Fixed Point Theory, 20(2019), No. 1, 203-210
DOI: 10.24193/fpt-ro.2019.1.13
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINTS OF GENERALIZED HYBRID MAPPINGS ON L_2-EMBEDDED SETS IN BANACH SPACES

A. JABBARI* AND R. KESHAVARZI**

*Department of Pure Mathematics, Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman, Kerman, Iran
E-mail: jabbari@uk.ac.ir

**Department of Pure Mathematics, Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman, Kerman, Iran
E-mail: keshavarzi88math@gmail.com

Abstract. In this paper, first we generalize the notion of L-embedded sets in Banach spaces, defined by A.T.-M. Lau and Y. Zhang in "Fixed point properties for semigroups of nonlinear mappings and amenability", *Journal of Functional Analysis,* 263 (2012), pp. 2949-2977, to the notion of L_p-embedded sets ($p > 0$). Then, for a given generalized hybrid mapping T, we introduce the concepts of T-Chebyshev radius and T-Chebyshev center, generalizing the concepts of Chebyshev radius and Chebyshev center for nonexpansive mappings. Finally, we study the existence of fixed points of generalized hybrid mappings on L_2-embedded subsets of a Banach space by using the notions of T-Chebyshev radius and T-Chebyshev center.
Key Words and Phrases: fixed point, generalized hybrid mapping, L_2-embedded set, Chebyshev center.
2010 Mathematics Subject Classification: 47H10, 37C25.

1. Introduction

For a nonempty subset K of a Hilbert space H, a mapping $T : K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in K.$$

In [8], Takahashi called $T : K \to K$ hybrid if

$$3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|Tx - y\|^2 + \|x - y\|^2, \ \forall x, y \in K.$$

In [6], Kohsaka and Takahashi called T nonspreading if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|Tx - y\|^2, \ \forall x, y \in K.$$

In 2010, Kocourek, Takahashi and Yao [5] introduced the concept of generalized hybrid mappings, which contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings as special cases. They called a mapping $T : K \to K$, $(\alpha, \beta)-$generalized hybrid if there exist real numbers α and β such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2, \forall x, y \in K.$$

Evidently, the notions of nonexpansive, nonspreading and hybrid mappings are equivalent to $(1,0)$-generalized hybrid, $(2,1)$-generalized hybrid and $(\frac{3}{2},\frac{1}{2})$-generalized hybrid mappings, respectively.
In the last decades, there has been considerable interest to the existence of fixed points of self-mappings or semigroups of self-mappings on a nonempty bounded closed convex subset of a Hilbert space . Among these mappings are nonexpansive, nonspreading and also hybrid mappings. The fixed points of nonexpansive mappings are extensively studied in [1] and [2]. The fixed points of hybrid mappings are studied in [5]. Fixed point of nonspreading mappings are also studied in [3] and [4]. Kocourek, Takahashi and Yao in [5] proved that if K is closed convex and bounded subset of a Hilbert space, then the (α,β)-generalized hybrid mapping $T : K \to K$ has a fixed point.
Let X be an arbitrary Banach space and let K be a nonempty subset of X. In this paper, first we introduce the concept of L_p-embedded subsets of a Banach space X (Definition 2.1) and we show that the notion of L_1-embedded sets coincides with that of L-embedded sets defined by Lau and Zhang [7]. Next, for a bounded subset B of X, we introduce the notions of T-Chebyshev radius and T-Chebyshev center of B in K. Finally, we study the existence of fixed points of generalized hybrid mappings on L_2-embedded subsets of X, by using the fact that the T-Chebyshev center of B in K is actually a weakly compact set.

Let us recall some preliminary definitions.
In [7], Lau and Zhang called a nonempty subset K of a Banach space X, $L-$embedded if there is a subspace X_s of X^{**} such that $X + X_s = X \oplus_1 X_s$ in X^{**} and $\overline{K}^{w^*} \subset K \oplus_1 X_s$, that is, for each $u \in \overline{K}^{w^*}$ there are $k \in K$ and $\xi \in X_s$ such that $u = k+\xi$ and $\|u\| = \|c\| + \|\xi\|$. They showed that every weakly compact subset of X is L-embedded, but not vice-versa.

Let K and B be two nonempty subsets of a Banach space X in which B is bounded. The Chebyshev radius of B in K is defined by

$$r_K(B) = \inf\{r \geq 0 : \exists x \in K, \sup_{b\in B} \|x - b\| \leq r\}.$$

Clearly, we have $0 \leq r_K(B) < \infty$. The Chebyshev center of B in K is defined to be

$$W_K(B) = \{x \in K : \sup_{b\in B} \|x - b\| \leq r_K(B)\}.$$

Note that, as a subset of K, $W_K(B)$ may be empty.

2. Main results

In the next definition, we introduce the concept of an L_p-embedded set.
Definition 2.1. Let K be a nonempty subset of a Banach space X. We say that K is L_p-embedded if there exists a subspace X_s of X^{**} such that $X + X_s = X \oplus_p X_s$ in X^{**} and $\overline{K}^{w^*} \subset K \oplus_p X_s$, ($\overline{K}^{w^*}$ is the closure of K in X^{**} in the weak* topology of X^{**},) that is, for each $u \in \overline{K}^{w^*}$ there are $k \in K$ and $\xi \in X_s$ such that $u = k + \xi$ and $\|u\|^p = \|c\|^p + \|\xi\|^p$.
Let K and B be two nonempty subsets of a Banach space X in which B is bounded.

Let $T : K \to K$ be any mapping. We define the T-Chebyshev radius of B in K as follows:

$$r_{T,K}(B) = \inf\{r \geq 0 : \exists x \in K,\ \beta \sup_{b\in B} \|Tx - b\|^2 + (1-\beta) \sup_{b\in B} \|x - b\|^2 \leq r^2\}.$$

Clearly, we have $0 \leq r_{T,K}(B) < \infty$. Also we define the T-Chebyshev center of B in K by

$$W_{T,K}(B) = \{x \in K : \beta \sup_{b\in B} \|Tx - b\|^2 + (1-\beta) \sup_{b\in B} \|x - b\|^2 \leq r^2_{T,K}(B)\}.$$

Clearly, the notions of Chebyshev radius and Chebyshev center for nonexpansive mappings coincide with the notions of T-Chebyshev radius and T-Chebyshev center, respectively.

Let K be an L_2-embedded subset of X, therefore there exists a subspace X_s of X^{**} such that $X + X_s = X \oplus_2 X_s$ in X^{**} and $\overline{K}^{w^*} \subset K \oplus_2 X_s$. Now put $\widetilde{K} = \overline{K}^{w^*}$ in X^{**}. We define $\widetilde{T} : \widetilde{K} \to X^{**}$ where $\widetilde{T}(c + \xi) = T(c) + \xi$, for all $c \in K$ and $\xi \in X_s$. Then we have the following lemma.

Lemma 2.2. *If $T : K \to K$ is an (α, β)-generalized hybrid mapping, then so is $\widetilde{T} : \widetilde{K} \to X^{**}$.*

Proof.

$$\begin{aligned}
&\alpha\|\widetilde{T}x - \widetilde{T}y\|^2 + (1-\alpha)\|x - \widetilde{T}y\|^2 \\
&= \alpha\|Tc_1 + \xi_1 - Tc_2 - \xi_2\| + (1-\alpha)\|c_1 + \xi_1 - Tc_2 - \xi_2\|^2 \\
&= \alpha\|Tc_1 - Tc_2\|^2 + \alpha\|\xi_1 - \xi_2\|^2 + (1-\alpha)\|c_1 - Tc_2\|^2 + (1-\alpha)\|\xi_1 - \xi_2\|^2 \\
&\leqslant \beta\|Tc_1 - c_2\|^2 + (1-\beta)\|c_1 - c_2\|^2 + \|\xi_1 - \xi_2\|^2 \\
&= \beta\|Tc_1 - c_2\|^2 + \beta\|\xi_1 - \xi_2\|^2 + (1-\beta)\|c_1 - c_2\|^2 + (1-\beta)\|\xi_1 - \xi_2\|^2 \\
&= \beta\|Tc_1 + \xi_1 - c_2 - \xi_2\|^2 + (1-\beta)\|c_1 + \xi_1 - c_2 - \xi_2\|^2 \\
&= \beta\|\widetilde{T}x - y\|^2 + (1-\beta)\|x - y\|^2
\end{aligned}$$

This shows that $\widetilde{T} : \widetilde{K} \to X^{**}$ is also an (α, β)-generalized hybrid mapping.

Recall that $T : K \to K$ is called affine if $T(ax + by) = aTx + bTy$ for all constant $a, b \geq 0$ with $a + b = 1$ and for all $x, y \in K$.

Theorem 2.3. *Let K be a nonempty L_2-embedded subset of a Banach space X and B a nonempty bounded subset of X. Let $T : K \to K$ be any $(\alpha.\beta)$-generalized hybrid mapping with $\alpha \geqslant 1$ and $0 \leq \beta \leqslant 1$. Then the following results hold.*

(i) $W_{T,K}(B) \neq \emptyset$.

(ii) $W_{T,K}(B)$ is weakly compact.

(iii) If $B \subset K$ with $B \subset T(B)$, then $W_{T,K}(B)$ is T-invariant, that is $T(W_{T,K}(B)) \subset W_{T,K}(B)$.

(iv) If K is convex and $T : K \to K$ is affine, then $W_{T,K}(B)$ is convex.

Proof. (i) From the definition of the T-Chebyshev radius $r_{T,C}(B)$, for each $n > 0$, there is $x_n \in C$ such that

$$\beta \sup_{b\in B} \|Tx - b\|^2 + (1-\beta) \sup_{b\in B} \|x - b\|^2 \leqslant r^2_{T,C}(B) + \frac{1}{n}$$

Let x^{**} be any w^*-limit point of (x_n) in X^{**}. Since K is L_2-embedded, therefore there exist $c \in K$ and $\xi \in X_s$ such that $x^{**} = c + \xi$ and also

$$\|x^{**}\|^2 = \|c\|^2 + \|\xi\|^2,$$

in which X_s is an arbitrary subspace of X^{**} such that $X + X_s = X \oplus_2 X_s$.
The mapping $\widetilde{T} : \widetilde{K} \to X^{**}$ is a generalized hybrid mapping by the above lemma and,

$$\begin{aligned}
&\beta\|Tc - b\|^2 + (1-\beta)\|c - b\|^2 \\
&\leqslant \beta\|Tc - b\|^2 + (1-\beta)\|c - b\|^2 + \|\xi\|^2 \\
&= \beta\|Tc - b\|^2 + \beta\|\xi\|^2 + (1-\beta)\|c - b\|^2 + (1-\beta)\|\xi\|^2 \\
&= \beta\|Tc + \xi - b\|^2 + (1-\beta)\|c + \xi - b\|^2 \\
&= \beta\|\widetilde{T}x^{**} - b\|^2 + (1-\beta)\|x^{**} - b\|^2 \\
&\quad \lim_{n\to\infty} r^2_{T,C}(B) + \frac{1}{n} = r^2_{T,C}(B)
\end{aligned}$$

for all $b \in B$. Thus $c \in W_{T,K}(B)$.

(ii) Let $\widetilde{K}$ be as in the above lemma. Then $\widetilde{K}$ is a nonempty weak* closed subset of X^{**} and $\widetilde{K} \subset K \oplus_2 X_s$. Consider B as a subset of X^{**}. We show that

$$W_{\widetilde{T},\widetilde{K}}(B) = W_{T,K}(B).$$

Take $x \in W_{\widetilde{T},\widetilde{K}}(B)$. Then there is $c \in K$ and $\xi \in X_s$ such that $x = c + \xi$. By using the fact that for each $b \in B$ we have

$$\|x - b\|^2 = \|c - b\|^2 + \|\xi\|^2,$$

one have:

$$\begin{aligned}
r_{\widetilde{T},\widetilde{K}}(B) &\geq \sup_{b\in B} \beta\|\widetilde{T}x - b\|^2 + (1-\beta)\sup_{b\in B}\|x - b\|^2 \\
&= \beta \sup_{b\in B}\|Tc + \xi - b\|^2 + (1-\beta)\sup_{b\in B}\|c + \xi - b\|^2 \\
&= \beta \sup_{b\in B}\|Tc - b\|^2 + (1-\beta)\sup_{b\in B}\|c - b\|^2 + \|\xi\|^2 \geq r^2_{T,K}(B) + \|\xi\|^2.
\end{aligned}$$

On the other hand, from the definition of the T-Chebyshev radius,

$$r_{\widetilde{T},\widetilde{K}}(B) \leq r_{T,K}(B).$$

Thus, $\xi = 0$ and $r_{\widetilde{T},\widetilde{K}}(B) = r_{T,K}(B)$. Hence $W_{\widetilde{T},\widetilde{K}}(B) = W_{T,K}(B)$. Now, it is obvious that $W_{\widetilde{T},\widetilde{K}}(B)$ is weak* closed and bounded. Therefore, it is weak* compact. Hence $W_{T,K}(B)$ is weakly compact, (since on $W_{T,K}(B)$ the *weak** topology of X^{**} coincides with the weak topology of X).

(iii) Suppose that $x \in W_{T,K}(B)$.

$$\begin{aligned}
&\beta \sup_{b\in B} \|T(Tx) - Tb\|^2 + (1-\beta) \sup_{b\in B} \|Tx - Tb\|^2 \\
&\leqslant \frac{\beta^2}{\alpha} \sup_{b\in B} \|T^2x - b\|^2 + \frac{\beta(1-\beta)}{\alpha} \sup_{b\in B} \|Tx - b\|^2 + \frac{\beta(\alpha-1)}{\alpha} \sup_{b\in B} \|Tx - Tb\|^2 \\
&+ \frac{\beta(1-\beta)}{\alpha} \sup_{b\in B} \|Tx - b\|^2 + \frac{(1-\beta)^2}{\alpha} \sup_{b\in B} \|x - b\|^2 + \frac{(\alpha-1)(1-\beta)}{\alpha} \sup_{b\in B} \|x - Tb\|^2 \\
&= \frac{\beta^2}{\alpha} \sup_{b\in B} \|T^2x - b\|^2 + \frac{\beta - 2\beta^2 + \beta\alpha}{\alpha} \sup_{b\in B} \|Tx - b\|^2 + \frac{\beta^2 - \alpha\beta + \alpha}{\alpha} \sup_{b\in B} \|x - Tb\|^2
\end{aligned}$$

therefore,

$$\begin{aligned}
&\beta - \frac{\beta^2}{\alpha} \sup_{b\in B} \|T^2x - b\|^2 + \left((1-\beta) - \frac{\beta - 2\beta^2 + \beta\alpha}{\alpha}\right) \sup_{b\in B} \|Tx - b\|^2 \\
&\leq \frac{\beta^2 - \alpha\beta + \alpha - \beta}{\alpha} \sup_{b\in B} \|x - b\|^2. \\
&\frac{\beta(\alpha-\beta)}{\alpha} \sup_{b\in B} \|T^2x - b\|^2 + \frac{(\alpha-\beta) - 2\beta(\alpha-\beta)}{\alpha} \sup_{b\in B} \|Tx - b\|^2 \\
&\leq \frac{\beta(\beta-\alpha) + (\alpha-\beta)}{\alpha} \sup_{b\in B} \|x - b\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\beta(\alpha-\beta) \sup_{b\in B} \|T^2x - b\|^2 + (\alpha-\beta)(1-2\beta) \sup_{b\in B} \|Tx - b\|^2 \\
&\leq (\alpha-\beta)(1-\beta) \sup_{b\in B} \|x - b\|^2.
\end{aligned}$$

Dividing by $(\alpha - \beta)$, we have

$$\begin{aligned}
&\beta \sup_{b\in B} \|T^2x - b\|^2 + (1-\beta-\beta) \sup_{b\in B} \|Tx - b\|^2 \\
&\leq (1-\beta) \sup_{b\in B} \|x - b\|^2.
\end{aligned}$$

So we obtain

$$\begin{aligned}
&\beta \sup_{b\in B} \|T^2x - b\|^2 + (1-\beta) \sup_{b\in B} \|Tx - b\|^2 \\
&\leq \beta \sup_{b\in B} \|Tx - b\|^2 + (1-\beta) \sup_{b\in B} \|x - b\|^2 \leq r^2_{T,K}(B).
\end{aligned}$$

Thus $Tx \in W_{T,K}(B)$ and $W_{T,K}(B)$ is T-invariant.

(iv) Assume that K is convex and T is affine. To show that $W_{T,K}(B)$ is convex, let $x, y \in K$ and $0 \leq \lambda \leq 1$, then

$$\begin{aligned}
&\beta \sup_{b\in B} \|T(\lambda x + (1-\lambda)y) - b\|^2 + (1-\beta) \sup_{b\in B} \|\lambda x + (1-\lambda)y - b\|^2 \\
&= \beta \sup_{b\in B} \|\lambda(Tx - b) + (1-\lambda)(Ty - b)\|^2 + (1-\beta) \sup_{b\in B} \|\lambda(x - b) + (1-\lambda)(y - b)\|^2 \\
&\leq \beta\lambda \sup_{b\in B} \|Tx - b\|^2 + \beta(1-\lambda) \sup_{b\in B} \|Ty - b\|^2 + \sup_{b\in B}\{-\beta\lambda(1-\lambda)\|Tx - Ty\|^2\} \\
&+ (1-\beta)\lambda \sup_{b\in B} \|x - b\|^2 + (1-\beta)(1-\lambda) \sup_{b\in B} \|y - b\|^2 \\
&+ \sup_{b\in B}\{-(1-\beta)\lambda(1-\lambda)\|x - b\|^2\} \\
&\leqslant \lambda r^2_{T,K}(B) + (1-\lambda) r^2_{T,K}(B) + \sup_{b\in B}\{-\beta\lambda(1-\lambda)\|Tx - Ty\|^2\} \\
&+ \sup_{b\in B}\{-(1-\beta)\lambda(1-\lambda)\|x - b\|^2\} \\
&< r^2_{T,K}(B).
\end{aligned}$$

This complete the proof. □

In [7], Lau and Zhang proved the following result, which is needed in the sequel.

Lemma 2.4. *Suppose that K is a weakly compact nonempty subset of a Banach space X. Let $T : K \to K$ be a weakly continuous on K. Then there is a weakly compact subset B in K such that $T(B) = B$.*

We are ready to state and prove our main result.

Theorem 2.5. *Let $T : K \to K$ be an (α, β)-generalized hybrid mapping on a nonempty L_2-embedded convex subset K of a Banach space X with T being weakly continuous on every weakly compact invariant convex subset of K. If K contains a nonempty bounded subset B such that $T(B) = B$, then T has a fixed point on K.*

Proof. From proposition 2 the T-Chebyshev center $W_{T,K}(B)$ is a nonempty invariant weakly compact convex subset of K. Hence T is weakly continuous on $W_{T,K}(B)$. Let U be a minimal nonempty weakly compact convex invariant subset of $W_{T,K}(B)$, and let F be a minimal nonempty weakly compact invariant subset of U. The existence of such U and F is guaranteed by Zorn's Lemma. Since the T restricted on $W_{T,K}(B)$ is weakly continuous, hence by Lemma 2, F satisfies $T(F) = F$. Also, F is norm compact and hence has a normal structure. Assume that F contains more than one point. Since F has a normal structure and $T(F) = F$, there is $y_0 \in coF \subset U$ (where coF denotes the closed convex hull of F) such that

$$\beta \sup_{x\in F} \|Tx - y_0\|^2 + (1-\beta) \sup_{x\in F} \|x - y_0\|^2 = \beta \sup_{x\in F} \|x - y_0\|^2 + (1-\beta) \sup_{x\in F} \|x - y_0\|^2 <$$

$$\beta \sup_{x,y\in F} \|x - y\|^2 + (1-\beta) \sup_{x,y\in F} \|x - y\|^2 = \beta \sup_{x,y\in F} \|Tx - y\|^2 + (1-\beta) \sup_{x\in F} \|x - y\|^2.$$

Then we define

$$r_0^2 = \{\beta \sup \|Tx - y_0\|^2 + (1-\beta) \sup \|x - y_0\|^2 : x \in F\} \text{ with } r_0 > 0.$$

Then
$$r_0^2 < \{\beta \sup \|Tx - y\|^2 + (1-\beta) \sup \|x - y\|^2 : x, y \in F, x \neq y\}.$$
Let $M = \{x \in U : \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \leqslant r_0^2 \text{ for all } y \in F\}$. Then M is a nonempty, norm closed convex (hence weakly closed) subset of U. Since T is (α, β)-generalized hybrid mapping and $T(F) = F$, hence M is invariant. But $F \nsubseteq M$. So $M \subset U$. This contradicts with the minimality of U. Therefore F must be a singleton. Thus T has a fixed point on K.

The next corollary is a direct consequence of our main result.

Corollary 2.6. *Let $T : K \to K$ be a hybrid mapping, a nonspreading mapping or a nonexpansive mapping on a nonempty L_2-embedded convex subset K of a Banach space X with T being weakly continuous on every weakly compact invariant convex subset of K. If K contains a nonempty bounded subset B such that $T(B) = B$, then T has a fixed point on K.*

Problem. Can Theorem 2.5 be generalized to semigroup of mappings as in [7], Theorem 3.7?

Acknowledgments. We would like to express our gratitude to the referee for the kind suggestions and also for posing the following problem.

References

[1] F.E. Browder, *Non-expansive nonlinear operators in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A., **54**(1965), 1041-1044.

[2] M. Edelstein, *On non-expansive mappings of Banach spaces*, Proc. Camb. Phil. Soc., **60**(1964), 439-447.

[3] S. Iemoto, W. Takahashi, *Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space*, Nonlinear Anal., **71**(2009), 2082-2089.

[4] W. Inthakon, A. Kaewkhao, N. Niyamosot, *Common fixed points for some generalized nonexpansive mappings and nonspreading-type mappings in uniformly convex Banach spaces*, Fixed Point Theory Appl., 2012.

[5] P. Kocourek, W. Takahashi, J.C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math., **14**(2010), 2497-2511.

[6] F. Kohsaka, W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel), **91**(2008), 166-177.

[7] A.T.-M. Lau, Y. Zhang, *Fixed point properties for semigroups of nonlinear mappings and amenability*, J. Functional Anal., **263**(2012), 2949-2977.

[8] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal., **11**(2010), 79-88.

Received: August 17, 2016; Accepted: October 16, 2016.