i = 1, ..., s$, are uniformly continuous on Ω, so there exists $\delta > 0$ such that

$$\forall x,\ \bar{x} \in \Omega,\ |x - \bar{x}| < \delta \Longrightarrow |u_i(x) - u_i(\bar{x})| + |D_1 u_i(x) - D_1 u_i(\bar{x})| < \tfrac{\varepsilon}{3},\ \forall i = 1, ..., s.$$

For all $u \in F$, note that $u \in B_{X_1}(u_i, \frac{\varepsilon}{3})$ for some $i = 1, ..., s$. Thus, for all $x,\ \bar{x} \in \Omega$, if $|x - \bar{x}| < \delta$ then we get

$$\begin{aligned}
&|u(x) - u(\bar{x})| + |D_1 u(x) - D_1 u(\bar{x})| \\
&\quad \le |u(x) - u_i(x)| + |u_i(x) - u_i(\bar{x})| + |u_i(\bar{x}) - u(\bar{x})| \\
&\quad + |D_1 u(x) - D_1 u_i(x)| + |D_1 u_i(x) - D_1 u_i(\bar{x})| + |D_1 u_i(\bar{x}) - D_1 u(\bar{x})| \\
&\quad = |u(x) - u_i(x)| + |D_1 u(x) - D_1 u_i(x)| + |u_i(x) - u_i(\bar{x})| \\
&\quad + |D_1 u_i(x) - D_1 u_i(\bar{x})| + |u_i(\bar{x}) - u(\bar{x})| + |D_1 u_i(\bar{x}) - D_1 u(\bar{x})| \\
&\quad \le 2\,\|u - u_i\|_{X_1} + |u_i(x) - u_i(\bar{x})| + |D_1 u_i(x) - D_1 u_i(\bar{x})| \\
&\quad < \tfrac{2\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This implies that (2.10)(ii) holds.

(b) Conversely, let (2.10) hold.

In order to prove that F is relatively compact in X_1, let $\{u_m\}$ be a sequence in F, we have to show that there exists a convergent subsequence of $\{u_m\}$.

By (2.10), $F_1 = \{u_m : m \in \mathbb{N}\}$ and $F_2 = \{D_1 u_m : m \in \mathbb{N}\}$ are uniformly bounded and equicontinuous in X. Hence an application of the Ascoli-Arzela theorem to F_1 implies that it is relatively compact in X, so there exists a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ and $u \in X$ such that

$$\|u_{m_k} - u\|_X \to 0, \text{ as } k \to \infty.$$

Remark that $\{D_1 u_{m_k} : k \in \mathbb{N}\} \subset F_2$ is also uniformly bounded and equicontinuous in X, so it is also relatively compact in X. We deduce the existence of a subsequence of $\{D_1 u_{m_k}\}$, denoted by the same symbol, and $v \in X$, such that

$$\|D_1 u_{m_k} - v\|_X \to 0, \text{ as } k \to \infty.$$

By the fact that

$$u_{m_k}(x_1, x') - u_{m_k}(0, x') = \int_0^{x_1} D_1 u_{m_k}(s, x')ds, \ \ \forall (x_1, x') \in \Omega,$$

furthermore by $\|u_{m_k} - u\|_X \to 0$ and $\|D_1 u_{m_k} - v\|_X \to 0$, we obtain

$$u(x_1, x') - u(0, x') = \int_0^{x_1} v(s, x')ds, \ \ \forall (x_1, x') \in \Omega.$$

As $v \in X$, we see that the right hand side is continuously differentiable with respect to x_1 and this leads to $D_1 u = v \in X$. Therefore $u \in X_1$ and $u_{m_k} \to u$ in X_1. Lemma 2.2 is proved.

3. The existence and the compactness of solutions set

In order to obtain the main result in section, we make the following assumptions

(**A1**) $g \in X_1$;

(**A2**) $H \in C(\Delta \times \mathbb{R}^2; \mathbb{R})$, $\frac{\partial H}{\partial x_1} \in C(\Delta \times \mathbb{R}^2; \mathbb{R})$, such that

There exist nonegative functions h_0, $h_1 : \Delta \to \mathbb{R}$ with the following properties

(i) $|H(x, y, u, v) - H(x, y, \bar{u}, \bar{v})| \leq h_0(x, y) \left(|u - \bar{u}| + |v - \bar{v}|\right)$
$\forall (x, y) \in \Delta$, $\forall u, v, \bar{u}, \bar{v} \in \mathbb{R}$,

(ii) $\left| \frac{\partial H}{\partial x_1}(x, y, u, v) - \frac{\partial H}{\partial x_1}(x, y, \bar{u}, \bar{v}) \right| \leq h_1(x, y) \left(|u - \bar{u}| + |v - \bar{v}|\right)$
$\forall (x, y) \in \Delta$, $\forall u, v, \bar{u}, \bar{v} \in \mathbb{R}$.

(**A3**) $K \in C(\Omega \times \Omega \times \mathbb{R}^2; \mathbb{R})$ such that $\frac{\partial K}{\partial x_1} \in C(\Omega \times \Omega \times \mathbb{R}^2; \mathbb{R})$ and there exist nonegative functions k_0, $k_1 : \Omega \times \Omega \to \mathbb{R}$ with the following properties

(i) $|K(x, y, u, v)| \leq k_0(x, y) \left(1 + |u| + |v|\right)$, $\forall (x, y) \in \Omega \times \Omega$, $\forall u, v \in \mathbb{R}$,

(ii) $\left| \frac{\partial K}{\partial x_1}(x, y, u, v) \right| \leq k_1(x, y) \left(1 + |u| + |v|\right)$, $\forall (x, y) \in \Omega \times \Omega$, $\forall u, v \in \mathbb{R}$.

(**A4**) $\bar{\beta}_1 + \bar{\beta}_2 < 1$, where

$$\bar{\beta}_1 = \sup_{x \in \Omega} \left[\int_0^{x_2} ... \int_0^{x_N} h_0(x, x_1, y')dy' + \int_{B_x} h_1(x, y)dy \right] + \sup_{x \in \Omega} \int_{B_x} h_0(x, y)dy,$$
$$\bar{\beta}_2 = \sup_{x \in \Omega} \int_\Omega k_0(x, y)dy + \sup_{x \in \Omega} \int_\Omega k_1(x, y)dy.$$

Theorem 3.1. *Let the functions g, H, K in* (1.1) *satisfy the assumptions* (**A1**) – (**A4**). *Then the equation* (1.1) *has a solution in X_1. Furthermore, the set of solutions is compact in X_1.*

Proof. We rewrite (1.1) as follows

$$u(x) = (Au)(x), x \in \Omega, \tag{3.1}$$

where

$$\begin{aligned} &(Au)(x) = (Uu)(x) + (Cu)(x), \\ &(Uu)(x) = g(x) + \int_{B_x} H(x, y, u(y), D_1 u(y))dy, \\ &(Cu)(x) = \int_\Omega K(x, y, u(y), D_1 u(y))dy, \\ &x \in \Omega, u \in X_1. \end{aligned} \tag{3.2}$$

A simple verification shows that Uu, $Cu \in X_1$, $\forall u \in X_1$.

For $M > 0$, we consider a closed ball in X_1 as follows

$$B_M = \{u \in X_1 : \|u\|_{X_1} \le M\}. \tag{3.3}$$

We will show that there exists $M > 0$ such that

(i) $Uu + Cv \in B_M$, for every u, $v \in B_M$,

and the operators U, C satisfy the conditions (ii)-(iv) below.

(ii) $U : B_M \to X_1$ is a contraction map,

(iii) $C : B_M \to X_1$ is continuous,

(iv) $F = C(B_M)$ is relatively compact in X_1.

Proof (i). Let $M > 0$. For every $u \in B_M$, for all $x \in \Omega$, we have

$$|(Uu)(x)| \le \|g\|_X + \int_{B_x} |H(x, y, u(y), D_1u(y))|\, dy$$
$$\le \|g\|_X + \int_{B_x} |H(x, y, u(y), D_1u(y)) - H(x, y, 0, 0)|\, dy + \int_{B_x} |H(x, y, 0, 0)|\, dy$$
$$\le \|g\|_X + \int_{B_x} h_0(x, y)\, (|u(y)| + |D_1u(y)|)\, dy + \int_{B_x} |H(x, y, 0, 0)|\, dy$$
$$\le \|g\|_X + \|u\|_{X_1} \int_{B_x} h_0(x, y) dy + \int_{B_x} |H(x, y, 0, 0)|\, dy$$
$$\le \|g\|_X + M \sup_{x\in\Omega} \int_{B_x} h_0(x, y) dy \ + \sup_{x\in\Omega} \int_{B_x} |H(x, y, 0, 0)|\, dy, \tag{3.4}$$

so

$$\|Uu\|_X \le \|g\|_X + M \sup_{x\in\Omega} \int_{B_x} h_0(x, y) dy \ + \sup_{x\in\Omega} \int_{B_x} |H(x, y, 0, 0)|\, dy. \tag{3.5}$$

On the other hand, for all $x = (x_1, x') \in \Omega$,

$$(Uu)(x) = g(x) + \int_0^{x_1} \left[\int_0^{x_2} ... \int_0^{x_N} H(x, y_1, y', u(y_1, y'), D_1u(y_1, y'))dy' \right] dy_1, \tag{3.6}$$

Therefore

$$D_1(Uu)(x) = D_1g(x) + \int_0^{x_2} ... \int_0^{x_N} H(x, x_1, y', u(x_1, y'), D_1u(x_1, y'))dy'$$
$$+ \int_{B_x} \frac{\partial H}{\partial x_1}(x, y, u(y), D_1u(y))dy$$
$$= D_1g(x) + \int_0^{x_2} ... \int_0^{x_N} [H(x, x_1, y', u(x_1, y'), D_1u(x_1, y')) - H(x, x_1, y', 0, 0)]\, dy'$$
$$+ \int_0^{x_2} ... \int_0^{x_N} H(x, x_1, y', 0, 0)dy' + \int_{B_x} \left[\frac{\partial H}{\partial x_1}(x, y, u(y), D_1u(y)) - \frac{\partial H}{\partial x_1}(x, y, 0, 0) \right] dy$$
$$+ \int_{B_x} \frac{\partial H}{\partial x_1}(x, y, 0, 0)dy, \text{ for all } x = (x_1, x') \in \Omega. \tag{3.7}$$

It implies that

$$|D_1(Au)(x)| \le \|D_1g\|_X$$

$$
\begin{aligned}
&+\int_0^{x_2} ... \int_0^{x_N} \left| H(x,x_1,y',u(x_1,y'),D_1u(x_1,y')) - H(x,x_1,y',0,0) \right| dy' \\
&\qquad + \int_0^{x_2} ... \int_0^{x_N} \left| H(x,x_1,y',0,0) \right| dy' \\
&+ \int_{B_x} \left| \frac{\partial H}{\partial x_1}(x,y,u(y),D_1u(y)) - \frac{\partial H}{\partial x_1}(x,y,0,0) \right| dy + \int_{B_x} \left| \frac{\partial H}{\partial x_1}(x,y,0,0) \right| dy \\
&\leq \|D_1 g\|_X + \int_0^{x_2} ... \int_0^{x_N} h_0(x,x_1,y') \left(|u(x_1,y')| + |D_1u(x_1,y')| \right) dy' \\
&\qquad + \int_0^{x_2} ... \int_0^{x_N} \left| H(x,x_1,y',0,0) \right| dy' \\
&+ \int_{B_x} h_1(x,y) \left(|u(y)| + |D_1u(y)| \right) dy + \int_{B_x} \left| \frac{\partial H}{\partial x_1}(x,y,0,0) \right| dy \\
&\leq \|D_1 g\|_X + \|u\|_{X_1} \int_0^{x_2} ... \int_0^{x_N} h_0(x,x_1,y') dy' + \int_0^{x_2} ... \int_0^{x_N} \left| H(x,x_1,y',0,0) \right| dy' \\
&\qquad + \|u\|_{X_1} \int_{B_x} h_1(x,y) dy + \int_{B_x} \left| \frac{\partial H}{\partial x_1}(x,y,0,0) \right| dy \\
&\leq \|D_1 g\|_X + M \left[\int_0^{x_2} ... \int_0^{x_N} h_0(x,x_1,y') dy' + \int_{B_x} h_1(x,y) dy \right] \\
&+ \int_0^{x_2} ... \int_0^{x_N} \left| H(x,x_1,y',0,0) \right| dy' + \int_{B_x} \left| \frac{\partial H}{\partial x_1}(x,y,0,0) \right| dy \\
&\leq \|D_1 g\|_X + M \sup_{x\in\Omega} \left[\int_0^{x_2} ... \int_0^{x_N} h_0(x,x_1,y') dy' + \int_{B_x} h_1(x,y) dy \right] \\
&+ \sup_{x\in\Omega} \left[\int_0^{x_2} ... \int_0^{x_N} \left| H(x,x_1,y',0,0) \right| dy' + \int_{B_x} \left| \frac{\partial H}{\partial x_1}(x,y,0,0) \right| dy \right]. \qquad (3.8)
\end{aligned}
$$

This yields

$$
\begin{aligned}
\|D_1(Uu)\|_X &\leq \|D_1 g\|_X + M \sup_{x\in\Omega} \left[\int_0^{x_2} ... \int_0^{x_N} h_0(x,x_1,y') dy' + \int_{B_x} h_1(x,y) dy \right] \\
&+ \sup_{x\in\Omega} \left[\int_0^{x_2} ... \int_0^{x_N} \left| H(x,x_1,y',0,0) \right| dy' + \int_{B_x} \left| \frac{\partial H}{\partial x_1}(x,y,0,0) \right| dy \right]. \qquad (3.9)
\end{aligned}
$$

From (3.5) and (3.9), it gives

$$
\|Uu\|_{X_1} = \|Uu\|_X + \|D_1(Uu)\|_X \leq \|g\|_{X_1} + \bar{\alpha}_1 + M\bar{\beta}_1, \qquad (3.10)
$$

where

$$
\begin{aligned}
\bar{\alpha}_1 &= \sup_{x\in\Omega} \int_{B_x} \left| H(x,y,0,0) \right| dy \\
&\quad + \sup_{x\in\Omega} \left[\int_{B_x} \left| \frac{\partial H}{\partial x_1}(x,y,0,0) \right| dy + \int_0^{x_2} ... \int_0^{x_N} \left| H(x,x_1,y',0,0) \right| dy' \right], \\
\bar{\beta}_1 &= \sup_{x\in\Omega} \int_{B_x} h_0(x,y) dy + \sup_{x\in\Omega} \left[\int_0^{x_2} ... \int_0^{x_N} h_0(x,x_1,y') dy' + \int_{B_x} h_1(x,y) dy \right]. \qquad (3.11)
\end{aligned}
$$

On the other hand, for every $v \in B_M$, for all $x \in \Omega$, we have

$$\begin{aligned} |(Cv)(x)| &\le \int_\Omega |K(x,y,v(y),D_1v(y))|\,dy \\ &\le \int_\Omega k_0(x,y)\,(1+|v(y)|+|D_1v(y)|)\,dy \\ &\le \int_\Omega k_0(x,y)\left(1+\|v\|_{X_1}\right)dy \le (1+M)\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy. \end{aligned} \tag{3.12}$$

Thus

$$\|Cv\|_X \le (1+M)\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy. \tag{3.13}$$

Similarly,

$$|D_1(Cv)(x)| \le \int_\Omega \left|\frac{\partial K}{\partial x_1}(x,y,v(y),D_1v(y))\right| dy \le (1+M)\sup_{x\in\Omega}\int_\Omega k_1(x,y)dy, \tag{3.14}$$

therefore

$$\|D_1(Cv)\|_X \le (1+M)\sup_{x\in\Omega}\int_\Omega k_1(x,y)dy. \tag{3.15}$$

From (3.12) and (3.15) we take

$$\|Cv\|_{X_1} \le (1+M)\left[\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy + \sup_{x\in\Omega}\int_\Omega k_1(x,y)dy\right] \equiv (1+M)\,\bar\beta_2. \tag{3.16}$$

From (3.10) and (3.16) we have that

$$\|Uu+Cv\|_{X_1} \le \|Uu\|_{X_1} + \|Cv\|_{X_1} \le \|g\|_{X_1} + \bar\alpha_1 + \bar\beta_2 + M\left(\bar\beta_1+\bar\beta_2\right). \tag{3.17}$$

Choose $M \ge \|g\|_{X_1} + \bar\alpha_1 + \bar\beta_2 + M\left(\bar\beta_1+\bar\beta_2\right)$, i.e.,

$$M \ge \frac{\|g\|_{X_1} + \bar\alpha_1 + \bar\beta_2}{1-\bar\beta_1-\bar\beta_2}.$$

Then $Uu + Cv \in B_M$, for all $u, v \in B_M$.
Proof (ii). In view of (**A4**), $U : B_M \to X_1$ is a contraction map, if we show that

$$\|Uu - U\bar u\|_{X_1} \le \bar\beta_1\,\|u-\bar u\|_{X_1},\ \forall u,\ \bar u \in B_M. \tag{3.18}$$

For every u, $\bar u \in B_M$, for all $(x,y) \in \Omega$, using (**A2**, i), (3.2) leads to

$$\begin{aligned} |(Uu)(x)-(U\bar u)(x)| &\le \int_{B_x} |H(x,y,u(y),D_1u(y)) - H(x,y,\bar u(y),D_1\bar u(y))|\,dy \\ &\le \int_{B_x} h_0(x,y)\,(|u(y)-\bar u(y)| + |D_1u(y)-D_1\bar u(y)|)\,dy \\ &\le \left(\sup_{x\in\Omega}\int_{B_x} h_0(x,y)dy\right)\|u-\bar u\|_{X_1}. \end{aligned}$$

Thus

$$\|Uu-U\bar u\|_X \le \left(\sup_{x\in\Omega}\int_{B_x} h_0(x,y)dy\right)\|u-\bar u\|_{X_1}. \tag{3.19}$$

Similarly, by (3.7) we deduce that, for all $x = (x_1, x') \in \Omega$,

$$\begin{aligned} &D_1(Uu)(x) - D_1(U\bar{u})(x) \\ &= \int_0^{x_2} ... \int_0^{x_N} [H(x, x_1, y', u(x_1, y'), D_1u(x_1, y')) \\ &\qquad -H(x, x_1, y', \bar{u}(x_1, y'), D_1\bar{u}(x_1, y'))]dy' \\ &+ \int_{B_x} \left[\frac{\partial H}{\partial x_1}(x, y, u(y), D_1u(y)) - \frac{\partial H}{\partial x_1}(x, y, \bar{u}(y), D_1\bar{u}(y)) \right] dy, \end{aligned} \tag{3.20}$$

using $(\mathbf{A2}, ii)$, we get

$$\begin{aligned} &|D_1(Uu)(x) - D_1(U\bar{u})(x)| \\ &\leq \int_0^{x_2} ... \int_0^{x_N} |H(x, x_1, y', u(x_1, y'), D_1u(x_1, y')) \\ &\qquad -H(x, x_1, y', \bar{u}(x_1, y'), D_1\bar{u}(x_1, y'))|dy' \\ &+ \int_{B_x} \left| \frac{\partial H}{\partial x_1}(x, y, u(y), D_1u(y)) - \frac{\partial H}{\partial x_1}(x, y, \bar{u}(y), D_1\bar{u}(y)) \right| dy \\ &\leq \int_0^{x_2} ... \int_0^{x_N} h_0(x, x_1, y') \left[|u(x_1, y') - \bar{u}(x_1, y')| + |D_1u(x_1, y') - D_1\bar{u}(x_1, y')| \right] dy' \\ &+ \int_{B_x} h_1(x, y) \left[|u(y) - \bar{u}(y)| + |D_1u(y) - D_1\bar{u}(y)| \right] dy \\ &\leq \sup_{x\in\Omega} \left[\int_0^{x_2} ... \int_0^{x_N} h_0(x, x_1, y')dy' + \int_{B_x} h_1(x, y)dy \right] \|u - \bar{u}\|_{X_1}. \end{aligned} \tag{3.21}$$

Then

$$\begin{aligned} &\|D_1(Uu) - D_1(U\bar{u})\|_X \\ &\leq \sup_{x\in\Omega} \left[\int_0^{x_2} ... \int_0^{x_N} h_0(x, x_1, y')dy' + \int_{B_x} h_1(x, y)dy \right] \|u - \bar{u}\|_{X_1}. \end{aligned} \tag{3.22}$$

From (3.19) and (3.22) we take

$$\|Uu - U\bar{u}\|_{X_1} \leq \bar{\beta}_1 \|u - \bar{u}\|_{X_1}, \tag{3.23}$$

where

$$\bar{\beta}_1 \leq \bar{\beta}_1 + \bar{\beta}_2 < 1, \tag{3.24}$$

by $(\mathbf{A4})$.

Proof (iii). To prove (iii), let $\{u_m\} \subset B_M$, $u \in B_M$, $\|u_m - u\|_{X_1} \to 0$, as $m \to \infty$, we have to prove that

$$\|Cu_m - Cu\|_X \to 0 \text{ and } \|D_1(Cu_m) - D_1(Cu)\|_X \to 0, \text{ as } m \to \infty. \tag{3.25}$$

Note that

$$|(Cu_m)(x) - (Cu)(x)| \leq \int_\Omega |K(x, y, u_m(y), D_1u_m(y)) - K(x, y, u(y), D_1u(y))|\, dy. \tag{3.26}$$

Give $\varepsilon > 0$. Since the function K is uniformly continuous on $\Omega \times \Omega \times [-M, M] \times [-M, M]$, there exists $\delta > 0$ such that

$$\forall (x, y) \in \Omega \times \Omega,\ \forall u, v, \bar{u}, \bar{v} \in [-M, M],$$
$$|u - \bar{u}| + |v - \bar{v}| < \delta \Longrightarrow |K(x, y, u, v) - K(x, y, \bar{u}, \bar{v})| < \varepsilon.$$

By $\|u_m - u\|_X \to 0$, and $\|D_1 u_m - D_1 u\|_X \to 0$, there is $m_0 \in \mathbb{N}$ such that

$$\forall m \in \mathbb{N},\ m \geq m_0 \Longrightarrow \|u_m - u\|_X + \|D_1 u_m - D_1 u\|_X < \delta.$$

It follows that $\forall m \in \mathbb{N}$, if $m \geq m_0$ then

$$|K(x, y, u_m(y), D_1 u_m(y)) - K(x, y, u(y), D_1 u(y))| < \varepsilon,$$

$\forall (x, y) \in \Omega \times \Omega$, so

$$|(Cu_m)(x) - (Cu)(x)| < \varepsilon,\ \forall x \in \Omega,\ \forall m \geq m_0,$$

it means that

$$\|Cu_m - Cu\|_X < \varepsilon,\ \forall m \geq m_0, \tag{3.27}$$

i.e., $\|Cu_m - Cu\|_X \to 0$, as $m \to \infty$.

By the same way, we get $\|D_1(Cu_m) - D_1(Cu)\|_X \to 0$, as $m \to \infty$.

Proof (iv). To prove (iv), we use Lemma 2.2. Condition (2.10)(i) holds because by (i), we have that $F = C(B_M) \subset B_M$. It remains to show (2.10)(ii). We have, for all $u \in B_M$, for all $x, \bar{x} \in \Omega$,

$$(Cu)(x) - (Cu)(\bar{x}) = \int_\Omega [K(x, y, u(y), D_1 u(y)) - K(\bar{x}, y, u(y), D_1 u(y))]\, dy, \tag{3.28}$$

Let $\varepsilon > 0$. By the fact that K is uniformly continuous on $\Omega \times \Omega \times [-M, M] \times [-M, M]$, there exists $\delta_1 > 0$ such that $\forall x,\ \bar{x} \in \Omega$,

$$|x - \bar{x}| < \delta_1 \Longrightarrow |K(x, y, \bar{u}, \bar{v}) - K(\bar{x}, y, \bar{u}, \bar{v})| < \tfrac{\varepsilon}{2},\ \forall (y, \bar{u}, \bar{v}) \in \Omega \times [-M, M].$$

Then

$$\forall x,\ \bar{x} \in \Omega,\ |x - \bar{x}| < \delta_1$$
$$\Longrightarrow |K(x, y, u(y), D_1 u(y)) - K(\bar{x}, y, u(y), D_1 u(y))| < \tfrac{\varepsilon}{2},\ \forall (y, u) \in \Omega \times B_M,$$

so

$$\forall x,\ \bar{x} \in \Omega,\ |x - \bar{x}| < \delta_1$$
$$\Longrightarrow \int_\Omega |K(x, y, u(y), D_1 u(y)) - K(\bar{x}, y, u(y), D_1 u(y))|\, dy < \frac{\varepsilon}{2},\ \forall u \in B_M.$$

This yields

$$\forall x,\ \bar{x} \in \Omega,\ |x - \bar{x}| < \delta_1 \Longrightarrow |(Cu)(x) - (Cu)(\bar{x})| < \tfrac{\varepsilon}{2},\ \forall u \in B_M. \tag{3.29}$$

Similarly, by the uniformly continuity of $\frac{\partial K}{\partial x_1}$, $D_1 g$, on their domains, we also have that for the $\varepsilon > 0$ considered there is a $\delta_2 > 0$ such that $\forall x,\ \bar{x} \in \Omega$,

$$|x - \bar{x}| < \delta_2 \Longrightarrow |D_1(Cu)(x) - D_1(Cu)(\bar{x})| < \tfrac{\varepsilon}{2},\ \forall u \in B_M. \tag{3.30}$$

Consequently, choosing $\delta = \min\{\delta_1, \delta_2\}$, we obtain $\forall x,\ \bar{x} \in \Omega$,

$$|x - \bar{x}| < \delta \Longrightarrow |(Cu)(x) - (Cu)(\bar{x})| + |D_1(Cu)(x) - D_1(Cu)(\bar{x})| < \varepsilon,\ \forall u \in B_M. \tag{3.31}$$

Lemma 2.2 implies that $F = C(B_M)$ is relatively compact in X_1. Applying the Krasnosel'skii fixed point theorem (Theorem 1.1), the existence of a solution is proved.

Now, we show that the set S of solutions is compact in X_1 :

$$S = \{u \in B_M : u = Uu + Cu\} = \{u \in B_M : u = (I - U)^{-1}Cu\}.$$

From the compactness of the operator $C : B_M \to B_M$ and the continuity of $(I-U)^{-1} : B_M \to B_M$, and $S = (I - U)^{-1}C(S)$, we only prove that S is closed. Let $\{u_m\} \subset S$, $u \in X_1$, $\|u_m - u\|_{X_1} \to 0$. The continuity of $(I - U)^{-1}C$ leads to

$$\begin{aligned}\left\|u - (I - U)^{-1}Cu\right\|_{X_1} &\le \|u - u_m\|_{X_1} + \left\|u_m - (I - U)^{-1}Cu\right\|_{X_1} \\ &= \|u - u_m\|_{X_1} + \left\|(I - U)^{-1}Cu_m - (I - U)^{-1}Cu\right\|_{X_1} \to 0,\end{aligned}$$

so $u = (I - U)^{-1}Cu \in S$. Theorem 3.1 is proved.

4. An example

Consider (1.1), with the functions H, K, g as follows

$$\begin{cases} H(x,y,u,v) = h(x)\left[y_1^\sigma ... y_N^\sigma \sin\left(\frac{\pi u}{2u_0(y)}\right) + y_1^\gamma ... y_N^\gamma \cos\left(\frac{2\pi v}{D_1 u_0(y)}\right)\right], \\ K(x,y,u,v) = h(x) K_1(y,u,v), \\ g(x) = u_0(x) - h(x)\left(\frac{2 + x_1^{\sigma+1} ... x_N^{\sigma+1}}{(\sigma+1)^N} + \frac{2 + x_1^{\gamma+1} ... x_N^{\gamma+1}}{(\gamma+1)^N}\right), \end{cases} \tag{4.1}$$

where

$$\begin{cases} K_1(y,u,v) = y_1^\sigma ... y_N^\sigma \left[\frac{|u|}{u_0(y)} + \left(\frac{u}{u_0(y)}\right)^{3/7}\right] + y_1^\gamma ... y_N^\gamma \left[\frac{|v|}{D_1 u_0(y)} + \left(\frac{v}{D_1 u_0(y)}\right)^{2/5}\right], \\ u_0(x) = e^{x_1} + x_1^{\gamma_1} \left|x_2 - \alpha\right|^{\gamma_2} + \sum_{i=3}^{N} \left|x_i - \alpha\right|, \\ h(x) = x_1^{\tilde{\gamma}_1} \left|x_2 - \tilde{\alpha}\right|^{\tilde{\gamma}_2} + \sum_{i=3}^{N} \left|x_i - \tilde{\alpha}\right|, \end{cases} \tag{4.2}$$

and σ, γ, α, γ_2, γ_1, $\tilde{\alpha}$, $\tilde{\gamma}_2$, $\tilde{\gamma}_1$ are positive constants satisfying

$$\begin{cases} 0 < \alpha < 1,\ 0 < \gamma_2 \le 1 < \gamma_1, \\ 0 < \tilde{\alpha} < 1,\ 0 < \tilde{\gamma}_2 \le 1 < \tilde{\gamma}_1, \\ 2\pi\left(\frac{1}{(\sigma+1)^{N-1}} + \frac{1}{(\gamma+1)^{N-1}}\right)\left[\max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\} + (N-2)\max\{\tilde{\alpha}, 1-\tilde{\alpha}\}\right] \\ +2(1+\pi)\left(\frac{1}{(\sigma+1)^N} + \frac{1}{(\gamma+1)^N}\right) \times \\ \times\left[(1+\tilde{\gamma}_1)\max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\} + (N-2)\max\{\tilde{\alpha}, 1-\tilde{\alpha}\}\ \right] < 1. \end{cases} \tag{4.3}$$

Note that

$$u_0(x) = e^{x_1} + x_1^{\gamma_1} \left|x_2 - \alpha\right|^{\gamma_2} + \sum_{i=3}^{N} \left|x_i - \alpha\right|,$$
$$D_1 u_0(x) = e^{x_1} + \gamma_1 x_1^{\gamma_1 - 1} \left|x_2 - \alpha\right|^{\gamma_2},$$

so $u_0 \in X_1$ and $u_0(x) \ge 1$, $D_1 u_0(x) \ge 1$.

We can prove that (**A1**), (**A2**) hold. It is easy to see that (**A1**) holds, since u_0, $h \in X_1$.

Assumption (**A2**) holds, by the following. First, $H \in C(\Omega \times \Omega \times \mathbb{R}^2; \mathbb{R})$. On the other hand,

$$\frac{\partial h}{\partial x_1} \in X$$

and

$$\frac{\partial H}{\partial x_1} = \frac{\partial h}{\partial x_1}(x) \left[y_1^\sigma ... y_N^\sigma \sin\left(\frac{\pi u}{2u_0(y)}\right) + y_1^\gamma ... y_N^\gamma \cos\left(\frac{2\pi v}{D_1 u_0(y)}\right)\right],$$

hence

$$\frac{\partial H}{\partial x_1} \in C(\Omega \times \Omega \times \mathbb{R}^2; \mathbb{R}).$$

Next, it is obviously that

$$\begin{aligned} |H(x,y,u,v) - H(x,y,\bar{u},\bar{v})| &\leq h(x) \left[y_1^\sigma ... y_N^\sigma \tfrac{\pi |u-\bar{u}|}{2u_0(y)} + y_1^\gamma ... y_N^\gamma \tfrac{2\pi |v-\bar{v}|}{D_1 u_0(y)} \right] \\ &\leq 2\pi h(x) \left(y_1^\sigma ... y_N^\sigma + y_1^\gamma ... y_N^\gamma \right) \left[|u-\bar{u}| + |v-\bar{v}| \right] \\ &\equiv h_0(x,y) \left(|u-\bar{u}| + |v-\bar{v}| \right), \end{aligned} \tag{4.4}$$

with

$$h_0(x,y) = 2\pi h(x) \left(y_1^\sigma ... y_N^\sigma + y_1^\gamma ... y_N^\gamma \right). \tag{4.5}$$

Similarly

$$\left| \tfrac{\partial H}{\partial x_1}(x,y,u,v) - \tfrac{\partial H}{\partial x_1}(x,y,\bar{u},\bar{v}) \right| \leq h_1(x,y) \left(|u-\bar{u}| + |v-\bar{v}| \right), \tag{4.6}$$

where

$$h_1(x,y) = 2\pi \left| \tfrac{\partial h}{\partial x_1}(x) \right| \left(y_1^\sigma ... y_N^\sigma + y_1^\gamma ... y_N^\gamma \right). \tag{4.7}$$

Assumption (**A3**) holds, by the following. First, $K \in C(\Omega \times \Omega \times \mathbb{R}^2; \mathbb{R})$. On the other hand,

$$\frac{\partial h}{\partial x_1} \in X, \quad \frac{\partial K}{\partial x_1} = \frac{\partial h}{\partial x_1}(x) K_1(y,u,v),$$

so

$$\frac{\partial K}{\partial x_1} \in C(\Omega \times \Omega \times \mathbb{R}^2; \mathbb{R}).$$

Next, applying the inequality

$$x \leq 1 + x^q, \ \forall x \geq 0, \ \forall q \geq 1, \tag{4.8}$$

we have

$$\begin{aligned} |K_1(y,u,v)| &\leq y_1^\sigma ... y_N^\sigma \left[1 + \tfrac{2|u|}{u_0(y)} \right] + y_1^\gamma ... y_N^\gamma \left[1 + \tfrac{2|v|}{D_1 u_0(y)} \right] \\ &\leq 2 \left(y_1^\sigma ... y_N^\sigma + y_1^\gamma ... y_N^\gamma \right) \left(1 + |u| + |v| \right). \end{aligned} \tag{4.9}$$

It follows that

$$|K(x,y,u,v)| = h(x) \left| K_1(y,u,v) \right| \leq k_0(x,y) \left(1 + |u| + |v| \right), \tag{4.10}$$

with

$$k_0(x,y) = 2h(x) \left(y_1^\sigma ... y_N^\sigma + y_1^\gamma ... y_N^\gamma \right). \tag{4.11}$$

Similarly

$$\left| \tfrac{\partial K}{\partial x_1}(x,y,u,v) \right| \leq k_1(x,y) \left(1 + |u| + |v| \right), \tag{4.12}$$

in which

$$k_1(x,y) = 2\left|\tfrac{\partial h}{\partial x_1}(x)\right|(y_1^\sigma ... y_N^\sigma + y_1^\gamma ... y_N^\gamma). \tag{4.13}$$

To see that assumption (**A4**) holds, we make the following estimations.

(i) *Estimating the term* $\bar{\beta}_1$.

We have

$$\int_{B_x} h_0(x,y)dy = 2\pi h(x)\int_{B_x}(y_1^\sigma ... y_N^\sigma + y_1^\gamma ... y_N^\gamma)\,dy$$

$$= 2\pi\left(\frac{x_1^{\sigma+1} ... x_N^{\sigma+1}}{(\sigma+1)^N} + \frac{x_1^{\gamma+1} ... x_N^{\gamma+1}}{(\gamma+1)^N}\right)h(x) \leq 2\pi\left(\frac{1}{(\sigma+1)^N} + \frac{1}{(\gamma+1)^N}\right)h(x),$$

$$\int_{B_x} h_1(x,y)dy = 2\pi\left|\frac{\partial h}{\partial x_1}(x)\right|\int_{B_x}(y_1^\sigma ... y_N^\sigma + y_1^\gamma ... y_N^\gamma)\,dy$$

$$\leq 2\pi\left(\frac{1}{(\sigma+1)^N} + \frac{1}{(\gamma+1)^N}\right)\left|\frac{\partial h}{\partial x_1}(x)\right|, \tag{4.14}$$

and

$$\int_0^{x_2} ... \int_0^{x_N} h_0(x, x_1, y')dy' = 2\pi h(x)\int_0^{x_2} ... \int_0^{x_N}(x_1^\sigma y_2^\sigma ... y_N^\sigma + x_1^\gamma y_2^\gamma ... y_N^\gamma)\,dy'$$

$$= 2\pi h(x)\left(\frac{x_1^\sigma x_2^{\sigma+1} ... x_N^{\sigma+1}}{(\sigma+1)^{N-1}} + \frac{x_1^\gamma x_2^{\gamma+1} ... x_N^{\gamma+1}}{(\gamma+1)^{N-1}}\right)$$

$$\leq 2\pi h(x)\left(\frac{1}{(\sigma+1)^{N-1}} + \frac{1}{(\gamma+1)^{N-1}}\right). \tag{4.15}$$

In order to continue, we need the following lemma, its proof is not difficult so we omit it.

Lemma 4.1. *Let positive constants* α, γ_2, γ_1 *satisfy* $0 < \alpha < 1$, $0 < \gamma_2 \leq 1 < \gamma_1$. *Then*

$$\begin{aligned} &0 \leq x_1^{\gamma_1}|x_2 - \alpha|^{\gamma_2} \leq \max\{\alpha^{\gamma_2}, (1-\alpha)^{\gamma_2}\},\ \forall x_1,\ x_2 \in [0,1],\\ &0 \leq x_1^{\gamma_1 - 1}|x_2 - \alpha|^{\gamma_2} \leq \max\{\alpha^{\gamma_2}, (1-\alpha)^{\gamma_2}\},\ \forall x_1,\ x_2 \in [0,1]. \end{aligned} \tag{4.16}$$

Using Lemma 4.1, we get

$$\begin{cases} 0 \leq h(x) = x_1^{\tilde{\gamma}_1}|x_2 - \tilde{\alpha}|^{\tilde{\gamma}_2} + \sum_{i=3}^N |x_i - \tilde{\alpha}|, \\ h(x) \leq \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\} + (N-2)\max\{\tilde{\alpha}, 1-\tilde{\alpha}\}; \\ 0 \leq \dfrac{\partial h}{\partial x_1}(x) = \tilde{\gamma}_1 x_1^{\tilde{\gamma}_1 - 1}|x_2 - \tilde{\alpha}|^{\tilde{\gamma}_2} \leq \tilde{\gamma}_1 \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\},\ \forall x \in \Omega. \end{cases} \tag{4.17}$$

Then

$$\begin{aligned}
&\int_{B_x} h_0(x,y)dy \\
&\le 2\pi\left(\tfrac{1}{(\sigma+1)^N}+\tfrac{1}{(\gamma+1)^N}\right)\left[\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}+(N-2)\max\{\tilde{\alpha},1-\tilde{\alpha}\}\ \right], \\
&\int_{B_x} h_1(x,y)dy \le 2\pi\left(\frac{1}{(\sigma+1)^N}+\frac{1}{(\gamma+1)^N}\right)\tilde{\gamma}_1\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}, \\
&\int_0^{x_2}...\int_0^{x_N} h_0(x,x_1,y')dy' \\
&\le 2\pi\left(\tfrac{1}{(\sigma+1)^{N-1}}+\tfrac{1}{(\gamma+1)^{N-1}}\right)\left[\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}+(N-2)\max\{\tilde{\alpha},1-\tilde{\alpha}\}\ \right].
\end{aligned} \tag{4.18}$$

Therefore

$$\begin{aligned}
&\bar{\beta}_1=\sup_{x\in\Omega}\int_{B_x}h_0(x,y)dy+\sup_{x\in\Omega}\left[\int_{B_x}h_1(x,y)dy+\int_0^{x_2}...\int_0^{x_N}h_0(x,x_1,y')dy'\right] \\
&\le 2\pi\left(\tfrac{1}{(\sigma+1)^{N-1}}+\tfrac{1}{(\gamma+1)^{N-1}}\right)\left[\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}+(N-2)\max\{\tilde{\alpha},1-\tilde{\alpha}\}\right] \\
&+2\pi\left(\tfrac{1}{(\sigma+1)^N}+\tfrac{1}{(\gamma+1)^N}\right)\left[(1+\tilde{\gamma}_1)\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}+(N-2)\max\{\tilde{\alpha},1-\tilde{\alpha}\}\right].
\end{aligned} \tag{4.19}$$

(ii) *Estimating the term* $\bar{\beta}_2$.

Similarly, using Lemma 4.1, we get

$$\begin{aligned}
&\int_\Omega k_0(x,y)dy=2h(x)\int_\Omega\left(y_1^\sigma...y_N^\sigma+y_1^\gamma...y_N^\gamma\right)dy=2\left(\frac{1}{(\sigma+1)^N}+\frac{1}{(\gamma+1)^N}\right)h(x) \\
&\le 2\left(\tfrac{1}{(\sigma+1)^N}+\tfrac{1}{(\gamma+1)^N}\right)\left[\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}+(N-2)\max\{\tilde{\alpha},1-\tilde{\alpha}\}\ \right], \\
&\int_\Omega k_1(x,y)dy=2\left|\frac{\partial h}{\partial x_1}(x)\right|\int_\Omega\left(y_1^\sigma...y_N^\sigma+y_1^\gamma...y_N^\gamma\right)dy \\
&=2\left(\tfrac{1}{(\sigma+1)^N}+\tfrac{1}{(\gamma+1)^N}\right)\left|\tfrac{\partial h}{\partial x_1}(x)\right| \\
&\le 2\left(\tfrac{1}{(\sigma+1)^N}+\tfrac{1}{(\gamma+1)^N}\right)\tilde{\gamma}_1\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2},
\end{aligned} \tag{4.20}$$

where

$$\begin{aligned}
&\bar{\beta}_2=\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy+\sup_{x\in\Omega}\int_\Omega k_1(x,y)dy \\
&\le 2\left(\tfrac{1}{(\sigma+1)^N}+\tfrac{1}{(\gamma+1)^N}\right)\left[(1+\tilde{\gamma}_1)\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}+(N-2)\max\{\tilde{\alpha},1-\tilde{\alpha}\}\ \right].
\end{aligned} \tag{4.21}$$

It follows from (4.3), (4.19) and (4.21) that

$$\bar{\beta}_1+\bar{\beta}_2<1. \tag{4.22}$$

Theorem 3.1 is fulfilled in this case. Furthermore, $u_0 \in X_1$ is also a solution of (1.1).

Acknowledgments. The authors wish to express their sincere thanks to the referees for the valuable comments and important remarks. Their comments uncovered several weaknesses in the presentation of the paper and helped us to clarify it. This research is funded by Vietnam National University Ho Chi Minh City (VNU-HCM) under Grant no. B2017-18-04.

References

[1] M.A. Abdou, W.G. El-Sayed, E.I. Deebs, *A solution of a nonlinear integral equation*, Applied Math. Computation, **160**(2005), 1-14.

[2] M.A. Abdou, A.A. Badr,M. M. El-Kojok, *On the solution of a mixed nonlinear integral equation*, Applied Math. Computation, **217**(12)(2011), 5466-5475.

[3] A. Aghajani, E. Pourhadi, M. Rivero, J. Trujillo, *Application of Perov's fixed point theorem to Fredholm type integro-differential equations in two variables*, Mathematica Slovaca, **66**(2016), 1207-1216.

[4] C. Avramescu, *Some remarks on a fixed point theorem of Krasnosel'skii*, E.J. Qualitative Theory of Diff. Equ., **5**(2003), 1-15.

[5] C. Avramescu, C. Vladimirescu, *An existence result of asymptotically stable solutions for an integral equation of mixed type*, Electronic J. Qualitative Theory of Diff. Equat., **25**(2005), 1-6.

[7] C. Corduneanu, *Integral equations and applications*, Cambridge University Press, New York, 1991.

[8] K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, 1985.

[9] B.C. Dhage, S.K. Ntouyas, *Existence results for nonlinear functional integral equations via a fixed point theorem of Krasnoselskii-Schaefer type*, Nonlinear Studies, **9**(2002), 307-317.

[6] M.M. El-Borai, M.A. Abdou, M.M. El-Kojok, *On a discussion of nonlinear integral equation of type Volterra-Hammerstein*, J. Korea Soc. Math. Educ., Ser. B, Pure Appl. Math., **15**(2008), no. 1, 1-17.

[10] M. Lauran, *Existence results for some nonlinear integral equations*, Miskolc Mathematical Notes, **13**(2012), no. 1, 67-74.

[11] L.T.P. Ngoc, N.T. Long, *Applying a fixed point theorem of Krasnosel'skii type to the existence of asymptotically stable solutions for a Volterra-Hammerstein integral equation*, Nonlinear Anal. TMA., **74**(11)(2011), 3769-3774.

[12] L.T.P. Ngoc, N.T. Long, *On a nonlinear Volterra-Hammerstein integral equation in two variables*, Acta Mathematica Scientia, **33B**(2013), no. 2, 484-494.

[13] L.T.P. Ngoc, N.T. Long, *Existence of asymptotically stable solutions for a mixed functional nonlinear integral equation in N variables*, Mathematische Nachrichten, **288**(2015), no. 5-6, 633-647.

[14] B.G. Pachpatte, *On Fredholm type integrodifferential equation*, Tamkang J. of Math., **39**(2008), no. 1, 85-94.

[15] B.G. Pachpatte, *On Fredholm type integral equation in two variables*, Differential Equations & Applications, **1**(2009), no. 1, 27-39.

[16] B.G. Pachpatte, *Volterra integral and integrodifferential equations in two variables*, J. Inequal. Pure and Appl. Math., **10**(2009), no. 4, Art. 108, 10 pp.

[17] I.K. Purnaras, *A note on the existence of solutions to some nonlinear functional integral equations*, Electronic J. Qualitative Theory of Diff. Equat., **17**(2006), 1-24.

[18] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Springer-Verlag New York Berlin Heidelberg Tokyo, Part I, 1986.

Received: May 11, 2016; Accepted: September 1st, 2016.