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# A NONLOCAL PROBLEM FOR PROJECTED DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH APPLICATIONS

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**Abstract.** We study a nonlocal problem for projected differential equations and inclusions in finite dimensional spaces. By applying the fixed point theory methods we obtain the existence of solutions to the considered problem for projected differential inclusions. For the case of the projected differential equations we prove, under some suitable conditions, the uniqueness of a solution and the Ulam-Hyers stability of solutions. It is shown how the abstract results can be applied to the study of a market model with the price intervention in the form of price floors and ceilings. An example with exponential demand and supply functions is presented.
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## 1. Introduction

Let $K \subset \mathbb{R}^n$ be a bounded, closed and convex subset. For $z \in K$ denote by $T_K(z)$ the tangent cone to $K$ at point $z$. This paper deals with the nonlocal problem of projected differential inclusion of the form:

$$\begin{cases} x'(t) \in P_{T_K(x(t))}\big(F(x(t))\big), \quad \text{for a.e. } \ t \in I := [0,T], \\ x(0) \in M(x), \end{cases} \tag{1.1}$$

where $x\colon I \to K$ is a unknown function, $F\colon K \multimap \mathbb{R}^n$ and $M\colon C(I,K) \multimap K$ are given multivalued maps, $P_{T_K(x(t))}\big(F(x(t))\big)$ is the projection of the set $F(x(t))$ onto the cone $T_K(x(t))$.

Projected differential inclusions in finite dimensional spaces were studied first by Henry [12] for the case when $Mx = x_0$ (the Cauchy problem). Then, Cornet [9] introduced the notion of slow solutions of projected differential inclusions. Further, Nagurney [17] proved the existence and uniqueness of solution of (1.1) for the case when $F$ is single-valued map and $Mx = x_0$. This result was extended to Hilbert infinite dimensional spaces by Cojocaru and Jonker [8]. The existence of periodic solutions of projected differential equations was given in [7]. The stability problem for projected dynamical systems was investigated in [19, 21]. For other results in this direction see [18] and the references therein.

In the present paper, we focus on the nonlocal problem for projected differential equations and inclusions. Let us mention that the differential equations with nonlocal conditions were studied first by Byszewski [6]. The paper is organized in the following way. In the next section we recall some notions and results from the convex analysis and the theory of multivalued maps. In Section 3, the existence of a solution to problem (1.1) is given. For a particular case of a projected differential inclusion and a single-valued linear operator $M$ the uniqueness result is presented. For the same case, the Ulam-Hyers stability for (1.1) is considered in Section 4. In the last section, the abstract results are applied for the study of a market model with price intervention in the form of price floors and ceilings. An example with exponential demand and supply functions is given for the illustration of the results.

## 2. Preliminaries

2.1. **Notation and general properties.** For simplicity, we will use the same notation $|\cdot|$ $[\langle\cdot,\cdot\rangle]$ to denote the norm [resp., the inner product] in finite-dimensional spaces. By $C(I,\mathbb{R}^n)$ $[L^p(I,\mathbb{R}^n)$ $(p \geq 1)]$ we denote the spaces of all continuous [respectively, $p-$summable] functions $u\colon I \to \mathbb{R}^n$ with usual norms:

$$\|u\|_C = \max_{t\in I} |u(t)| \text{ and } \|u\|_p = \left(\int_0^T |u(t)|^p dt\right)^{\frac{1}{p}}.$$

Denote by $AC(I,\mathbb{R}^n)$ the space of all absolutely continuous functions $u\colon I \to \mathbb{R}^n$. Consider the space of all functions $u \in AC(I,\mathbb{R}^n)$ whose derivatives belong to $L^p(I,\mathbb{R}^n)$. It is known (see, e.g., [3]) that this space can be identified with the Sobolev space $W^{1,p}(I,\mathbb{R}^n)$ with the norm

$$\|u\|_W = \left(\|u\|_p^p + \|u'\|_p^p\right)^{\frac{1}{p}}.$$

Recall that (see, e.g., [3]) the embedding $W^{1,2}(I,\mathbb{R}^n) \hookrightarrow C(I,\mathbb{R}^n)$ is compact.

Now, let $H$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\| \cdot \|_H$. Let $\mathcal{K} \subset H$ be a nonempty, closed and convex subset. It is known that for any $z \in H$, there exists a unique element $w \in \mathcal{K}$ such that

$$\|w - z\|_H = \inf_{y \in \mathcal{K}} \|y - z\|_H.$$

The element $w$ is called the projection of $z$ onto $\mathcal{K}$ and is denoted by $P_{\mathcal{K}}(z)$.
The set

$$T_{\mathcal{K}}(x) = \overline{\bigcup_{h>0} \frac{1}{h}(K - x)}$$

is called the tangent cone to $\mathcal{K}$ at the point $x \in \mathcal{K}$. The normal cone to the set $\mathcal{K}$ at the point $x$ is given by

$$N_{\mathcal{K}}(x) := \left\{ p \in H \colon \langle p, x - w \rangle_H \geq 0, \ \forall w \in \mathcal{K} \right\}.$$

Notice that $T_{\mathcal{K}}(x)$ and $N_{\mathcal{K}}(x)$ are closed, convex cones.
From the definition of $N_{\mathcal{K}}(x)$ one can easily deduce that

**Lemma 2.1.** *The multimap $x \multimap N_{\mathcal{K}}(x)$ is monotone, i.e., for any $x, y \in \mathcal{K}$ and any $n_x \in N_{\mathcal{K}}(x)$ and $n_y \in N_{\mathcal{K}}(y)$, we have*

$$\langle x - y, n_x - n_y \rangle_H \geq 0.$$

The following result can be found in [13, Theorem 2.23].

**Theorem 2.2.** *(J.J. Moreau). If $C \subset H$ is a closed convex cone, $C^-$ its polar cone and $x, y, z \in H$, then the following properties are equivalent:*

*(a) $z = x + y$, $x \in C$, $y \in C^-$ and $\langle x, y \rangle_H = 0$;*
*(b) $x = P_C(z)$ and $y = P_{C^-}(z)$.*

Let us mention the following corollary of this result.

**Lemma 2.3.** *For each $x \in \mathcal{K}$ there exists $n \in N_{\mathcal{K}}(x)$ such that $P_{T_{\mathcal{K}}(x)}(v) = v - n$.*

2.2. **Multivalued maps.** We will recall some notions from the theory of multivalued maps (see, e.g., [1, 4, 11, 15, 20]).

Let $X, Y$ be metric spaces. Denote by $P(Y)$ $[K(Y)]$ the collection of all nonempty [respectively, nonempty compact] subsets of $Y$. For a Banach space $E$ by symbol $Kv(E)$ we denote the collection of all nonempty convex compact subsets of $E$.

**Definition 2.4.** A multivalued map (multimap) $F \colon X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) if for every open subset $V \subset Y$ the set

$$F_+^{-1}(V) = \{x \in X \colon F(x) \subset V\}$$

is open in $X$. An u.s.c. multimap $F$ is said to be compact, if the set $\overline{F(X)}$ is compact in $Y$.

We will need the following property of u.s.c. multimaps.

**Lemma 2.5.** *If a multimap $F \colon X \to K(Y)$ is u.s.c. and a set $A \subset X$ is compact in $X$ then its image $F(A)$ is compact in $Y$.*

**Definition 2.6.** A set $M \in K(Y)$ is said to be *aspheric* (or $UV^\infty$, or $\infty$-*proximally connected*) (see, e.g., [11, 16]), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that each continuous map $\sigma\colon S^n \to O_\delta(M)$, $n = 0, 1, 2, \cdots$, can be extended to a continuous map $\widetilde{\sigma}\colon B^{n+1} \to O_\varepsilon(M)$, where

$$S^n = \{x \in \mathbb{R}^{n+1}\colon |x| = 1\}, \quad B^{n+1} = \{x \in \mathbb{R}^{n+1}\colon |x| \leq 1\},$$

and $O_\delta(M)$ $[O_\varepsilon(M)]$ denotes the $\delta$-neighborhood [resp. $\varepsilon$-neighborhood] of the set $M$.

**Definition 2.7.** (see [14]). A nonempty compact space $A$ is said to be an $R_\delta$-*set* if it can be represented as the intersection of a decreasing sequence of compact, contractible spaces.

**Definition 2.8.** (see [11]). A u.s.c. multimap $\Sigma : X \to K(Y)$ is said to be a $J$-*multimap* ($\Sigma \in J(X, Y)$) if every value $\Sigma(x)$, $x \in X$, is an aspheric set.

Now let us recall (see, e.g., [5, 11]) that a metric space $X$ is called *the absolute retract (the AR-space)* [resp., *the absolute neighborhood retract (the ANR-space)]* provided for each homeomorphism $h$ taking it onto a closed subset of a metric space $X'$, the set $h(X)$ is the retract of $X'$ [resp., of its open neighborhood in $X'$]. Notice that the class of $ANR$-spaces is broad enough: in particular, a finite-dimensional compact set is the $ANR$-space if and only if it is locally contractible. In turn, it means that compact polyhedrons and compact finite-dimensional manifolds are the $ANR$-spaces. The union of a finite number of convex closed subsets in a normed space is also the $ANR$-space.

**Proposition 2.9.** *(see* [11]*). Let $Z$ be an $ANR$-space. In each of the following cases an u.s.c. multimap $\Sigma : X \to K(Z)$ is a $J$-multimap:*
*for each $x \in X$ the value $\Sigma(x)$ is*
*a) a convex set;*
*b) a contractible set;*
*c) an $R_\delta$-set;*
*d) an AR-space.*
*In particular, every continuous map $\sigma : X \to Z$ is a $J$-multimap.*

**Definition 2.10.** (cf. [2]) Let $X, Y$ be normed spaces, $X' \subseteq X$, $Y' \subseteq Y$. By $J^c(X', Y')$ we will denote the collection of all multimaps $F\colon X' \to K(Y')$ that may be represented in the form of composition $F = \Sigma_k \circ \cdots \circ \Sigma_1$, where $\Sigma_i \in J(U_{i-1}, U_i)$, $i = 1 \cdots k$, $U_0 = X'$, $U_k = Y'$, and $U_i$ $(0 < i < k)$ are open subsets of normed spaces.

The next fixed point theorem is a direct consequence of the fixed point index theory for $J^c-$multimaps (see [2]).

**Theorem 2.11.** *Let $X$ be a Banach space and $V \subset X$ a convex closed set and $\mathcal{F}\colon V \to K(V)$ a compact $J^c-$multimap. Then there exists $x_* \in V$ such that $x_* \in \mathcal{F}(x_*)$.*

Recall now a result on the structure of the solution set of differential inclusions and on the dependence of this set on the initial condition.

**Lemma 2.12.** *(see Theorem 70.6 and Proposition 70.9 of* [11]*). Let $\mathcal{F}\colon I \times \mathbb{R}^n \to Kv(\mathbb{R}^n)$ be such that*

($\mathcal{F}1$) *for every* $z \in \mathbb{R}^n$ *multifunction* $\mathcal{F}(\cdot, z)\colon I \to Kv(\mathbb{R}^n)$ *is measurable;*
($\mathcal{F}2$) *for a.e.* $t \in I$ *multimap* $\mathcal{F}(t, \cdot)\colon \mathbb{R}^n \to Kv(\mathbb{R}^n)$ *is u.s.c.;*
($\mathcal{F}3$) *there exists* $a \in L^1(I; \mathbb{R}_+)$ *such that*

$$\|\mathcal{F}(t,z)\| := \max\{|y|\colon y \in \mathcal{F}(t,z)\} \le a(t)(1+|z|),$$

*for all* $z \in \mathbb{R}^n$ *and a.e.* $t \in I$.

*Then for every* $z \in \mathbb{R}^n$ *the solution set* $\Sigma(z)$ *of the Cauchy problem*

$$\begin{cases} x'(t) \in \mathcal{F}(t, x(t)), & \textit{for a.e. } \ t \in I, \\ x(0) = z, \end{cases}$$

*is an* $R_\delta$*−set in* $C(I, \mathbb{R}^n)$*. Moreover, the multimap* $z \multimap \Sigma(z)$ *is u.s.c..*

## 3. Existence and uniqueness

Assume that

($A1$) $F\colon K \to Kv(\mathbb{R}^n)$ is u.s.c.;
($A2$) $M \in J^c\big(C(I, K); K\big)$.

**Remark 3.1.** The class of boundary value problems with the operator $M$ satisfying condition ($A2$) is sufficiently large. In particular, it includes the following problems:

($i$) $Mx = x_0 \in K$ (Cauchy condition);
($ii$) $Mx = x(T)$ (periodic problem);
($iii$) $Mx = \frac{1}{T}\int_0^T x(t)dt$ (mean value problem);
($iv$) $Mx = \sum_{i=1}^{k_0} \alpha_i x(t_i)$ with $\alpha_i \ge 0$ and $\sum_{i=1}^{k_0} \alpha_i = 1$, where $0 \le t_1 < \cdots < t_{k_0} \le T$ (multi-point problem). For the case when $K$ contains 0, we can take $\alpha_i \ge 0$ and $\sum_{i=1}^{k_0} \alpha_i \le 1$.
($v$) $Mx = M_r x := \{y \in K : \|y - x(T)\| \le r\}$, where $r > 0$ is a given number (problem of finding a solution going not far away from the beginning).

**Definition 3.2.** By a solution to (1.1) we mean a function $x \in AC(I, K)$ that satisfies (1.1).

For $z \in \mathbb{R}^n$ let $\overline{z}$ be the projection of $z$ onto $K$. Define the map $H\colon \mathbb{R}^n \to P(\mathbb{R}^n)$,

$$H(z) = P_{T_K(\overline{z})}\big(F(\overline{z})\big).$$

and consider the problem

$$\begin{cases} x'(t) \in H(x(t)), & \text{for a.e. } \ t \in I, \\ x(0) \in M(x). \end{cases} \tag{3.1}$$

It is clear that every absolutely continuous function $x\colon I \to K$ that satisfies problem (3.1) will be a solution of problem (1.1). Now, let $G\colon \mathbb{R}^n \to Kv(\mathbb{R}^n)$ be a multimap defined as

$$G(z) = \bigcap_{\varepsilon > 0} \overline{co} H\big(z + \varepsilon B\big),$$

where $\overline{co}$ denotes the closure of the convex hull of a set and $B$ is the unit closed ball in $\mathbb{R}^n$.
Applying Lemma 2.5 we conclude that there exists $\alpha > 0$ such that

$$\|F(z)\| := \sup\{|y\| : y \in F(z)\} \leq \alpha, \ \forall z \in K.$$

Then, it is easy to verify (cf., for example, Theorem 3.2.15 of [15]) that $G$ is u.s.c. and, moreover,

$$\|G(z)\| \leq \alpha \tag{3.2}$$

for all $z \in \mathbb{R}^n$.

**Lemma 3.3.** *(see* [12, Lemmas 1 and 2]*). Let $x \in AC(I, \mathbb{R}^n)$, $x(0) = x_0 \in K$ be such that $x'(t) \in G(x(t))$ for a.e. $t \in I$. Then $x(t) \in K$ for all $t \in I$ and $x'(t) \in H(x(t))$ for a.e. $t \in I$.*

**Theorem 3.4.** *Let conditions* $(A1) - (A2)$ *hold. Then problem* (1.1) *has a solution.*

*Proof.* Set $Q = C(I, K)$ and for every $y \in Q$ and arbitrary $x_0 \in M(y)$ consider the Cauchy problem

$$\begin{cases} x'(t) \in G(x(t)), & \text{for a.e. } \ t \in I, \\ x(0) = x_0 \in M(y). \end{cases} \tag{3.3}$$

From Lemma 2.12 it follows that the solution set $\Sigma(x_0)$ of (3.3) is an $R_\delta-$set in $C(I, \mathbb{R}^n)$ and the multimap $\Sigma\colon M(y) \to K\big(C(I, \mathbb{R}^n)\big)$,

$$x_0 \in M(y) \multimap \Sigma(x_0)$$

is a $J-$multimap.
Define the multimap $\widetilde{\Sigma}\colon Q \to K\big(C(I, \mathbb{R}^n)\big)$, $\widetilde{\Sigma} = \Sigma \circ M$. It is clear that $\widetilde{\Sigma} \in J^c\big(Q, C(I, \mathbb{R}^n)\big)$.
On the other hand, from $(A2)$ it follows that $M(y) \subset K$. Hence, by applying Lemma 2.1 we get that $x(t) \in K$, $\forall t \in I$ for all $x \in \widetilde{\Sigma}(Q)$, i.e., $\widetilde{\Sigma}(Q) \subseteq Q$. Now, let us show that $\widetilde{\Sigma}(Q)$ is compact in $C(I, \mathbb{R}^n)$. In fact, for arbitrary $x \in \widetilde{\Sigma}(Q)$, we have $x \in Q$ and hence there exists $L > 0$ such that $\|x\|_2 < L$. From the other side, $x$ is a solution of (3.3) for some $y \in Q$ and hence, by virtue of (3.2) we have $\|x'\|_2 < \alpha\sqrt{T}$. Consequently, the set $\widetilde{\Sigma}(Q)$ is bounded in $W^{1,2}(I, \mathbb{R}^n)$. From the compactness of the embedding $W^{1,2}(I, \mathbb{R}^n) \hookrightarrow C(I, \mathbb{R}^n)$ we obtain the compactness of the set $\overline{\widetilde{\Sigma}(Q)}$ in $C(I, \mathbb{R}^n)$.
Thus, from Theorem 2.11 it follows that there exists $x \in Q$ such that $x \in \widetilde{\Sigma}(x)$, i.e., the function $x$ is a solution to the problem

$$\begin{cases} x'(t) \in G(x(t)), & \text{for a.e. } \ t \in I, \\ x(0) \in M(x) \end{cases}$$

and hence, by Lemma 2.1 it is a solution for (1.1). $\square$

Consider now the uniqueness problem. Assume that

$(A1)'$ $F\colon K \to \mathbb{R}^n$ is a single-valued map satisfying the Lipschitz condition

$$|F(x) - F(y)| \le L|x - y|,$$

for all $x, y \in K$ for some $L > 0$;

$(A2)'$ $M\colon C(I, \mathbb{R}^n) \to \mathbb{R}^n$ is a linear continuous operator such that $M\big(C(I,K)\big) \subseteq K$.

It means that problem (1.1) turns into

$$\begin{cases} x'(t) = P_{T_K(x(t))}\big(F(x(t))\big), & \text{for a.e. } \ t \in I := [0,T], \\ x(0) = M(x). \end{cases} \tag{3.4}$$

**Theorem 3.5.** *Let conditions $(A1)' - (A2)'$ hold. If*

$$\|M\|e^{LT} < 1, \tag{3.5}$$

*then problem* (3.4) *has a unique solution,*

*Proof.* The existence of a solution to (1.1) clearly follows from Theorem 3.4. Let us show that this solution is unique. To the contrary, assume that $x_1(\cdot)$ and $x_2(\cdot)$ are two different solutions to problem (1.1). For a.e. $t \in (0,T]$ we have

$$\begin{aligned} \frac{d}{dt}\big(|x_1(t) - x_2(t)|^2\big) &= 2\big\langle x_1(t) - x_2(t), x_1'(t) - x_2'(t)\big\rangle \\ &= 2\big\langle x_1(t) - x_2(t), P_{T_{K(x_1(t))}}\big(F(x_1(t))\big) - P_{T_{K(x_2(t))}}\big(F(x_2(t))\big)\big\rangle \\ &= 2\big\langle x_1(t) - x_2(t), F(x_1(t)) - n_1 - \big(F(x_2(t)) - n_2\big)\big\rangle \\ &\le 2\big\langle x_1(t) - x_2(t), F(x_1(t)) - F(x_2(t))\big\rangle \\ &\le 2L|x_1(t) - x_2(t)|^2, \end{aligned}$$

where $n_1 \in N_K(x_1(t))$ and $n_2 \in N_K(x_2(t))$.

Applying the theorem on differential inequalities to the function $t \to |x_1(t) - x_2(t)|^2$ we obtain

$$|x_1(t) - x_2(t)|^2 \le |x_1(0) - x_2(0)|^2 e^{2Lt} \le |Mx_1 - Mx_2|^2 e^{2LT},$$

for all $t \in I$.

Hence, by applying (3.5) we have

$$\|x_1 - x_2\|_C = \max_{t \in I} |x_1(t) - x_2(t)| \le \|M\|e^{LT}\|x_1 - x_2\|_C < \|x_1 - x_2\|_C,$$

giving the contradiction. □

## 4. The Ulam-Hyers stability of solutions to projected differential equations

Consider now the problem of Ulam-Hyers stability of solutions to problem (3.4). Toward this goal, let us first introduce its definition.

**Definition 4.1.** Problem (3.4) is said to be Ulam-Hyers stable if there exists $c > 0$ such that for each $\varepsilon > 0$ and each solution $y(\cdot) \in AC(I, \mathbb{R}^n)$ of the problem

$$\begin{cases} \left|y'(t) - P_{T_K(y(t))}\big(F(y(t))\big)\right| \leq \varepsilon, \text{ for a.e. } t \in I, \\ y(0) = My, \end{cases} \tag{4.1}$$

there exists a solution $x(\cdot)$ of (3.4) with

$$|x(t) - y(t)| \leq c\varepsilon, \ \forall t \in I.$$

In sequel we need the following result which is a generalized version of the Gronwall lemma.

**Lemma 4.2.** *(see, e.g.,* [10, Theorem 21]*). Let $u(t)$ be a nonnegative function that satisfies the integral inequality*

$$u(t) \leq c + \int_{t_0}^{t} \big(a(s)u(s) + b(s)u^{\alpha}(s)\big)ds, \ c \geq 0, \ \alpha \in [0,1),$$

*where $a(t)$ and $b(t)$ are continuous nonnegative functions for $t \geq t_0$. Then*

$$u(t) \leq \left(c^{1-\alpha}\, e^{(1-\alpha)\int_{t_0}^{t} a(s)ds} + (1-\alpha)\int_{t_0}^{t} b(s)\, e^{(1-\alpha)\int_{s}^{t} a(\tau)d\tau} ds\right)^{\frac{1}{1-\alpha}}.$$

**Theorem 4.3.** *Let conditions $(A1)'$, $(A2)'$ and* (3.5) *hold. Then problem* (3.4) *is Ulam-Hyers stable.*

*Proof.* From Theorem 3.5 it follows that problem (3.4) has a unique solution $x_*(\cdot)$. Now, let $y(\cdot) \in AC(I, \mathbb{R}^n)$ be an arbitrary solution of (4.1). Then for a.e. $t \in (0, T]$ we have

$$\begin{aligned}
\frac{d}{dt}\big(|x_*(t) - y(t)|^2\big) &= 2\big\langle x_*(t) - y(t), x_*'(t) - y'(t)\big\rangle \\
&= 2\Big\langle x_*(t) - y(t), P_{T_{K(x_*(t))}}\big(F(x_*(t))\big) - P_{T_{K(y(t))}}\big(F(y(t))\big)\Big\rangle \\
&\quad -2\Big\langle x_*(t) - y(t), y'(t) - P_{T_{K(y(t))}}\big(F(y(t))\big)\Big\rangle \\
&= 2\Big\langle x_*(t) - y(t), F(x_*(t)) - n_1 - \big(F(y(t)) - n_2\big)\Big\rangle \\
&\quad -2\Big\langle x_*(t) - y(t), y'(t) - P_{T_{K(y(t))}}\big(F(y(t))\big)\Big\rangle \\
&\leq 2\big\langle x_*(t) - y(t), F(x_*(t)) - F(y(t))\big\rangle \\
&\quad +2|x_*(t) - y(t)|\,\Big|y'(t) - P_{T_{K(y(t))}}\big(F(y(t))\big)\Big| \\
&\leq 2L|x_*(t) - y(t)|^2 + 2\varepsilon|x_*(t) - y(t)|,
\end{aligned}$$

where $n_1 \in N_K(x_*(t))$ and $n_2 \in N_K(y(t))$.
Therefore, for any fixed $t \in I$:

$$\int_0^t \frac{d}{ds}\big(|x_*(s) - y(s)|^2\big) \leq \int_0^t \big(2L|x_*(s) - y(s)|^2 + 2\varepsilon|x_*(s) - y(s)|\big)ds,$$

or equivalently,

$$\begin{aligned}|x_*(t)-y(t)|^2 &\le |x_*(0)-y(0)|^2+\int_0^t \big(2L|x_*(s)-y(s)|^2+2\varepsilon|x_*(s)-y(s)|\big)ds\\ &\le \|M\|^2\|x_*-y\|_C^2+\int_0^t \big(2L|x_*(s)-y(s)|^2+2\varepsilon|x_*(s)-y(s)|\big)ds.\end{aligned}$$

Taking $u(t)=|x_*(t)-y(t)|^2$, we can rewrite the above inequality as

$$u(t)\le \|M\|^2\|x_*-y\|_C^2+\int_0^t \big(2Lu(s)+2\varepsilon u^{\frac{1}{2}}(s)\big)ds.$$

Applying Lemma 4.2 we obtain

$$\begin{aligned}u(t) &\le \Big(\|M\|\|x_*-y\|_C\, e^{\frac{1}{2}\int_0^t 2Lds}+\frac{1}{2}\int_0^t 2\varepsilon\, e^{\frac{1}{2}\int_s^t 2Ld\tau}ds\Big)^2\\ &\le \big(\|M\|\|x_*-y\|_C\, e^{LT}+\frac{\varepsilon}{L}(e^{LT}-1)\big)^2,\end{aligned}$$

for all $t\in I$.
Therefore,

$$\|x_*-y\|_C^2=\max_{t\in I}u(t)\le \big(\|M\|\|x_*-y\|_C\, e^{LT}+\frac{\varepsilon}{L}(e^{LT}-1)\big)^2,$$

or equivalently,

$$\|x_*-y\|_C\le \frac{e^{LT}-1}{L(1-\|M\|\, e^{LT})}\,\varepsilon.$$

Thus, problem (3.4) is Ulam-Hyers stable. □

## 5. Application to a market model

Consider a market model of with the price intervention in the form of price floors and ceilings. In this model there are $n$ commodities. Let $p_i^C$ and $p_i^F$ $(0\le p_i^F<p_i^C)$ denote the imposed price ceiling and price floor on the price of commodity $i$ $(i=1\cdots n)$, respectively. For this problem, the adjustment process for each commodity $i$ is defined as:

$$p_i'(t)=\begin{cases}d_i(p(t))-s_i(p(t)), & \text{if } p_i^F<p_i(t)<p_i^C,\\ \max\{0,d_i(p(t))-s_i(p(t))\}, & \text{if } p_i(t)=p_i^F,\\ \min\{0,d_i(p(t))-s_i(p(t))\}, & \text{if } p_i(t)=p_i^C,\end{cases} \tag{5.1}$$

for a.e. $t\in[0,T]$, where $p(t)=\big(p_1(t),\cdots,p_n(t)\big)$, $p_i(t)$ denotes the price of commodity $i$ at the time $t$, $d_i(\cdot)\colon\mathbb{R}^n\to\mathbb{R}$ and $s_i(\cdot)\colon\mathbb{R}^n\to\mathbb{R}$ are the demand and supply functions for commodity $i$, respectively.
In other words, the model can be explained as following. We study the adjustment process on a given time interval $I=[0,T]$. The price of a commodity $i$ will increase (decrease) if the demand for that commodity exceeds (respectively, is less that) the supply of that commodity. If the price of an commodity is equal to the price floor (price ceiling), and the supply of that commodity exceeds (respectively, is less than) the demand, then the price will not change.

Set $K = \{z \in \mathbb{R}^n \colon p^F \leq z \leq p^C\}$, where $p^F$ and $p^C$ denote the $n-$dimensional vectors of imposed price floors and ceilings, respectively. Then problem (5.1) can be substituted with the following projected differential equation

$$p'(t) = P_{T_K(p(t))}\big(d(p(t)) - s(p(t))\big), \quad \text{for a.e. } t \in I, \tag{5.2}$$

where $d(\cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$ and $s(\cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$ are the $n-$dimensional vectors of demand and supply functions, respectively.

Let us note that projected differential equation (5.2) was considered in [17] for the finding of market equilibrium, i.e. the point $p^* \in K$ such that

$$P_{T_K(p^*)}\big(d(p^* - s(p^*)\big) = 0,$$

or equivalently, for each commodity $i$ $(i = 1, \cdots, n)$:

$$d_i(p_*) - s_i(p_*) \begin{cases} \geq 0, & \text{if } p_i^* = p_i^C, \\ = 0, & \text{if } p_i^F < p_i^* < p_i^C, \\ \leq 0, & \text{if } p_i^* = p_i^F. \end{cases}$$

Here, by applying Theorems 3.4, 3.5 and 4.3 we obtain

**Theorem 5.1.** *Let the demand function $d(\cdot)$ and the supply function $s(\cdot)$ be continuous and a multimap $M$ satisfy condition $(A2)$. Then the price regulation problem*

$$\begin{cases} p'(t) = P_{T_K(p(t))}\big(d(p(t)) - s(p(t))\big), & \textit{for a.e. } t \in I, \\ p(0) \in Mp, \end{cases} \tag{5.3}$$

*has a solution.*

Notice that if we will take, in particular, for a given $r > 0$ the multimap $M_r$ defined as in Remark 3.1(v), we obtain the existence of a price trajectory $p(t)$, $t \in I$ such that

$$\|p(T) - p(0)\| \leq r.$$

**Theorem 5.2.** *Let the demand function $d(\cdot)$ and the supply function $s(\cdot)$ be Lipschitz continuous with constants $l_d$ and $l_s$ respectively and a linear operator $M$ satisfy conditions $(A2)'$ and (3.5) with $L = l_d + l_s$. Then problem (5.3) with the boundary condition*

$$p(0) = Mp$$

*has a unique solution. Moreover, this problem is Ulam-Hyers stable.*

**Example 5.3.** Consider a market model with one commodity. Assume $p_L = 0$ and $p_C = 3$ are price floor and ceiling for this commodity, respectively. In addition, assume that $d(p(t)) = e^{-p(t)} + 2$ and $s(p(t)) = e^{p(t)}$ are the demand and supply functions with respect to the price $p(t)$, $t \in I$. Then there exists a periodic price trajectory (i.e., $p(0) = p(T)$) for the adjustment process

$$\begin{aligned} p'(t) &= P_{T_K(p(t))}\big(e^{-p(t)} + 2 - e^{p(t)}\big) \\ &= \begin{cases} e^{-p(t)} + 2 - e^{p(t)}, & \text{if } 0 < p(t) < 3, \\ \max\{0, e^{-p(t)} + 2 - e^{p(t)}\} = 2, & \text{if } p(t) = 0, \\ \min\{0, e^{-p(t)} + 2 - e^{p(t)}\} = e^{-3} + 2 - e^3, & \text{if } p(t) = 3, \end{cases} \end{aligned} \tag{5.4}$$

where $K = [0, 3]$.
It is clear (see Figure 1[1]) that in this case the periodic price trajectories are the market equilibriums of (5.4). Moreover, we can replace the periodic condition $p(0) = p(T)$ by other nonlocal conditions (see Remark 3.1) to obtain new existence theorems for problem (5.4).

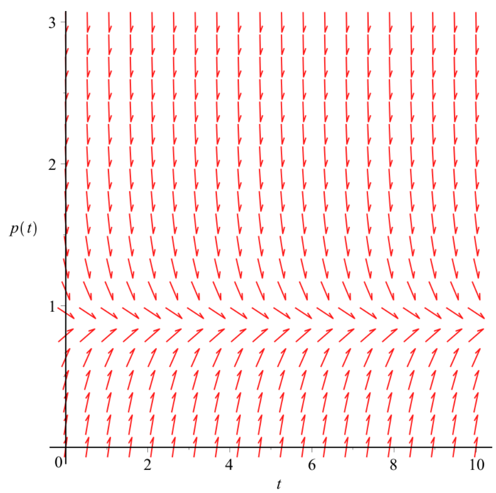


FIGURE 1. Direction field of $p'(t) = e^{-p(t)} + 2 - e^{p(t)}$.

Further, put $F\colon [0, 3] \to \mathbb{R}$, $F(z) = e^{-z} + 2 - e^{z} = 2(1 - \sinh z)$. Since

$$\max_{z\in[0,3]} \left|\frac{d}{dz}F(z)\right| = \max_{z\in[0,3]} (e^{z} + e^{-z}) = e^{3} + e^{-3},$$

we have

$$|F(z) - F(w)| \leq (e^{3} + e^{-3})|z - w|,\ \forall z, w \in [0, 3].$$

Consider again problem (5.4) with the multi-point condition

$$p(0) = \sum_{i=1}^{n} c_1 p(t_i), \tag{5.5}$$

where $0 \leq t_1 < t_2 < \cdots < t_n \leq 3$ and $c_i \geq 0$ such that

$$\sum_{i=1}^{n} c_i < \frac{1}{e^{T(e^{3}+e^{-3})}}.$$

By applying Theorem 5.2 we obtain that problem (5.4)-(5.5) is Ulam-Hyers stable and has a unique price function solution $p\colon I \to [0, 3]$.

[1]Figure was done with Maple 17

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