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COMMON SOLUTION TO A SPLIT EQUALITY MONOTONE VARIATIONAL INCLUSION PROBLEM, A SPLIT EQUALITY GENERALIZED GENERAL VARIATIONAL-LIKE INEQUALITY PROBLEM AND A SPLIT EQUALITY FIXED POINT PROBLEM

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Abstract. This paper deals with a strong convergence theorem for an iterative method for approximating a common solution to a split equality monotone variational inclusion problem, a split equality generalized general variational-like inequality problem and a split equality fixed point problem for quasi-nonexpansive mappings in real Hilbert spaces. Some consequences are derived from the main result. Finally, we give a numerical example to justify the main result. The main result extends and unifies some recent known results in the literature.
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1. Introduction

Let H_1, H_2 and H_3 be real Hilbert spaces, let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex sets. We denote the inner product and norm of H_1, H_2 and H_3 by notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. The split feasibility problem (in short, $\mathrm{S_pFP}$) is to find a point

$$\bar{x} \in C \text{ such that } A\bar{x} \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The $\mathrm{S_pFP}(1.1)$ in finite dimensional Hilbert space was introduced by Censor and Elfving [7] for modeling inverse problem which arise from retrievals and in medical image reconstruction [5]. Since then various iterative methods have been proposed to solve $\mathrm{S_pFP}(1.1)$; see for instance [1, 4, 10, 27].

Recently, Moudafi [20] introduced and studied the following split equality problem which is a natural generalization of $\mathrm{S_pFP}(1.1)$: find

$$\bar{x} \in C, \ \bar{y} \in Q \text{ such that } A\bar{x} = B\bar{y}, \tag{1.2}$$

where $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators. For related work, see [21, 16]. Note that the problem (1.2) reduces to problem (1.1) if $H_2 = H_3$ and $B = I$, where I stands for the identity operator on H_2, in (1.2).

Further, Moudafi [22] introduced and studied the following split equality fixed point problem (in short, $\mathrm{S_p}$EFPP): find $(\bar{x}, \bar{y}) \in C \times Q$ such that

$$\bar{x} \in \mathrm{Fix}(S),\ \bar{y} \in \mathrm{Fix}(T) \text{ and } A\bar{x} = B\bar{y}, \tag{1.3}$$

where $S : C \to C$ and $T : Q \to Q$ be nonlinear mappings and $\mathrm{Fix}(S) := \{x \in C : Sx = x\}$. The solution set of $\mathrm{S_p}$EFPP(1.3) is denoted by Θ. We note as given in Zhao *et al.* [30] (see also Dong *et al.* [11], Moudafi [22]) that $\mathrm{S_p}$EFPP(1.3) and related problems allow asymmetric and partial relations between the variables x and y. The interest is to cover many situations, for instance in decomposition methods for partial differential equations, applications in game theory and in intensity-modulated radiation therapy (in short, IMRT). In decision sciences, this allows consideration of agents that interplay only via some components of their decision variables (see, [2]). In (IMRT), this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity (see, [8]).

Recently, Zhao [29] introduced and studied a simultaneous iterative method and proved a weak convergence theorem for $\mathrm{S_p}$EFPP (1.3) for quasi-nonexpansive operators. For further related work, see Zhao *et al.* [30] and Dong *et al.* [11].

It is well known that the theory of variational inequalities plays an important role in optimization, economics and engineering sciences. Because of its vast range applicability, various extensions and generalizations of variational inequality problems have been made and analyzed in various directions for past several years. One of the important generalizations is variational-like inequality problem introduced by Parida *et al.* [25] which has applications in optimization.

In 2006, Preda *et al.* [26] introduced and studied the general variational-like inequality problem (in, short GVLIP) of finding $\bar{x} \in C$ such that

$$F(x, \bar{x}; \bar{x}) \geq 0,\ \forall x \in C, \tag{1.4}$$

which has applications in mathematical and equilibrium programming, see for example [28].

Very recently, Kazmi and Ali [15] introduced the generalized general variational-like inequality problem (in, short GGVLIP) which is to find $\bar{x} \in C$ such that

$$F(x, \bar{x}; \bar{x}) + \phi(x, \bar{x}) - \phi(\bar{x}, \bar{x}) \geq 0,\ \forall x \in C. \tag{1.5}$$

They proved an existence theorem for GGVLIP(1.5) and proved strong convergence theorem for an iterative method for approximating a common solution to a system of GGVLIPs and a common fixed point problem in Banach space.

If we set $F(x, \bar{x}; \bar{x}) = \langle f\bar{x} + g\bar{x}, \eta_1(x, \bar{x})\rangle$ where $f, g : C \to H_1$ and $\eta_1 : C \times C \to H_1$ then GGVLIP(1.5) is reduced to the mixed variational-like inequality problem introduced and studied by Noor [23].

Further, if we set $F(x,\bar{x};\bar{x}) = \langle f\bar{x}, \eta_1(x,\bar{x})\rangle$ where $f : C \to H_1$ and $\eta_1 : C \times C \to H_1$ and $\phi = 0$, then GGVLIP(1.5) is reduced to the variational-like inequality problem of finding $\bar{x} \in C$ such that

$$\langle f\bar{x}, \eta_1(x,\bar{x})\rangle \geq 0, \ \forall x \in C,$$

introduced and studied by Parida *et al.* [25], which has applications in mathematical programming problems.

Moreover if $\eta_1(x,\bar{x}) = x - \bar{x}$ for all $x, \bar{x} \in C$, then variational-like inequality problem is reduced to the classical variational inequality problem of finding $\bar{x} \in C$ such that

$$\langle f\bar{x}, x - \bar{x}\rangle \geq 0, \ \forall x \in C,$$

introduced and studied by Hartman and Stampacchia [12].

In this paper, we introduce the following split equality generalized general variational-like inequality problem (in short, $\mathrm{S_pEGGVLIP}$) which is an extension of GGVLIP(1.5): find $\bar{x} \in C$ and $\bar{y} \in Q$ such that

$$F(x,\bar{x};\bar{x}) + \phi(x,\bar{x}) - \phi(\bar{x},\bar{x}) \geq 0, \ \forall x \in C, \tag{1.6}$$

$$G(y,\bar{y};\bar{y}) + \psi(y,\bar{y}) - \psi(\bar{y},\bar{y}) \geq 0, \ \forall y \in Q \tag{1.7}$$

$$\text{and } A\bar{x} = B\bar{y},$$

where $F : C\times C\times C \to \mathbb{R}$ and $G : Q\times Q\times Q \to \mathbb{R}$ are trifunctions. When looked separately, (1.6) is GGVLIP and its solution set is denoted by Sol(GGVLIP(1.6)). Solution set of $\mathrm{S_pEGGVLIP}$(1.6)-(1.7) is denoted by Sol($\mathrm{S_pEGGVLIP}$(1.6)-(1.7))=$\{(\bar{x},\bar{y}) \in C \times Q : \bar{x} \in \mathrm{Sol(GGVLIP(1.6))}, \bar{y} \in \mathrm{Sol(GGVLIP(1.7))}$
and $A\bar{x} = B\bar{y}\}$.

If we set $\phi, \psi = 0$; $H_1 = \mathbb{R}^n, H_2 = \mathbb{R}^m, H_3 = \mathbb{R}^k$; $F(x,\bar{x};\bar{x}) = \langle \bigtriangledown f\bar{x}, \eta_1(x,\bar{x})\rangle$ and $G(y,\bar{y};\bar{y}) = \langle \bigtriangledown g\bar{y}, \eta_2(y,\bar{y})\rangle$ where $\eta_1 : C \times C \to \mathbb{R}^n$, $\eta_2 : Q \times Q \to \mathbb{R}^m$ are continuous, and $f : C \to \mathbb{R}^n$ and $g : Q \to \mathbb{R}^m$ are differentiable and respectively, η_1- and η_2-convex [25], then $\mathrm{S_pEGGVLIP}$(1.6)-(1.7) is reduced to the following new mathematical programming problem:

$$\begin{aligned} &\min_{\bar{x}\in C} f(\bar{x}), \\ &\min_{\bar{y}\in Q} g(\bar{y}), \\ \text{and } &A\bar{x} = B\bar{y}. \end{aligned} \tag{1.8}$$

Further, we consider the following split equality monotone variational inclusion problem (in short, $\mathrm{S_pEMVIP}$): find $\bar{x} \in H_1$, $\bar{y} \in H_2$ such that

$$0 \in U(\bar{x}) + M(\bar{x}), \tag{1.9}$$

$$0 \in V(\bar{y}) + N(\bar{y}), \tag{1.10}$$

$$\text{and } A\bar{x} = B\bar{y},$$

where $M : H_1 \to 2^{H_1}$ and $N : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings. When looked separately, (1.9) is called monotone variational inclusion problem (in short, MVIP) and its solution set is denoted by Sol(MVIP(1.9)). Solution set of $\mathrm{S_p}$EMVIP(1.9)-(1.10) is denoted by Sol($\mathrm{S_p}$EMVIP(1.9)-(1.10)).

If we set $U = 0$ and $V = 0$, then $\mathrm{S_p}$EMVIP(1.9)-(1.10) is reduced to the following problem: find $\bar{x} \in H_1$ and $\bar{y} \in H_2$ such that

$$0 \in M(\bar{x}), \tag{1.11}$$

$$0 \in N(\bar{y}), \tag{1.12}$$

$$\text{and } A\bar{x} = B\bar{y}.$$

Problem (1.11)-(1.12) is called the split equality null point problem (in short, $\mathrm{S_p}$ENPP). Solution set of $\mathrm{S_p}$ENPP(1.11)-(1.12) is denoted by Sol($\mathrm{S_p}$ENPP(1.11)-(1.12)). $\mathrm{S_p}$ENPP(1.11)-(1.12) generalizes split null point problem (in short, $\mathrm{S_p}$NPP) studied by [6, 14].

Also, $\mathrm{S_p}$EMVIP(1.9)-(1.10) is a natural generalization of split monotone variational inclusion problem (in short, $\mathrm{S_p}$MVIP) given by Moudafi [19]. Moudafi [19] proved a weak convergence theorem for solving $\mathrm{S_p}$MVIP. It is worth to mention that the weak and strong convergence are different in setting of general Hilbert spaces and in the most cases, strong convergence is more desirable than weak convergence. However, there is a very little progress in strong convergence results for iterative methods for solving $\mathrm{S_p}$MVIP. Therefore, to prove a strong convergence theorem for finding a common solution to $\mathrm{S_p}$EMVIP(1.9)-(1.10) (a more general problem than $\mathrm{S_p}$MVIP), $\mathrm{S_p}$EGGVLIP(1.6)-(1.7) and $\mathrm{S_p}$EFPP(1.3) is the main interest of this paper.

Motivated by the ongoing work in this direction, we propose and analyze an iterative method for solving $\mathrm{S_p}$EMVIP(1.9)-(1.10), $\mathrm{S_p}$EGGVLIP(1.6)-(1.7) and $\mathrm{S_p}$EFPP(1.3) and prove a strong convergence theorem for the proposed iterative algorithm to approximate a common solution to $\mathrm{S_p}$EMVIP(1.9)-(1.10), $\mathrm{S_p}$EGGVLIP(1.6)-(1.7) and $\mathrm{S_p}$EFPP(1.3). Further, we derive some consequences from the main result. Finally, we give a numerical example to justify the main result. The result presented here extends and unifies some known results in the literature, see for instance, [29].

2. Preliminaries

Throughout the paper, we denote the strong and weak convergence of a sequence $\{x_n\}$ to a point $x \in X$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. For every point $x \in H_1$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$, $\forall y \in C$. The mapping P_C is called the metric projection from H_1 onto C. It is well known that P_C is a firmly nonexpansive mapping from H_1 to C, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \forall x, y \in H_1.$$

Further, for any $x \in H_1$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y\rangle \geq 0, \forall y \in C. \tag{2.1}$$

Definition 2.1. *A mapping $S : H_1 \to H_1$ is said to be*

(i) *nonexpansive, if*

$$\|Sx - Sy\| \leq \|x - y\|,\ \forall x \in H_1,\ y \in H_1;$$

(ii) *quasi-nonexpansive, if*

$$\|Sx - Sq\| \leq \|x - q\|,\ \forall x \in H_1,\ q \in \mathrm{Fix}(S);$$

(iii) *firmly quasi-nonexpansive, if*

$$\|Sx - q\|^2 \leq \|x - q\|^2 - \|x - Sx\|^2,\ \forall x \in H_1,\ q \in \mathrm{Fix}(S)).$$

Lemma 2.1. [Corollary 4.15 [3]] *Let $C \subset H_1$ be a nonempty, closed and convex set and let $S : C \to H_1$ be a nonexpansive mapping. Then* $\mathrm{Fix}(S)$ *is closed and convex.*

Lemma 2.2. [18] *Let $S : H_1 \to H_1$ be quasi-nonexpansive mapping. Set $S_\beta = \beta I + (1-\beta)S$, for $\beta \in [0, 1)$. Then the following properties are reached for all $x \in H_1,\ q \in \mathrm{Fix}(S)$:*

(i) $\langle x - Sx, x - q\rangle \geq \frac{1}{2}\|x - Sx\|^2$ and $\langle x - Sx, q - Sx\rangle \leq \frac{1}{2}\|x - Sx\|^2$;
(ii) $\|S_\beta x - q\|^2 \leq \|x - q\|^2 - \beta(1-\beta)\|Sx - x\|^2$;
(iii) $\langle x - S_\beta x, x - q\rangle \geq \frac{1-\beta}{2}\|x - Sx\|^2$.

Remark 2.1. [18] Let $S_\beta = \beta I+(1-\beta)S$, where $S : H_1 \to H_1$ is a quasi-nonexpansive mapping and $\beta \in [0, 1)$. We have $\mathrm{Fix}(S_\beta) = \mathrm{Fix}(S)$ and

$$\|S_\beta x - x\|^2 = (1-\beta)^2\|Sx - x\|^2.$$

It follows from (ii) of Lemma 2.2 that

$$\|S_\beta x - q\|^2 = \|x - q\|^2 - \frac{\beta}{1-\beta}\|S_\beta x - x\|^2,$$

which implies that S_β is firmly nonexpansive when $\beta = \frac{1}{2}$. On the other hand, if $\widehat{S}$ is a firmly quasi-nonexpansive mapping, we can easily obtain $\widehat{S} = \frac{1}{2}I + \frac{1}{2}S$, where S is quasi-nonexpansive.

Definition 2.2. *A mapping $U : H_1 \to H_1$ is said to be*

(i) *monotone, if* $\langle Ux - Uy, x - y\rangle \geq 0,\ \forall x, y \in H_1$;
(ii) *strongly monotone, if there exists a constant $\beta > 0$ such that*

$$\langle Ux - Uy, x - y\rangle \geq\ \beta\|x - y\|^2,\ \forall x, y \in H_1;$$

(iii) *β-inverse strongly monotone, if there exists a constant $\beta > 0$ such that*

$$\langle Ux - Uy, x - y\rangle\ \geq \beta\|Ux - Uy\|^2,\ \forall x, y \in H_1.$$

Definition 2.3. *A multi-valued mapping $M : H_1 \to 2^{H_1}$ is called monotone if for all $x, y \in H_1$, $u \in Mx$ and $v \in My$ such that*

$$\langle x - y, u - v\rangle \geq 0.$$

Definition 2.4. *A monotone mapping $M : H_1 \to 2^{H_1}$ is maximal if the*

$$Graph(M) := \{(x, y) : x \in H_1, y \in M(x)\}$$

is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping M is maximal if and only if for $(x, u) \in H_1 \times H_1$, $\langle x - y, u - v\rangle \geq 0$, for every $(y, v) \in \text{Graph}(M)$ implies that $u \in Mx$.

Let A be a monotone mapping of C into H_1 and $N_C v$ the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H_1 : \langle v - u, w\rangle \geq 0, \ \forall u \in C\},$$

and define a mapping M on C by

$$Mv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C, \end{cases}$$

then M is maximal monotone and $0 \in Mv$ if and only if $\langle Av, u - v\rangle \geq 0$ for all $u \in C$.

Definition 2.5. *Let $M : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. Then, the resolvent mapping $J_\lambda^M : H_1 \to H_1$ associated with M, is defined by*

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \ \forall x \in H_1.$$

Remark 2.2.
(i) For all $\lambda > 0$, the resolvent operator J_λ^M is single-valued and firmly nonexpansive.
(ii) If we take $M = \partial I_C$, the subdifferential of the indicator function I_C of C, where I_C is defined by

$$I_C(x) = \begin{cases} 0, & x \in C \\ +\infty, & x \notin C, \end{cases}$$

then

$$y = J_\lambda^{\partial I_C}(x) = (I + \lambda \partial I_C)^{-1} x \Leftrightarrow y = P_C x.$$

(iii) It is easy to see that I_C is a proper and lower semicontinuous convex function on H_1 and the subdifferential ∂I_C of the indicator function I_C is maximal monotone.

Assumption 2.1. Let F and ϕ satisfy the following conditions:

(i) $F(x, y; z) = 0$ if $x = y$ for any $x, y, z \in C$;
(ii) F is generalized relaxed α-monotone, i.e., for any $x, y \in C$ and $t \in (0, 1]$, we have

$$F(y, x; y) - F(y, x; x) \geq \alpha(x, y),$$

where $\alpha : H_1 \times H_1 \to \mathbb{R}$ such that

$$\lim_{t \to 0} \frac{\alpha(x, ty + (1 - t)x)}{t} = 0;$$

(iii) $F(y, x; \cdot)$ is hemicontinuous for any fixed $x, y \in C$;
(iv) $F(\cdot, x; z)$ is convex and lower semicontinuous for any fixed $x, y \in C$;
(v) $F(x, y; z) + F(y, x; z) = 0$ for any $x, y, z \in C$;
(vi) $\phi(\cdot, \cdot)$ is weakly continuous and $\phi(\cdot, y)$ is convex for any fixed $y \in C$;
(vii) ϕ is skew-symmetric, i.e., $\phi(x, x) - \phi(x, y) + \phi(y, y) - \phi(y, x) \geq 0, \ \forall x, y \in C$.

For a given $r \geq 0$, define a mapping $T_r^F : H_1 \to C$ as follows:

$$T_r^F(x) = \left\{ z \in C : F(y,z;z) + \frac{1}{r}\langle y-z, z-x\rangle + \phi(z,y) - \phi(z,z) \geq 0, \forall y \in C \right\}, \quad \forall x \in H_1. \tag{2.2}$$

The following lemma is a special case of Lemma 3.1-3.3 due to [15] in real Hilbert space.

Lemma 2.3. [15] *Assume that $F : C \times C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ satisfy Assumption 2.1. Suppose the mapping $T_r^F : H_1 \to C$ be defined as in (2.2). Then the following holds:*

(i) $T_r^F(x) \neq \emptyset$ *for each* $x \in H_1$;
(ii) T_r^F *is single valued;*
(iii) T_r^F *is firmly nonexpansive, i.e.,*

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x-y\rangle, \ \forall x,y \in H_1;$$

(iv) $\mathrm{Fix}(T_r^F) = \mathrm{Sol}(\mathrm{GGVLIP}(1.6))$;
(v) $\mathrm{Sol}(\mathrm{GGVLIP}(1.6))$ *is closed and convex.*

Assume that $G : Q \times Q \times Q \to \mathbb{R}$, $\psi : Q \times Q \to \mathbb{R}$ satisfy Assumption 2.1. For $s \geq 0$ and $u \in H_2$, define a mapping $T_s^G : H_2 \to Q$ as follows

$$T_s^G u = \left\{ v \in Q : G(w,v;v) + \psi(w,v) - \psi(v,v) + \frac{1}{s}\langle w-v, v-u\rangle \geq 0, \forall w \in Q \right\}. \tag{2.3}$$

Then it follows from Lemma 2.3 that T_s^G satisfies (i)-(v) of Lemma 2.3, and

$$\mathrm{Fix}(T_s^G) = \mathrm{Sol}(\mathrm{GGVLIP}(1.7)).$$

Definition 2.6. *Let H_1 be a real Hilbert space. A mapping $S : H_1 \to H_1$ is said to be:*

(i) *demiclosed at origin if, for any sequence $\{x_n\} \subset H_1$ with $x_n \rightharpoonup \bar{x}$ and if the sequence $\{Sx_n\}$ strongly converges to x^*, we have $S\bar{x} = x^*$;*
(ii) *semi-compact if, for any bounded sequence $\{x_n\} \subset H_1$ with $\|x_n - Sx_n\| \to 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to a point $\bar{x} \in H_1$;*
(iii) *weakly continuous at x if for any sequence $\{x_n\}$ which converges weakly to x, the sequence $\{Sx_n\}$ converges weakly to Sx.*

Lemma 2.4. [17]

(i) *For all $x, y \in H_1$, we have*

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle; \tag{2.4}$$

(ii) *For any $x, y \in H_1$, we have*

$$2\langle x,y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2, \forall x,y \in H_1. \tag{2.5}$$

Lemma 2.5. [24] *(Opial's lemma) Let H_1 be a Hilbert space and $\{\mu_n\}$ be a sequence in H_1 such that there exists a nonempty set $W \subset H_1$ satisfying:*

(i) *For every* $\mu^* \in W$, $\lim_{n\to\infty} \|\mu_n - \mu^*\|$ *exists.*

(ii) *Any weak-cluster point of the sequence* $\{\mu_n\}$ *belongs to* W;

Then there exists $\mu^* \in W$ *such that* $\{\mu_n\}$ *weakly converges to* μ^*.

3. Main results

We prove a strong convergence theorem to approximate a common solution to $\mathrm{S_pEMVIP}$(1.9)-(1.10), $\mathrm{S_pEGGVLIP}$(1.6)-(1.7) and $\mathrm{S_pEFPP}$(1.3) for quasi-nonexpansive mappings by selecting the step size in such a way that the implementation of the algorithm does not require the calculation or estimation of the operator norms.

Theorem 3.1. *Let* H_1, H_2 *and* H_3 *be real Hilbert spaces,* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *be nonempty closed and convex sets. Assume that* $F : C\times C\times C \to \mathbb{R}$, $G : Q\times Q\times Q \to \mathbb{R}$ *are trifunctions and* $\phi : C \times C \to \mathbb{R}$, $\psi : Q \times Q \to \mathbb{R}$ *are bifunctions satisfying Assumption* 2.1 *with* $F(x, \cdot; x)$ *and* $G(y, \cdot; y)$ *are weakly continuous, and let* $A : H_1 \to H_3$, $B : H_2 \to H_3$ *be two bounded linear operators. Let* $U : C \to H_1$ *be an* σ*-inverse strongly monotone mapping and let* $M : H_1 \to 2^{H_1}$ *be a maximal monotone mapping. Let* $V : Q \to H_2$ *be an* β*-inverse strongly monotone mapping and let* $N : H_2 \to 2^{H_2}$ *be a maximal monotone mapping. Let* $(x_1, y_1) \in C \times Q$ *be given and the iteration sequence* $\{(x_n, y_n)\}$ *be generated by the scheme:*

$$\begin{cases} F(u,u_n;u_n)+\phi(u,u_n)-\phi(u_n,u_n) \\ \qquad +\dfrac{1}{s_n}\left\langle u-u_n, u_n - J^M_{r_n}(x_n - r_nUx_n)\right\rangle \geq 0,\ \forall u \in C; \\ G(v,v_n;v_n)+\psi(v,v_n)-\psi(v_n,v_n) \\ \qquad +\dfrac{1}{s_n}\left\langle v-v_n, v_n - J^N_{r_n}(y_n - r_nVy_n)\right\rangle \geq 0,\ \forall v \in Q; \\ z_n = P_C(u_n - \gamma_n A^*(Au_n - Bv_n)); \\ x_{n+1} = \alpha_n z_n + (1-\alpha_n)Sz_n; \\ w_n = P_Q(v_n + \gamma_n B^*(Au_n - Bv_n)); \\ y_{n+1} = \alpha_n w_n + (1-\alpha_n)Tw_n, \end{cases} \tag{3.1}$$

where $S : C \to C$ *and* $T : Q \to Q$ *be quasi-nonexpansive mappings and the step size* γ_n *is chosen in such a way that for some* $\epsilon > 0$,

$$\gamma_n \in \left(\epsilon, \frac{2\|Au_n - Bv_n\|^2}{\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2} - \epsilon\right),\ n \in \Lambda \tag{3.2}$$

otherwise $\gamma_n = \gamma$ ($\gamma \geq 0$), *where the index set* $\Lambda = \{n : Au_n - Bv_n \neq 0\}$, $\alpha_n \subset (\delta, 1-\delta)$ *for some small enough* $\delta > 0$ *and* $\{r_n\}$, $\{s_n\} \subset (0, \infty)$. *Assume that the control sequences* $\{r_n\}$ *and* $\{s_n\}$ *satisfy the following conditions:*

(i) $0 < r \leq r_n \leq r' < 2\min\{\sigma, \beta\}$;

(ii) $\liminf_{n\to\infty} s_n > 0$;

(iii) $S - I$ *and* $T - I$ *are demiclosed at* 0.

If $\Gamma := \mathrm{Sol}(\mathrm{S_pEMVIP}(1.9)-(1.10)) \cap \mathrm{Sol}(\mathrm{S_pEGGVLIP}(1.6)-(1.7)) \cap \Theta \neq \emptyset$, *then the sequence* $\{(x_n, y_n)\}$ *converges weakly to a point* $(\bar{x}, \bar{y})$ *of* Γ. *In addition if* S *and* T *are semi-compact, then* $\{(x_n, y_n)\}$ *converges strongly to the point* $(\bar{x}, \bar{y})$ *of* Γ.

Proof. Since the mappings $U : C \to H_1$ and $V : Q \to H_2$ are σ-inverse strongly monotone and β-inverse strongly monotone mapping, respectively, and $r_n \leq r' < 2\min\{\sigma, \beta\}$, then we can easily show that $(I - r_nU)$ and $(I - r_nV)$ are nonexpansive. Hence $J^M_{r_n}(I - r_nU)$ and $J^N_{r_n}(I - r_nV)$ are nonexpansive. Since $\Gamma \neq \emptyset$, it follows from Lemma 2.1 that $\mathrm{Fix}(J^M_{r_n}(I - r_nU)) = (U + M)^{-1}(0)$ and $\mathrm{Fix}(J^N_{r_n}(I - r_nV)) = (V + N)^{-1}(0)$ are closed and convex sets. Further, it follows from Lemma 2.3 that $T^F_{s_n}$ and $T^G_{s_n}$ are nonexpansive and hence $\mathrm{Fix}(T^F_{s_n})$ and $\mathrm{Fix}(T^G_{s_n})$ are closed and convex sets. Thus Γ is nonempty closed and convex. Let $(x, y) \in \Gamma$, it follows from Lemma 2.3 that $x = T^F_{s_n}x$ and $y = T^G_{s_n}y$. Also, we observe that $x = J^M_{r_n}(I - r_nU)x$ and $y = J^N_{r_n}(I - r_nV)y$. Since $T^F_{s_n}t_n$, where $t_n = J^M_{r_n}(I - r_nU)$, is nonexpansive, we have

$$\begin{aligned}\|u_n - x\| &= \|T^F_{s_n}J^M_{r_n}(x_n - r_nUx_n) - T^F_{s_n}J^M_{r_n}(I - r_nU)x\| \\ &\leq \|x_n - x\|. \end{aligned} \tag{3.3}$$

Similarly, we obtain

$$\|v_n - y\| \leq \|y_n - y\|. \tag{3.4}$$

Since $(x, y) \in \Gamma$, then $x \in C$ and hence $P_Cx = x$. Now, we estimate

$$\begin{aligned}\|z_n - x\|^2 &= \|P_C(u_n - \gamma_nA^*(Au_n - Bv_n)) - P_Cx\|^2 \\ &\leq \|u_n - \gamma_nA^*(Au_n - Bv_n) - x\|^2 \\ &\leq \|u_n - x\|^2 - 2\gamma_n\langle u_n - x, A^*(Au_n - Bv_n)\rangle + \gamma_n^2\|A^*(Au_n - Bv_n)\|^2 \\ &\leq \|u_n - x\|^2 - 2\gamma_n\langle Au_n - Ax, Au_n - Bv_n\rangle + \gamma_n^2\|A^*(Au_n - Bv_n)\|^2 \qquad (3.5) \\ &\leq \|u_n - x\|^2 + 2\gamma_n\|Au_n - Ax\|\|Au_n - Bv_n\| + \gamma_n^2\|A^*(Au_n - Bv_n)\|^2. \qquad (3.6)\end{aligned}$$

Now, using (2.5) in (3.5), we get

$$\begin{aligned}\|z_n - x\|^2 \leq{}& \|u_n - x\|^2 - \gamma_n\|Au_n - Ax\|^2 - \gamma_n\|Au_n - Bv_n\|^2 + \gamma_n\|Bv_n - Ax\|^2 \\ &+ \gamma_n^2\|A^*(Au_n - Bv_n)\|^2. \end{aligned} \tag{3.7}$$

By similar step as in (3.7), we obtain

$$\begin{aligned}\|w_n - y\|^2 \leq{}& \|v_n - y\|^2 - \gamma_n\|Bv_n - By\|^2 - \gamma_n\|Au_n - Bv_n\|^2 + \gamma_n\|Au_n - By\|^2 \\ &+ \gamma_n^2\|B^*(Au_n - Bv_n)\|^2. \end{aligned} \tag{3.8}$$

Adding (3.7) and (3.8), and using the fact that $Ax = By$, we get

$$\begin{aligned}\|z_n - x\|^2 + \|w_n - y\|^2 \leq{}& \|u_n - x\|^2 + \|v_n - y\|^2 - \gamma_n[2\|Au_n - Bv_n\|^2 \\ &- \gamma_n(\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2)]. \end{aligned} \tag{3.9}$$

Now, from assumption on γ_n, we get

$$\|z_n - x\|^2 + \|w_n - y\|^2 \leq \|u_n - x\|^2 + \|v_n - y\|^2. \tag{3.10}$$

Since S and T are quasi-nonexpansive mappings, it follows from Lemma 2.2(ii) that

$$\begin{aligned}\|x_{n+1}-x\|^2 &= \|\alpha_n z_n+(1-\alpha_n)S(z_n)-x\|^2\\ &\leq \|z_n-x\|^2-\alpha_n(1-\alpha_n)\|S(z_n)-z_n\|^2. \qquad (3.11)\end{aligned}$$

Similarly, we obtain

$$\|y_{n+1}-y\|^2 \leq \|w_n-y\|^2-\alpha_n(1-\alpha_n)\|T(w_n)-w_n\|^2. \qquad (3.12)$$

Adding (3.11) and (3.12), we get

$$\begin{aligned}\|x_{n+1}-x\|^2+\|y_{n+1}-y\|^2 &\leq \|z_n-x\|^2+\|w_n-y\|^2\\ &\quad -\alpha_n(1-\alpha_n)(\|S(z_n)-z_n\|^2+\|T(w_n)-w_n\|^2).\end{aligned}$$

Using (3.3), (3.4) and (3.9) in above inequalities, we get

$$\begin{aligned}\|x_{n+1}-x\|^2+\|y_{n+1}-y\|^2 &\leq \|x_n-x\|^2+\|y_n-y\|^2-\gamma_n[2\|Au_n-Bv_n\|^2\\ &\quad -\gamma_n(\|A^*(Au_n-Bv_n)\|^2+\|B^*(Au_n-Bv_n)\|^2)]\\ &\quad -\alpha_n(1-\alpha_n)(\|S(z_n)-z_n\|^2+\|T(w_n)-w_n\|^2). \qquad (3.13)\end{aligned}$$

Now, setting $\rho_n(x,y):=\|x_n-x\|^2+\|y_n-y\|^2$ in (3.13), we obtain

$$\begin{aligned}\rho_{n+1}(x,y) &\leq \rho_n(x,y)-\gamma_n[2\|Au_n-Bv_n\|^2\\ &\quad -\gamma_n(\|A^*(Au_n-Bv_n)\|^2+\|B^*(Au_n-Bv_n)\|^2)]\\ &\quad -\alpha_n(1-\alpha_n)(\|S(z_n)-z_n\|^2+\|T(w_n)-w_n\|^2). \qquad (3.14)\end{aligned}$$

From the condition (3.2) on γ_n, we observe that the sequence $\{\rho_n(x,y)\}$ being decreasing and lower bounded by 0, therefore it converges to some finite limit, say $\rho(x,y)$. Thus condition (i) of Lemma 2.5 is satisfied with $\mu_n=(x_n,y_n)$, $\mu^*=(x,y)$ and $W=\Gamma$.

Since $\|x_n-x\|^2\leq\rho_n(x,y)$, $\|y_n-y\|^2\leq\rho_n(x,y)$ and $\lim_{n\to\infty}\rho_n(x,y)$ exists, we observe that $\{x_n\}$ and $\{y_n\}$ are bounded and $\limsup_{n\to\infty}\|x_n-x\|$ and $\limsup_{n\to\infty}\|y_n-y\|$ exist. From (3.3) and (3.4), we have that $\limsup_{n\to\infty}\|u_n-x\|$ and $\limsup_{n\to\infty}\|v_n-y\|$ also exist. Now, let $\bar{x}$ and $\bar{y}$ be weak cluster points of the sequences $\{x_n\}$ and $\{y_n\}$, respectively. From Lemma 2.4(i), we have

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &= \|x_{n+1}-x-x_n+x\|^2\\ &= \|x_{n+1}-x\|^2-\|x_n-x\|^2-2\langle x_{n+1}-x_n,x_n-x\rangle\\ &= \|x_{n+1}-x\|^2-\|x_n-x\|^2-2\langle x_{n+1}-\bar{x},x_n-x\rangle+2\langle x_n-\bar{x},x_n-x\rangle.\end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (3.15)$$

Similarly, we have

$$\limsup_{n\to\infty}\|y_{n+1}-y_n\|=0. \qquad (3.16)$$

Further, it follows from (3.15) and (3.16) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0, \qquad (3.17)$$

and

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.18}$$

For $n \in \Lambda$, again from (3.14), we have

$$\begin{aligned}\rho_{n+1}(x,y) \quad &\leq \quad \rho_n(x,y) - \gamma_n[2\|Au_n - Bv_n\|^2 \\ &\quad -\gamma_n(\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2)].\end{aligned}$$

Since $\lim_{n\to\infty} \rho_n(x,y)$ exists, it follows from condition (3.2) that

$$\lim_{n\to\infty} (\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2) = 0. \tag{3.19}$$

(Note that $Au_n - Bv_n = 0$ if $n \notin \Lambda$). Hence, we obtain

$$\lim_{n\to\infty} \|A^*(Au_n - Bv_n)\| = \lim_{n\to\infty} \|B^*(Au_n - Bv_n)\| = 0. \tag{3.20}$$

Similarly, from assumption $\{\alpha_n\} \subset (\delta, 1-\delta),\ \delta > 0$ and (3.14), we observe that

$$\lim_{n\to\infty} \|z_n - S(z_n)\| = \lim_{n\to\infty} \|w_n - T(w_n)\| = 0. \tag{3.21}$$

Since γ_n is bounded and $\lim_{n\to\infty} \rho_n(x,y)$ exists, it follows from (3.14), (3.20) and (3.21) that

$$\lim_{n\to\infty} \|Au_n - Bv_n\| = 0. \tag{3.22}$$

Now, we estimate

$$\begin{aligned}\|z_n - x\|^2 \quad &= \quad \|P_C(u_n - \gamma_n A^*(Au_n - Bv_n)) - P_C x\|^2 \\ &\leq \quad \langle z_n - x, u_n - \gamma_n A^*(Au_n - Bv_n) - x\rangle \\ &= \quad \frac{1}{2}\Bigg\{ \|z_n - x\|^2 + \|u_n - \gamma_n A^*(Au_n - Bv_n) - x\|^2 \\ &\qquad -\|z_n - u_n + \gamma_n A^*(Au_n - Bv_n)\|^2 \Bigg\}.\end{aligned}$$

This implies that

$$\begin{aligned}\|z_n - x\|^2 &\leq \|u_n - x\|^2 - 2\gamma_n\langle u_n - x, A^*(Au_n - Bv_n)\rangle + \gamma_n^2\|A^*(Au_n - Bv_n)\|^2 \\ &\quad - \|z_n - u_n\|^2 - \gamma_n^2\|A^*(Au_n - Bv_n)\|^2 - 2\gamma_n\langle z_n - u_n, A^*(Au_n - Bv_n)\rangle \\ &\leq \|u_n - x\|^2 + 2\gamma_n\|Au_n - Ax\|\|Au_n - Bv_n\| - \|z_n - u_n\|^2 \\ &\quad + 2\gamma_n\|Az_n - Au_n\|\|Au_n - Bv_n\|.\end{aligned}$$

Using (3.3) and above inequality in (3.11), we get

$$\begin{aligned}\|x_{n+1} - x\|^2 \quad &\leq \quad \|x_n - x\|^2 + 2\gamma_n(\|Au_n - Ax\| + \|Az_n - Au_n\|)\|Au_n - Bv_n\| \\ &\quad -\|z_n - u_n\|^2 - \alpha_n(1-\alpha_n)\|S(z_n) - z_n\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\|z_n - u_n\|^2 \quad &\leq \quad (\|x_n - x\| + \|x_{n+1} - x\|)\|x_n - x_{n+1}\| \\ &\quad +2\gamma_n(\|Au_n - Ax\| + \|Az_n - Au_n\|)\|Au_n - Bv_n\| \\ &\quad -\alpha_n(1-\alpha_n)\|S(z_n) - z_n\|^2.\end{aligned} \tag{3.23}$$

Using (3.17), (3.21) and (3.22) in (3.23), we get

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{3.24}$$

Similarly, we get

$$\lim_{n\to\infty} \|w_n - v_n\| = 0. \tag{3.25}$$

Since $J^M_{r_n}$ is firmly nonexpansive, we find that

$$\begin{aligned}
\|t_n - x\|^2 &\leq \langle (x_n - r_nUx_n) - (x - r_nUx), t_n - x\rangle \\
&= \frac{1}{2}\{\|(x_n - r_nUx_n) - (x - r_nUx)\|^2 + \|t_n - x\|^2 \\
&\quad -\|(x_n - r_nUx_n) - (x - r_nUx) - (t_n - x)\|^2\} \\
&\leq \frac{1}{2}\{\|x_n - x\|^2 + \|r_n(Ux_n - Ux)\|^2 - 2r_n\sigma\|Ux_n - Ux\|^2 + \|t_n - x\|^2 \\
&\quad -\|x_n - t_n - r_n(Ux_n - Ux)\|^2\} \\
&\leq \frac{1}{2}\{\|x_n - x\|^2 + \|r_n(Ux_n - Ux)\|^2 - 2r_n\sigma\|Ux_n - Ux\|^2 + \|t_n - x\|^2 \\
&\quad -\|x_n - t_n\|^2 - \|r_n(Ux_n - Ux)\|^2 + 2\|x_n - t_n\|\|r_n(Ux_n - Ux)\|\}.
\end{aligned}$$

It follows that

$$\|t_n - x\|^2 \leq \|x_n - x\|^2 + 2r_n\|x_n - t_n\|\|Ux_n - Ux\| - \|x_n - t_n\|^2. \tag{3.26}$$

Since $T^F_{s_n}$ is nonexpansive and $u_n = T^F_{s_n}t_n$ and $x = T^F_{s_n}x$, then we have

$$\|u_n - x\| \leq \|t_n - x\|.$$

Using (3.6) and above relation in (3.11), we get

$$\begin{aligned}
\|x_{n+1} - x\|^2 &\leq \|u_n - x\|^2 + 2\gamma_n\|Au_n - Ax\|\|Au_n - Bv_n\| + \gamma_n^2\|A^*(Au_n - Bv_n)\|^2 \\
&\leq \|t_n - x\|^2 + 2\gamma_n\|Au_n - Ax\|\|Au_n - Bv_n\| \\
&\quad + \gamma_n^2\|A^*(Au_n - Bv_n)\|^2.
\end{aligned} \tag{3.27}$$

Using (3.26) in (3.27), we have

$$\begin{aligned}
\|x_{n+1} - x\|^2 &\leq \|x_n - x\|^2 + 2r_n\|x_n - t_n\|\|Ux_n - Ux\| - \|x_n - t_n\|^2 \\
&\quad +2\gamma_n\|Au_n - Ax\|\|Au_n - Bv_n\| + \gamma_n^2\|A^*(Au_n - Bv_n)\|^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_n - t_n\|^2 &\leq (\|x_n - x\| + \|x_{n+1} - x\|)\|x_n - x_{n+1}\| + 2r_n\|x_n - t_n\|\|Ux_n - Ux\| \\
&\quad +2\gamma_n\|Au_n - Ax\|\|Au_n - Bv_n\| + \gamma_n^2\|A^*(Au_n - Bv_n)\|^2.
\end{aligned} \tag{3.28}$$

Again, since $t_n = J^M_{r_n}(x_n - r_nUx_n)$, we have

$$\begin{aligned}
\|t_n - x\|^2 &= \|J^M_{r_n}(x_n - r_nUx_n) - J^M_{r_n}(I - r_nU)x\|^2 \\
&\leq \|(x_n - r_nUx_n) - (x - r_nUx)\|^2 \\
&\leq \|(x_n - x) - r_n(Ux_n - Ux)\|^2 \\
&\leq \|x_n - x\|^2 - r_n(2\sigma - r_n)\|Ux_n - Ux\|^2.
\end{aligned} \tag{3.29}$$

Using (3.29) in (3.27), we have

$$\begin{aligned}\|x_{n+1}-x\|^2 &\leq \|x_n-x\|^2-r_n(2\sigma-r_n)\|Ux_n-Ux\|^2+\gamma_n^2\|A^*(Au_n-Bv_n)\|^2\\ &\quad +2\gamma_n\|Au_n-Ax\|\|Au_n-Bv_n\|, \qquad (3.30)\end{aligned}$$

which can be written as

$$\begin{aligned}r_n(2\sigma-r_n)\|Ux_n-Ux\|^2 &\leq (\|x_n-x\|+\|x_{n+1}-x\|)\|x_n-x_{n+1}\|\\ &\quad +2\gamma_n\|Au_n-Ax\|\|Au_n-Bv_n\|\\ &\quad +\gamma_n^2\|A^*(Au_n-Bv_n)\|^2. \qquad (3.31)\end{aligned}$$

Taking $n\to\infty$ and condition (i), using (3.17), (3.20) and (3.22) in (3.31), we have

$$\lim_{n\to\infty}\|Ux_n-Ux\|=0. \qquad (3.32)$$

Again, taking $n\to\infty$, using (3.17), (3.20) (3.22) and (3.32) in (3.28), we get

$$\lim_{n\to\infty}\|x_n-t_n\|=0. \qquad (3.33)$$

Similarly, we get

$$\lim_{n\to\infty}\|Vy_n-Vy\|=0 \qquad (3.34)$$

and

$$\lim_{n\to\infty}\|y_n-t_n^{'}\|=0, \qquad (3.35)$$

where $t_n^{'}=J_{r_n}^N(y_n-r_nVy_n)$. Since $T_{s_n}^F$ is a firmly nonexpansive, therefore

$$\begin{aligned}\|u_n-x\|^2 &= \|T_{s_n}^F t_n-x\|^2\\ &\leq \langle t_n-x,u_n-x\rangle\\ &= \frac{1}{2}(\|t_n-x\|^2+\|u_n-x\|^2-\|u_n-t_n\|^2),\end{aligned}$$

i.e.,

$$\|u_n-x\|^2\leq\|t_n-x\|^2-\|u_n-t_n\|^2. \qquad (3.36)$$

$$\|u_n-x\|^2\leq\|x_n-x\|^2-r_n(2\sigma-r_n)\|Ux_n-Ux\|^2-\|u_n-t_n\|^2. \qquad (3.37)$$

Similarly, we can find

$$\|v_n-y\|^2\leq\|y_n-y\|^2-r_n(2\beta-r_n)\|Vy_n-Vy\|^2-\|v_n-t_n^{'}\|^2.$$

Using (3.6), (3.37) in (3.11), we get

$$\begin{aligned}\|x_{n+1}-x\|^2 &\leq \|u_n-x\|^2+2\gamma_n\|Au_n-Ax\|\|Au_n-Bv_n\|+\gamma_n^2\|A^*(Au_n-Bv_n)\|^2\\ &\leq \|x_n-x\|^2-r_n(2\sigma-r_n)\|Ux_n-Ux\|^2-\|u_n-t_n\|^2\\ &\quad +\gamma_n^2\|A^*(Au_n-Bv_n)\|^2+2\gamma_n\|Au_n-Ax\|\|Au_n-Bv_n\|,\end{aligned}$$

which can be written as

$$\begin{aligned}\|u_n-t_n\|^2 &\leq (\|x_n-x\|+\|x_{n+1}-x\|)\|x_n-x_{n+1}\|-r_n(2\sigma-r_n)\|Ux_n-Ux\|^2\\ &\quad +\gamma_n^2\|A^*(Au_n-Bv_n)\|^2+2\gamma_n\|Au_n-Ax\|\|Au_n-Bv_n\|.\end{aligned}$$

Now, using (3.20), (3.22), (3.17) and (3.32) in above inequality, we get

$$\lim_{n\to\infty}\|u_n-t_n\|=0. \qquad (3.38)$$

Now,

$$\|u_n - x_n\| \leq \|u_n - t_n\| + \|t_n - x_n\|.$$

Using (3.33) and (3.38), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.39}$$

Again, since

$$\|z_n - x_n\| \leq \|z_n - u_n\| + \|u_n - x_n\|.$$

Using (3.24) and (3.39), we get

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.40}$$

Similarly, we can also obtain

$$\lim_{n\to\infty} \|v_n - t_n^{'}\| = 0, \tag{3.41}$$

and

$$\lim_{n\to\infty} \|v_n - y_n\| = 0, \tag{3.42}$$

$$\lim_{n\to\infty} \|w_n - v_n\| = 0, \tag{3.43}$$

$$\lim_{n\to\infty} \|w_n - y_n\| = 0. \tag{3.44}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x}$ and hence it follows from (3.40) that there is a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup \bar{x}$. Further, demiclosedness of $S-I$ at 0 and (3.21) imply that $\bar{x} \in \text{Fix}(S)$. Also, it follows from boundedness of $\{y_n\}$ and (3.44) that there exist subsequences $\{y_{n_i}\}$ of $\{y_n\}$ and $\{w_{n_i}\}$ of $\{w_n\}$ such that $y_{n_i} \rightharpoonup \bar{y}$ and $w_{n_i} \rightharpoonup \bar{y}$ and hence demiclosedness of $T-I$ at 0 and (3.21) yield that $\bar{y} \in \text{Fix}(T)$. Since every Hilbert space satisfies Opial's condition which ensures that the weakly subsequential limit of $\{(x_n, y_n)\}$ is unique. Since $\{x_n\}$ and $\{u_n\}$ both have the same asymptotic behaviour, then there is a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup \bar{x}$.

Now, we show that $\bar{x} \in \text{Sol(GGVLIP(1.6))}$ and $\bar{y} \in \text{Sol(GGVLIP(1.7))}$.
Since $u_n = T^F_{s_n} t_n$, where $t_n = J^M_{r_n}(x_n - r_n U x_n)$, we have

$$F(u, u_n; u_n) + \phi(u, u_n) - \phi(u_n, u_n) + \frac{1}{s_n}\langle u - u_n, u_n - t_n\rangle \geq 0, \ \forall u \in C.$$

It follows from generalized relaxed α-monotonicity of F, above inequality implies that

$$\phi(u, u_{n_i}) - \phi(u_{n_i}, u_{n_i}) + \langle u - u_{n_i}, \frac{u_{n_i} - t_{n_i}}{s_{n_i}}\rangle \geq -F(u, u_{n_i}; u) + \alpha(u_{n_i}, u), \ \forall u \in C. \tag{3.45}$$

Since $\liminf\limits_{n\to\infty} s_n > 0$, then there exists a real number $s > 0$ such that $s_n \geq s$, $\forall\ n$ and hence we have

$$\frac{\|u_{n_i} - t_{n_i}\|}{s_{n_i}} \leq \frac{\|u_{n_i} - t_{n_i}\|}{s}.$$

It follows from (3.38) that $\lim\limits_{i\to\infty} \|u_{n_i} - t_{n_i}\| = 0$ and hence

$$\lim_{i\to\infty} \frac{\|u_{n_i} - t_{n_i}\|}{s_{n_i}} \leq \frac{1}{s} \lim_{i\to\infty} \|u_{n_i} - t_{n_i}\| = 0.$$

Since α is lower semicontinuous in the first argument, ϕ is weakly continuous and $F(u,\cdot;u)$ is weakly continuous then on taking $n\to\infty$ in (3.45), we get

$$\alpha(\bar{x},u)-F(u,\bar{x};u)-\phi(u,\bar{x})+\phi(\bar{x},\bar{x})\leq 0,\ \forall u\in C. \tag{3.46}$$

For t with $0<t\leq 1$ and $u\in C$, set $u_t=tu+(1-t)\bar{x}$. Since C is convex set, $u_t\in C$, then from (3.46), we have

$$\alpha(\bar{x},u_t)-F(u_t,\bar{x};u_t)-\phi(u_t,\bar{x})+\phi(\bar{x},\bar{x})\leq 0, \tag{3.47}$$

which implies that

$$\begin{aligned}\alpha(\bar{x},u_t) &\leq F(u_t,\bar{x};u_t)-\phi(\bar{x},\bar{x})+\phi(u_t,\bar{x})\\ &\leq tF(u,\bar{x};u_t)+(1-t)F(\bar{x},\bar{x};u_t)-\phi(\bar{x},\bar{x})+t\phi(u,\bar{x})+(1-t)\phi(\bar{x},\bar{x})\\ &\leq t[F(u,\bar{x};u_t)+\phi(u,\bar{x})-\phi(\bar{x},\bar{x})]. \end{aligned}\tag{3.48}$$

Since $F(u,\bar{x};\cdot)$ is hemicontinuous and letting $t\to 0$, we have

$$\lim_{t\to 0}\{F(u,\bar{x};u_t)+\phi(u,\bar{x})-\phi(\bar{x},\bar{x})\}\geq \lim_{t\to 0}\frac{\alpha(\bar{x},u_t)}{t}, \tag{3.49}$$

which implies

$$F(u,\bar{x};\bar{x})+\phi(u,\bar{x})-\phi(\bar{x},\bar{x})\geq 0. \tag{3.50}$$

This implies that $\bar{x}\in \mathrm{Sol}(\mathrm{GGVLIP}(1.6))$. Following a similar argument as the proof of above, we have $\bar{y}\in \mathrm{Sol}(\mathrm{GGVLIPP}(1.7))$.

Next, we show that $(\bar{x},\bar{y})\in \mathrm{Sol}(\mathrm{S_pEMVIP}(1.9)-(1.10))$. Since

$$t_{n_i}=J^M_{r_{n_i}}(x_{n_i}-r_{n_i}Ux_{n_i})$$

can be written as

$$\frac{x_{n_i}-t_{n_i}}{r_{n_i}}-Ux_{n_i}\in Mt_{n_i}.$$

Let $\mu\in Mv$. Since M is monotone, we have

$$\left\langle \frac{x_{n_i}-t_{n_i}}{r_{n_i}}-Ux_{n_i}-\mu, t_{n_i}-v\right\rangle\geq 0.$$

It follows from (3.33) and condition (i) that $\langle -U\bar{x}-\mu,\bar{x}-v\rangle\geq 0$. This implies that $-U\bar{x}\in M\bar{x}$, that is, $\bar{x}\in (U+M)^{-1}(0)$. Similarly, $\bar{y}\in (V+N)^{-1}(0)$.

Since $\|\cdot\|^2$ is weakly lower semicontinuous, we have

$$\|A\bar{x}-B\bar{y}\|^2\leq \lim_{n\to\infty}\inf\|Au_n-Bv_n\|^2=0, \tag{3.51}$$

i.e., $A\bar{x}=B\bar{y}$. Thus, $(\bar{x},\bar{y})\in\Gamma$ and hence $w_w(x_{n_i},y_{n_i})\subset\Gamma$. Now, it follows from Lemma 2.5 that the sequence $\{(x_n,y_n)\}$ generated by iterative algorithm (3.1) converges weakly to $(\bar{x},\bar{y})\in\Gamma$.

Further, since S and T are semi-compact, $\{x_n\}$ and $\{y_n\}$ are bounded, and $S-I$ and $T-I$ are demiclosed at 0 then there exist subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{y_{n_i}\}$ of $\{y_n\}$ such that $\{x_{n_i}\}$ and $\{y_{n_i}\}$ converge strongly to some $\bar{u}\in H_1$ and $\bar{v}\in H_2$, respectively. Since $\{x_{n_i}\}$ and $\{y_{n_i}\}$ converge weakly to $\bar{x}$ and $\bar{y}$, respectively then we have $\bar{u}=\bar{x}$, $\bar{v}=\bar{y}$, $\bar{x}\in \mathrm{Fix}(S)$ and $\bar{y}\in\mathrm{Fix}(T)$. Finally, using the same argument as the proof of above, we have $\bar{x}\in \mathrm{Sol}(\mathrm{GGVLIP}(1.6))$ and $\bar{y}\in\mathrm{Sol}(\mathrm{GGVLIP}(1.7))$,

$\bar{x} \in \mathrm{Sol(MVIP(1.9))}$ and $\bar{y} \in \mathrm{Sol(MVIP(1.10))}$. Since $Au_{n_i} - Bv_{n_i} \to A\bar{x} - B\bar{y}$, we have

$$\|A\bar{x} - B\bar{y}\|^2 \leq \lim_{i\to\infty} \inf \|Au_{n_i} - Bv_{n_i}\|^2 = 0, \tag{3.52}$$

which implies $A\bar{x} = B\bar{y}$ and hence $(\bar{x}, \bar{y}) \in \Gamma$.
On the other hand, since $\rho_n(x,y) = \|x_n - x\|^2 + \|y_n - y\|^2$, for any $(x,y) \in \Gamma$ then $\lim_{i\to\infty} \rho_{n_i}(\bar{x},\bar{y}) = 0$. Further, since $\lim_{n\to\infty} \rho_n(\bar{x},\bar{y})$ exists then $\lim_{n\to\infty} \rho_n(\bar{x},\bar{y}) = 0$ and hence $\lim_{n\to\infty} \|x_n - \bar{x}\| = 0$ and $\lim_{n\to\infty} \|y_n - \bar{y}\| = 0$. Thus $\{(x_n, y_n)\}$ converges strongly to $(\bar{x},\bar{y}) \in \Gamma$. This completes the proof. □

4. Consequences

We now give some consequences of Theorem 3.1. First, we have the following convergence result to approximate a common solution of $\mathrm{S_pENPP}$(1.11)-(1.12), $\mathrm{S_pEGGVLIP}$(1.6)-(1.7) and $\mathrm{S_pEFPP}$(1.3).

Corollary 4.1. *Let H_1, H_2 and H_3 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed and convex sets. Assume that $F : C\times C\times C \to \mathbb{R}$, $G : Q\times Q\times Q \to \mathbb{R}$ are trifunctions and $\phi : C \times C \to \mathbb{R}$, $\psi : Q \times Q \to \mathbb{R}$ are bifunctions satisfying Assumption 2.1 with $F(x,\cdot;x)$ and $G(y,\cdot;y)$ are weakly continuous, and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Let $M : H_1 \to 2^{H_1}$, $N : H_2 \to 2^{H_2}$ be a maximal monotone mappings. Let $(x_1, y_1) \in C \times Q$ be given and the iteration sequence $\{(x_n, y_n)\}$ be generated by the scheme:*

$$\begin{cases} F(u,u_n;u_n) + \phi(u,u_n) - \phi(u_n,u_n) + \dfrac{1}{s_n}\left\langle u - u_n, u_n - J^M_{r_n} x_n\right\rangle \geq 0, \ \forall u \in C; \\ G(v,v_n;v_n) + \psi(v,v_n) - \psi(v_n,v_n) + \dfrac{1}{s_n}\left\langle v - v_n, v_n - J^N_{r_n} y_n\right\rangle \geq 0, \ \forall v \in Q; \\ z_n = P_C(u_n - \gamma_n A^*(Au_n - Bv_n)); \\ x_{n+1} = \alpha_n z_n + (1-\alpha_n) S z_n; \\ w_n = P_Q(v_n + \gamma_n B^*(Au_n - Bv_n)); \\ y_{n+1} = \alpha_n w_n + (1-\alpha_n) T w_n, \end{cases} \tag{4.1}$$

where $S : C \to C$ and $T : Q \to Q$ be quasi-nonexpansive mappings and the step size γ_n is chosen in such a way that for some $\epsilon > 0$,

$$\gamma_n \in \left(\epsilon, \frac{2\|Au_n - Bv_n\|^2}{\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2} - \epsilon\right), \ n \in \Lambda \tag{4.2}$$

otherwise $\gamma_n = \gamma$ ($\gamma \geq 0$), where the index set $\Lambda = \{n : Au_n - Bv_n \neq 0\}$, $\alpha_n \subset (\delta, 1-\delta)$ for some small enough $\delta > 0$ and $\{r_n\}$, $\{s_n\} \subset (0,\infty)$. Assume that the control sequences $\{r_n\}$ and $\{s_n\}$ satisfy the following conditions:

(i) $\liminf_{n\to\infty} r_n > 0$, $\liminf_{n\to\infty} s_n > 0$;
(ii) *$S - I$ and $T - I$ are demiclosed at 0.*

If $\Gamma := \mathrm{Sol}(\mathrm{S_pENPP}(1.11) - (1.12)) \cap \mathrm{Sol}(\mathrm{S_pEGGVLIP}(1.6) - (1.7)) \cap \Theta \neq \emptyset$, then the sequence $\{(x_n, y_n)\}$ converges weakly to a point $(\bar{x},\bar{y})$ of Γ. In addition if S and T are semi-compact, then $\{(x_n, y_n)\}$ converges strongly to the point $(\bar{x},\bar{y})$ of Γ.

Proof. Take $U = 0$ and $V = 0$ in Theorem 3.1. □

Further, if we take $M = \partial I_C$ and $N = \partial I_Q$ then $\mathrm{S_pEMVIP}$(1.9)-(1.10) is reduced to the following problem: f ind $\bar{x} \in C$ and $\bar{y} \in Q$ such that

$$\langle U(\bar{x}), x - \bar{x}\rangle \geq 0, \forall x \in C \tag{4.3}$$

$$\langle V(\bar{y}), y - \bar{y}\rangle \geq 0, \forall y \in Q \tag{4.4}$$

$$\text{and } A\bar{x} = B\bar{y}.$$

Problem (4.3)-(4.4) is called the split equality variational inequality problem (in short, $\mathrm{S_pEVIP}$). Solution set of $\mathrm{S_pEVIP}$(4.3)-(4.4) is denoted by Sol($\mathrm{S_pEVIP}$(4.3)-(4.4)). $\mathrm{S_pEVIP}$(4.3)-(4.4) generalizes split variational inequality problem (in short, $\mathrm{S_pVIP}$) studied in [9].

Finally, we have the following convergence result to approximate a common solution of $\mathrm{S_pEVIP}$(4.3)-(4.4), $\mathrm{S_pEGGVLIP}$(1.6)-(1.7) and $\mathrm{S_pEFPP}$(1.3).

Corollary 4.2. *Let H_1, H_2 and H_3 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed and convex sets. Assume that $F : C \times C \times C \to \mathbb{R}$, $G : Q \times Q \times Q \to \mathbb{R}$ are trifunctions and $\phi : C \times C \to \mathbb{R}$, $\psi : Q \times Q \to \mathbb{R}$ are bifunctions satisfying Assumption 2.1 with $F(x, \cdot; x)$ and $G(y, \cdot; y)$ are weakly continuous, and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Let $U : C \to H_1$ be an σ-inverse strongly monotone mapping and $V : Q \to H_2$ be an β-inverse strongly monotone mapping. Let $(x_1, y_1) \in C \times Q$ be given and the iteration sequence $\{(x_n, y_n)\}$ be generated by the scheme:*

$$\begin{cases} F(u, u_n; u_n) + \phi(u, u_n) - \phi(u_n, u_n) \\ \qquad\qquad + \dfrac{1}{s_n}\langle u - u_n, u_n - P_C(x_n - r_n U x_n)\rangle \geq 0, \forall u \in C; \\ G(v, v_n; v_n) + \psi(v, v_n) - \psi(v_n, v_n) \\ \qquad\qquad + \dfrac{1}{s_n}\langle v - v_n, v_n - P_Q(y_n - r_n V y_n)\rangle \geq 0, \forall v \in Q; \\ z_n = P_C(u_n - \gamma_n A^*(Au_n - Bv_n)); \\ x_{n+1} = \alpha_n z_n + (1 - \alpha_n) S z_n; \\ w_n = P_Q(v_n + \gamma_n B^*(Au_n - Bv_n)); \\ y_{n+1} = \alpha_n w_n + (1 - \alpha_n) T w_n, \end{cases} \tag{4.5}$$

where $S : C \to C$ and $T : Q \to Q$ be quasi-nonexpansive mappings and the step size γ_n is chosen in such a way that for some $\epsilon > 0$,

$$\gamma_n \in \left(\epsilon, \frac{2\|Au_n - Bv_n\|^2}{\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2} - \epsilon\right), \ n \in \Lambda \tag{4.6}$$

otherwise $\gamma_n = \gamma$ ($\gamma \geq 0$), where the index set $\Lambda = \{n : Au_n - Bv_n \neq 0\}$, $\alpha_n \subset (\delta, 1-\delta)$ for some small enough $\delta > 0$ and $\{r_n\}$, $\{s_n\} \subset (0, \infty)$. Assume that the control sequences $\{r_n\}$ and $\{s_n\}$ satisfy the following conditions:

(i) $0 < r \leq r_n \leq r' < 2\min\{\sigma, \beta\}$;

(ii) $\liminf_{n\to\infty} s_n > 0$;

(iii) $S - I$ and $T - I$ are demiclosed at 0.

If $\Gamma := \text{Sol}(\text{S}_\text{p}\text{EVIP}(4.3) - (4.4)) \bigcap \text{Sol}(\text{S}_\text{p}\text{EGGVLIP}(1.6) - (1.7)) \bigcap \Theta \neq \emptyset$, *then the sequence* $\{(x_n, y_n)\}$ *converges weakly to a point* $(\bar{x}, \bar{y})$ *of* Γ. *In addition if* S *and* T *are semi-compact, then* $\{(x_n, y_n)\}$ *converges strongly to the point* $(\bar{x}, \bar{y})$ *of* Γ.

Proof. Take $M = \partial I_C$ and $N = \partial I_Q$ in Theorem 3.1. □

Remark 4.1. Further effort is needed to extend the iterative method presented in this paper to the viscosity iterative method to approximate a common solution to S_pEMVIP(1.9) – (1.10), S_pEGGVLIP(1.6) – (1.7) and S_pEFPP(1.3) for quasi-nonexpansive mappings by selecting the step size in such a way that the implementation of the algorithm does not require the calculation or estimation of the operator norms.

5. Numerical example

Now, we give a numerical example which justify Theorem 3.1.

Example 5.1. Let $H_1 = H_2 = H_3 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, $\forall\ x, y \in \mathbb{R}$, and induced usual norm $|\cdot|$. Let $C = [0, +\infty)$ and $Q = (-\infty, 0]$; let $F : C \times C \times C \to \mathbb{R}$ and $G : Q \times Q \times Q \to \mathbb{R}$ be defined by $F(y, x; x) = (x - \frac{5}{2})(y - x)$, with $\alpha(x, y) = (y - x)^2$, $\forall\ x, y \in C$ and $G(w, u : u) = (u + 10)(w - u)$, with $\alpha(u, w) = (w - u)^2\ \forall\ u, w \in Q$; let $\phi : C \times C \to \mathbb{R}$ and $\psi : Q \times Q \to \mathbb{R}$ be defined by $\phi(x, y) = xy$, $\forall x, y \in C$ and $\psi(u, w) = uw$, $\forall u, w \in Q$; let the mappings $U : C \to H_1$ and $V : Q \to H_2$ be defined by $U(x) = 2x-5$, $\forall x \in C$ and $V(y) = y + 25$, $\forall y \in Q$, respectively; let $M, N : \mathbb{R} \to \mathbb{R}$ be defined by $Mx = 2x$, $\forall x \in \mathbb{R}$ and $Ny = 4y$, $\forall y \in \mathbb{R}$; let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be defined by $A(x) = 4x$, $\forall x \in C$, $B(y) = -y$, $\forall y \in Q$ and let the mappings $S : C \to C$ and $T : Q \to Q$ be defined by $Sx = \dfrac{x + 5}{5}$, $\forall x \in C$, $Ty = \dfrac{y^2 + 5}{y - 1}$, $\forall y \in Q$, respectively. If we set $\alpha_n = \dfrac{1}{2}$, $\forall n$, then there are unique sequences $\{x_n\}$, $\{y_n\}$ generated by the iterative schemes:

$$\begin{cases} t_n = J^M_{r_n}(x_n - r_n U x_n);\ u_n = \left(\dfrac{5}{2} + \dfrac{t_n}{s_n}\right)\dfrac{s_n}{2s_n + 1}; \\ t'_n = J^N_{r_n}(y_n - r_n V y_n);\ v_n = \left(\dfrac{t'_n}{s_n} - 10\right)\dfrac{s_n}{2s_n + 1}; \\ z_n = (1 - 16\gamma_n)u_n - 4\gamma_n v_n;\ w_n = -4\gamma_n u_n + (1 - \gamma_n)v_n; \\ x_{n+1} = \dfrac{1}{2} + \dfrac{3}{5}z_n;\ y_{n+1} = w_n + \dfrac{5 + w_n}{2(w_n - 1)}; \end{cases} \tag{5.1}$$

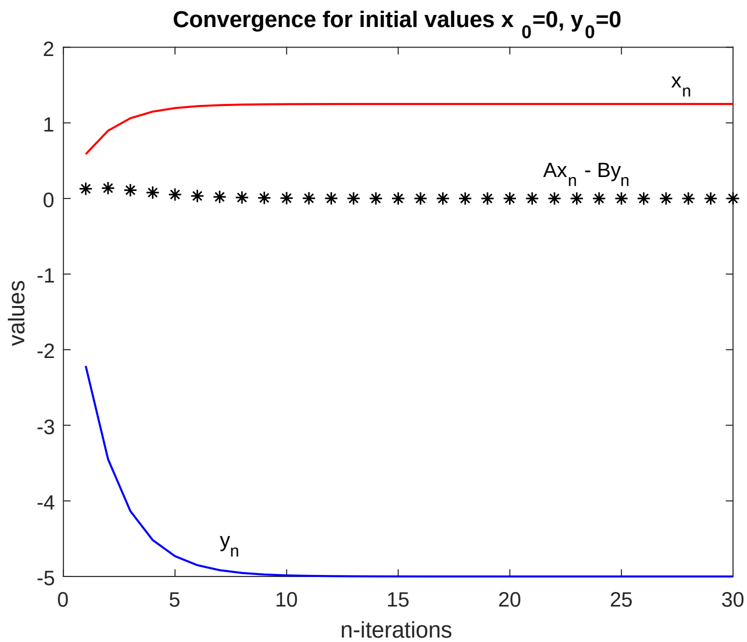
Then the sequence $\{(x_n, y_n)\}$ converges to a point $(\bar{x}, \bar{y}) \in \Gamma$.

Proof. It is easy to prove that the trifunctions F,G and bifunctions ϕ, ψ satisfy Assumption 2.1 and G is upper semicontinuous. A and B are bounded linear operators on $\mathbb{R}$ with adjoint operators A^*, B^* and $\|A\| = \|A^*\| = 4$, $\|B\| = \|B^*\| = 1$ and hence $\gamma_n \in \left(\epsilon, \frac{2}{17} - \epsilon\right)$. Therefore, for $\epsilon = \frac{1}{100}$, we can choose $\gamma_n = \frac{1}{25}$. Further, we observe that U, V are respectively, $\frac{1}{2}$- and 1-inverse strongly monotone mappings. Since $\{r_n\}, \{s_n\} \subset (0, \infty)$ such that $0 < r \leq r_n \leq r' < 2\min\{\sigma, \beta\}$, so we set $r_n = s_n = 0.4$, $\forall n$. Also, we can easily verify that M, N are maximal monotone mappings. Furthermore, we observe that S, T are quasi-nonexpansive mappings with $\mathrm{Fix}(S) = \left\{\frac{5}{4}\right\}$, $\mathrm{Fix}(T) = \{-5\}$ and $(S - I)$, $(T - I)$ are demiclosed at 0. Indeed, if $x_n \to \bar{x}$ and $Sx_n - x_n \to 0$ then by continuity of S, we have $\bar{x} = S\bar{x}$, i.e., $\bar{x} \in \mathrm{Fix}(S) = \left\{\frac{5}{4}\right\}$. Finally, we observe that $\Gamma := \mathrm{Sol}(\mathrm{S_pEMVIP}(1.9) - (1.10)) \cap \mathrm{Sol}(\mathrm{S_pEGGVLIP}$ $(1.6) - (1.7)) \cap \Theta = \left(\frac{5}{4}, -5\right) \neq \emptyset$. After simplification, iterative schemes (5.1) are reduced to the following:

$$\begin{cases} x_{n+1} &= \dfrac{1}{2} + \dfrac{3}{125}\left(\dfrac{5}{9}x_n - \dfrac{2}{3.9}y_n + \dfrac{95}{9} + \dfrac{204}{11.7}\right); \\ y_{n+1} &= \dfrac{4}{25}\left(\dfrac{-5}{81}x_n + \dfrac{9}{11.7}y_n - \dfrac{306}{11.7} + \dfrac{95}{81}\right) \\ & \quad + \dfrac{5 + \frac{4}{25}\left(\frac{-5}{81}x_n + \frac{9}{11.7}y_n - \frac{306}{11.7} + \frac{95}{81}\right)}{2\left(\frac{4}{25}\left(\frac{-5}{81}x_n + \frac{9}{11.7}y_n - \frac{306}{11.7} + \frac{95}{81}\right) - 1\right)}; \end{cases} \tag{5.2}$$

Next, using the software Matlab 7.8.0, we have following table and figure which shows that $\{(x_n, y_n)\}$ converges to a point $(\bar{x}, \bar{y}) = \left(\frac{5}{4}, -5\right)$.

No. of iterations	x_n $x_1 = 0$	y_n $y_1 = 0$	$Ax_n - By_n$		**No. of iterations**	x_n	y_n	$Ax_n - By_n$
1	1.171795	-4.430693	0.128243		**16**	1.249989	-4.999912	0.000173
2	1.206873	-4.681621	0.137057		**17**	1.249994	-4.999951	0.000099
3	1.226190	-4.822298	0.109758		**18**	1.249997	-4.999973	0.000056
4	1.236842	-4.900912	0.078108		**19**	1.249998	-4.999985	0.000032
5	1.242724	-4.944775	0.052114		**20**	1.249999	-4.999991	0.000018
6	1.245974	-4.969229	0.033389		**21**	1.249999	-4.999995	0.000010
7	1.247771	-4.982858	0.020807		**22**	1.250000	-4.999997	0.000006
8	1.248765	-4.990451	0.012708		**23**	1.250000	-4.999999	0.000003
9	1.249315	-4.994681	0.007644		**24**	1.250000	-4.999999	0.000002
10	1.249620	-4.997037	0.004545		**25**	1.250000	-5.000000	0.000001
11	1.249789	-4.998350	0.002676		**26**	1.250000	-5.000000	0.000001
12	1.249883	-4.999081	0.001564		**27**	1.250000	-5.000000	0.000000
13	1.249935	-4.999488	0.000908		**28**	1.250000	-5.000000	0.000000
14	1.249964	-4.999715	0.000525		**29**	1.250000	-5.000000	0.000000
15	1.249980	-4.999841	0.000302		**30**	1.250000	-5.000000	0.000000



This completes the proof. □

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References

[1] A. Aleyner, S. Reich, *Block iterative algorithms for solving convex feasibility problems in Hilbert and in Banach*, J. Math. Anal. Appl., **343**(2008), 427-435.

[2] H. Attouch, J. Bolte, P. Redont, A. Soubeyran, *Alternating proximal algorithms for weakly coupled minimization problems, Applications to dynamical games and PDEs*, J. Convex Anal., **15**(2008), 485-506.

[3] H.H. Bauschke, P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[4] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Probl., **18**(2002), 441-453.

[5] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl., **20**(2004), 103-120.

[6] C. Byrne, Y. Censor, A. Gibali, S. Reich, *Weak and strong convergence of algorithms for the split common null point problem*, J. Nonlinear Convex Anal., **13**(2012), 759-775.

[7] Y. Censor, T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms, **8**(1994), 221-239.

[8] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, *A unified approach for inversion problems in intensity modulated radiation therapy*, Physics in Medicine and Biology, **51**(2006), 2353-2365.

[9] Y. Censor, A. Gibali, S. Reich, *Algorithms for the split variational inequality problem*, Numer. Algorithms, **59**(2)(2012), 301-323.

[10] P.L. Combettes, *Hilbertian convex feasibility problem: convergence of projection methods*, Appl. Math. Optim., **35**(1997), 311-330.

[11] Q.L. Dong, S. He, J. Zhao, *Solving the split equality problem without prior knowledge of operator norms*, Optimization, **64**(9)(2015), 1887-1906.
[12] P. Hartman, G. Stampacchia, *On some non-linear elliptic differential-functional equation*, Acta Mathenatica, **115**(1966), 271-310.
[13] K.R. Kazmi, S.H. Rizvi, *Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem*, J. Egyptian Math. Soc., **21**(2013), 44-51.
[14] K.R. Kazmi, S.H. Rizvi, *An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping*, Optim. Lett., **8**(2014), 1113-1124.
[15] K.R. Kazmi, R. Ali, *Hybrid projection method for a system of unrelated generalized mixed variational-like inequality problems*, to appear in Georgian Math. J., https://doi.org/10.1515/gmj-2017-0027.
[16] Z. Ma, L. Wang, S.-S. Chang, W. Duan, *Convergence theorems for split equality mixed equilibrium problems with applications*, Fixed Point Theory Appl., **2015**, 2015:31, https://doi.org/10.1186/s13663-015-0281-x.
[17] G. Marino, H.K. Xu, *Weak and strong convergence theorems for strict pseudocontractions in Hilbert space*, J. Math. Anal. Appl., **329**(2007), 336-346.
[18] A. Moudafi, *A note on the split common fixed point problem for quasi-nonexpansive operators*, Nonlinear Anal., **74**(2008), 4083-4087.
[19] A. Moudafi, *Split monotone variational inclusions*, J. Optim. Theory Appl., **150**(2011), 275-283.
[20] A. Moudafi, *A relaxed alternating CQ-algorithm for convex feasibility problems*, Nonlinear Anal., **79**(2013), 117-121.
[21] A. Moudafi, E. Al-Shemas, *Simultaneous iterative methods for split equality problems*, Trans. Math. Program. Appl., **1**(2)(2013), 1-11.
[22] A. Moudafi, *Alternating CQ-algorithm for convex feasibility and split fixed point problems*, J. Nonlinear Convex Anal., **15**(2014), 809-818.
[23] M.A. Noor, *General nonlinear mixed variational-like inequalities*, Optimization, **37**(1996), 357-367.
[24] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73**(1967), 591-597.
[25] J. Parida, M. Sahoo, A. Kumar, *A variational-like inequalitiy problem*, Bull. Austral. Math. Soc., **39**(1989), 225-231.
[26] V. Preda, M. Beldiman, A. Batatoresou, *On variational-like inequalities with generalized monotone mappings*, In: Generalized Convexity and Related Topics, Lecture Notes in Economics and Mathematical Systems, **583**(2006), 415-431.
[27] B. Qu, N. Xu, *A note on the CQ algorithm for the split feasibility problem*, Inverse Probl., **21**(2005), 1655-1665.
[28] J.C. Yao, *The Generalized quasi-variational inequality problem with applications*, J. Math. Anal. Appl., **158**(1991), 139-160.
[29] J. Zhao, *Solving split equality fixed point problem of quasi-nonexpansive mappings without prior knowledge of operator norms*, Optimization, **64**(2015), 2619-2630.
[30] J. Zhao, S. Wang, *Mixed iterative algorithms for the multiple-set split equality common fixed-point problems without prior knowledge of operator norms*, Optimization, **65**(2016), 1069-1083.

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