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# FIXED POINTS, MULTI-VALUED UNIFORMLY LIPSCHITZIAN MAPPINGS AND UNIFORM NORMAL STRUCTURE

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**Abstract.** Khamsi and Kirk [11] gave the definition of multi-valued uniformly Lipschitzian mappings via generalized orbits. We will prove a fixed point theorem in metric spaces with uniform normal structure which extends Lim and Xu's theorems [13] to multi-valued uniformly Lipschitzian mappings. We also give a fixed point result for multi-valued uniformly Lipschitzian mappings in hyperconvex spaces.
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## 1. Introduction

Let $C$ be a nonempty subset of a metric space $(E, d)$. A mapping $T : C \to C$ is called *uniformly $k-$Lipschitzian* if there exists a constant $k > 0$ such that

$$d(T^n x, T^n y) \leqslant k \cdot d(x, y)$$

for all points $x, y \in C$ and any positive integer $n \geqslant 1$. It is clear that nonexpansive mappings are uniformly $1-$Lipschitzian mappings. Again, when $E$ has some "nice" geometric properties and $k > 1$ is not too large, we can assure that $T$ has a fixed point (see [1]).

The first fixed point theorem for uniformly Lipschitzian mappings in uniformly convex Banach spaces was initiated by Goebel and Kirk [9]. A different and more general approach is proposed by Lifshitz [12] in metric spaces. The next fixed point theorem for uniformly Lipschitzian mappings in Banach spaces with uniform normal structure is due to Casini and Maluta [5]. Finally the work by Lim and Xu [13] links the existence of fixed points of such mappings to uniform normal structure property using the ideas developed by Baillon [3].

In [11] Khamsi and Kirk introduced the definition of multi-valued uniformly Lipschitzian mappings in metric spaces and extended the famous Lifshitz's fixed point theorem to multi-valued mappings.

In this note we will prove a fixed point theorem for multi-valued uniformly Lipschitzian mappings in metric spaces with uniform normal stucture which extend Lim and Xu's theorem [13].

## 2. Preliminaries

We start by giving the definition of multi-valued uniformly Lipschitzian mappings via the generalized orbits [11].

**Definition 2.1.** Let $(E, d)$ be a metric space and $T : E \to \mathcal{N}(E)$, where $\mathcal{N}(E)$ denotes the collection of all nonempty subset of $E$, be a multi-valued map. For any $x \in E$, the sequence $\{x_n\}$ is called a *generalized orbit of* $x$ if $x_0 = x$ and $x_{n+1} \in T(x_n)$ for any $n \geqslant 0$. We will use the notion $O_T(x)$ to denote such a sequence.

It is clear that for a given $x \in E$, the map $T$ may have many different orbits generated by $x$. Therefore the notation $O_T(x)$ should be used carefully.

**Definition 2.2.** Let $(E, d)$ be a metric space. A multi-valued mapping $T : E \to \mathcal{N}(E)$ is called a *uniformly* $k-$*Lipschitzian* mapping (with $k > 0$) if and only if for any $x, y \in E$, and any generalized orbit $\{x_n\} \in O_T(x)$, there exists a generalized orbit $\{y_n\} \in O_T(y)$ such that

$$d(x_{n+h}, y_n) \leqslant k \cdot d(x_h, y),$$

for any $n \geqslant 1$, $h \geqslant 0$, where $x_0 = x$.

Note that when $T$ is single-valued, then the above definition coincides with the traditional definition since any $x$ will have only one orbit generated by iterating $T$.

Now we recall some convexity structure on metric spaces.

Let $\mathcal{F}(E)$ denote a nonempty family of subsets of a metric space $(E, d)$. We say that $\mathcal{F}(E)$ defines *convexity stucture* on $E$ if $\mathcal{F}(E)$ is satble by intersection and that $\mathcal{F}(E)$ has *property* $(R)$ if any decreasing sequence $\{C_n\}$ of nonempty closed bounded subset of $E$ with $C_n \in \mathcal{F}(E)$ has nonvoid intersection.

A subset of $E$ is said to be *admissible* if it is an intersection of closed balls [7]. We denote by $\mathcal{A}(E)$ the family of all admissible subsets of $E$. It is obvious that $\mathcal{A}(E)$ defines a convexity structure on $E$. In this paper any other convexity structure $\mathcal{F}(E)$ is always assumed to contain $\mathcal{A}(E)$.

For $C \subset E$, we denote the following:

$$r_x(C) = \sup\{d(x, y) : y \in C\}, \ x \in C,$$

$$R(C) = \inf\{r_x(C) : x \in C\}.$$

For a bounded subset $C$ of $E$, we define the admissible hull of $C$, denoted $ad(C)$, as the intersection of all those admissible subsets of $E$ that contains $C$, i.e.

$$ad(C) = \bigcap\{B \subset E : C \subset B \text{ with } B \text{ admissible}\}.$$

We know [1]: *Let $C$ be a bounded subset of a metric space $E$ and $x \in E$. Then $r_x(ad(C)) = r_x(C)$.*

We introduce uniformly normal structure with respect to convexity structure $\mathcal{F}(E)$ in metric space $E$.

**Definition 2.3.** A metric space $(E, d)$ is said to have *uniformly normal structure* if there exists a convexity structure $\mathcal{F}(E)$ such that

$$R(C) \leqslant \alpha \cdot diam(C)$$

for some constant $\alpha \in (0,1)$ and for all $C \in \mathcal{F}(E)$ that is bounded and consist of more than one point. We also say that $\mathcal{F}(E)$ is uniformly normal.

The number $N(E)$ is said to be the *normal structure coefficient* [4] if

$$N(E) = \inf\Big\{\frac{diam(C)}{R(C)} : C \in \mathcal{F}(E) \text{ bounded with } diam(C) > 0\Big\}.$$

It is easy to see that $E$ has uniformly normal structure if and only if $N(E) > 1$.

It is known that *every convexity structure with uniformly normal structure has property* $(R)$, [1].

In [13], Lim and Xu introduced so-called property $(P)$ for metric space.

**Definition 2.4.** A metric space $(E, d)$ is said to have *property* $(P)$ if given two bounded sequences $\{x_n\}$ and $\{z_n\}$ in $E$, one can find some $z \in \bigcap_{n\geqslant 1} ad(\{z_j : j \geqslant n\})$ such that

$$\limsup_{n\to+\infty} d(z, x_n) \leqslant \limsup_{j\to+\infty} \limsup_{n\to+\infty} d(z_j, x_n).$$

If $E$ has property $(R)$, then $\bigcap_{n\geqslant 1} ad(\{z_j : j \geqslant n\}) \neq \emptyset$. Also, if $E$ is a weakly compact convex subset of a normed linear space then admissible hulls are closed convex and $\bigcap_{n\geqslant 1} ad(\{z_j : j \geqslant n\}) \neq \emptyset$ by weak compactness of $E$, and that $E$ possesses property $(P)$ follows directly from the weak lower semicontinuity of the function

$$\limsup_{n\to+\infty} \|x_n - x\|.$$

## 3. Main result

Lim and Xu [13] established the following key lemma.

**Lemma 3.1.** *Let $(E, d)$ be a complete bounded metric space with both property $(P)$ and uniformly normal structure. Let $N(E)$ be the normal structure coefficient with respect to the given convexity structure $\mathcal{F}(E)$. Then for any sequence $\{x_n\}$ in $E$ and any constant $\alpha > \tilde{N}(E) = \frac{1}{N(E)}$, there exists a point $z \in E$ satisfying the properties:*

(a) $d(z, y) \leqslant \limsup_{n\to+\infty} d(x_n, y)$ *for all $y \in E$,*

(b) $\limsup_{n\to+\infty} d(z,x_n) \leqslant \alpha \cdot diam(\{x_n\})$.

We now present the existence theorem for multi-valued uniformly Lipschitzian mappings in a metric space.

**Theorem 3.2.** *Let $(E,d)$ be a complete bounded metric space with both property $(P)$ and uniformly normal structure and $T : E \to \mathcal{C}(E)$, i.e. $T(x)$ is a nonempty closed convex subset of $E$, for any $x \in E$. If $T$ is uniformly $k-$Lipschitzian with*

$$k < [\tilde{N}(E)]^{-\frac{1}{2}} = \sqrt{N(E)},$$

*then $T$ has a fixed point, i.e. there exists $x \in E$ such that $x \in T(x)$.*

*Proof.* We may assume $k \geqslant 1$ since then $k < 1$, i.e. $T$ is a set-valued contraction in the usual sense and hence has a fixed point in a complete space $E$, [14].

Choose constant $\alpha \in (\tilde{N}(E), 1)$ such that $k < \frac{1}{\sqrt{\alpha}}$.

Let $x \in E$ and $\{x_n\} \in O_T(x)$ be a generalized orbit of $x$. Using Lemma 3.1, one can construct a point $z$ found in Lemma 3.1 associated with the sequence $\{x_n\}$. Set $\sigma^1(x) = z$. This is the notation used in [3]. Note that $\sigma^2(x)$ is a point found in Lemma 3.1 associated with the sequence $\{\sigma^1(x)_n\}$ which is the generalized orbit of $\sigma^1(x)$.

Fix $x \in E$ and $\{x_n\}$ be a generalized orbit of $x$. By induction, one will construct a sequence $\{\sigma^m(x)\}$ and generalized orbit $\{\sigma^m(x)_n\}_{n\geqslant 1}$ of $\sigma^m(x)$ for any $m \geqslant 1$, such that $\sigma^{m+1}(x)$ satisfy (as a point $z$) the conditions from Lemma 3.1:

(a) $d(\sigma^{m+1}(x), y) \leqslant \limsup_{i\to+\infty} d(\sigma^m(x)_i, y)$ for all $y \in E$,

(b) $\limsup_{i\to+\infty} d(\sigma^{m+1}(x), \sigma^m(x)_i) \leqslant \alpha \cdot diam(\{\sigma^m(x)_i\})$.

Set

$$r_m = \limsup_{i\to+\infty} d(\sigma^{m+1}(x), \sigma^m(x)_i).$$

Note for each $i > j \geqslant 0$,

$$\begin{aligned} d(\sigma^m(x)_i, \sigma^m(x)_j) &\leqslant k \cdot d(\sigma^m(x)_{i-j}, \sigma^m(x)) \leqslant \text{ (by (a))} \\ &\leqslant k \cdot \limsup_{n\to+\infty} d(\sigma^{m-1}(x)_n, \sigma^m(x)_{i-j}) \\ &\leqslant k^2 \cdot \limsup_{n\to+\infty} d(\sigma^{m-1}(x)_n, \sigma^m(x)) = k^2 \cdot r_{m-1}. \end{aligned} \tag{3.1}$$

Observe that

$$\begin{aligned} r_m &= \limsup_{i\to+\infty} d(\sigma^{m+1}(x), \sigma^m(x)_i) \leqslant \text{ (by (b))} \\ &\leqslant \alpha \cdot diam(\{\sigma^m(x)_i\}) \leqslant \text{ (by (3.1))} \\ &\leqslant \alpha \cdot k^2 \cdot r_{m-1} = h \cdot r_{m-1} \end{aligned}$$

and

$$r_m \leqslant h \cdot r_{m-1} \leqslant \ldots \leqslant h^{m-1} \cdot r_1.$$

Our assumption on $k$ leads to $h = \alpha \cdot k^2 < 1$.
Hence for each $i \geqslant 1$,

$$d(\sigma^{m+1}(x), \sigma^m(x)) \leqslant d(\sigma^{m+1}(x), \sigma^m(x)_i) + d(\sigma^m(x)_i, \sigma^m(x)) \leqslant \text{ (by (a))}$$

$$d(\sigma^{m+1}(x),\sigma^m(x)_i)+\limsup_{j\to+\infty} d(\sigma^{m-1}(x)_j,\sigma^m(x)_i)\leq$$

$$\leq d(\sigma^{m+1}(x),\sigma^m(x)_i)+k\cdot\limsup_{j\to+\infty} d(\sigma^{m-1}(x)_j,\sigma^m(x))=$$

$$=d(\sigma^{m+1}(x),\sigma^m(x)_i)+k\cdot r_{m-1}.$$

Taking the limit superior as $i\to+\infty$, we get

$$d(\sigma^{m+1}(x),\sigma^m(x))\leqslant\limsup_{i\to+\infty} d(\sigma^{m+1}(x),\sigma^m(x)_i)+k\cdot r_{m-1}\leq$$

$$\leqslant r_m+k\cdot r_{m-1}\leqslant(h^{m-1}+k\cdot h^{m-2})\cdot r_1.$$

This implies that $\{\sigma^m(x)\}_{m\geqslant1}$ is Cauchy and $\sigma^m(x)\to z\in C$ as $m\to+\infty$.
Next we prove that $z$ is a fixed point of $T$, i.e. $z\in T(z)$. Indeed, we have (by (a))

$$d(\sigma^m(x),\sigma^m(x)_1)\leqslant\limsup_{n\to+\infty} d(\sigma^{m-1}(x)_n,\sigma^m(x)_1)\leqslant$$

$$\leqslant k\cdot\limsup_{n\to+\infty} d(\sigma^{m-1}(x)_n,\sigma^m(x))=k\cdot r_{m-1},$$

which implies

$$d(\sigma^m(x),\sigma^m(x)_1)\leqslant k\cdot h^{m-2}\cdot r_1$$

for any $m\geqslant3$. Hence $\{\sigma^m(x)_1\}$ also converges to $z$:

$$d(z,\sigma^m(x)_1)\leqslant d(z,\sigma^m(x))+d(\sigma^m(x),\sigma^m(x)_1)\to0,$$

as $m\to+\infty$. Using the uniform Lipschitz behavior of $T$, for any $m\geqslant1$, there exists a generalized orbit $\{z_n^m\}$ of $z$ such that

$$d(\sigma^m(x)_n,z_n^m)\leqslant k\cdot d(\sigma^m(x),z)$$

for any $n\geqslant1$. In particular, we have

$$d(\sigma^m(x)_1,z_1^m)\leqslant k\cdot d(\sigma^m(x),z)\leq$$

$$\leqslant k\cdot\Big[d(\sigma^m(x),\sigma^m(x)_1)+d(\sigma^m(x)_1,z)\Big]\to0$$

as $m\to+\infty$. Hence $\{z_1^m\}$ also converges to $z$:

$$d(z,z_1^m)\leqslant d(z,\sigma^m(x))+d(\sigma^m(x),\sigma^m(x)_1)+d(\sigma^m(x)_1,z_1^m)\to0$$

as $m\to+\infty$. But $z_1^m\in T(z)$ for any $m\geqslant1$ and $T(z)$ is closed. This will force $z\in T(z)$ as claimed. □

In Banach spaces we have the following multi-valued version of classical Casini and Maluta's theorem [5].

**Corollary 3.3.** *Let $C$ be a nonempty bounded closed convex subset of a Banach space with uniform normal structure and $T:C\to\mathcal{C}(C)$, i.e. $T(x)$ is a nonempty closed convex subset of $C$, for any $x\in C$. If $T$ is uniformly $k-$Lipschitzian with $k<\sqrt{N(E)}$, then $T$ has a fixed point, i.e. there exists $x\in C$ such that $x\in T(x)$.*

## 4. Remarks

**A.** Let $(M,d)$ be a metric space. We recall that the *Lifshitz charcteristic* $\kappa(M)$ is the supremum of all positive real numbers $b$ such that there exists $a>1$ such that for every $x,y\in M$ and $r>0$ with $d(x,y)>r$ there exists $z\in M$ satisfying $\overline{B}(x,br)\cap\overline{B}(y,ar)\subset\overline{B}(z,r)$. It is clear that $\kappa(M)\geqslant 1$. In Banach space $E$ we denote by $\kappa_0(E)$ the infimum of the numbers $\kappa(C)$ where $C$ is a closed convex bounded subset of $E$. It is known [5] that

$$\kappa_0(E)\leqslant N(E).$$

Let $E_\beta$, $1\leqslant\beta\leqslant 2$, be the space $l^2$ renormed by

$$\|x\|=\max\{\|x\|_2,\beta\cdot\|x\|_\infty\},$$

where $\|\cdot\|_2$ is the Euclidean norm and $\|\cdot\|_\infty$ the supremum norm. If $1\leqslant\beta\leqslant\frac{1}{2}\sqrt{5}$, then [6], [2],

$$\kappa_0(E_\beta)=\sqrt{1+\frac{1}{\beta^2}-\frac{2}{\beta^2}\sqrt{\beta^2-1}}.$$

If $\beta\geqslant\frac{1}{2}\sqrt{5}$ it is known [2] that $\kappa_0(E_\beta)=1$.
Since [5] $N(E_\beta)=\frac{1}{\beta}\sqrt{2}$, so $\sqrt{N(E_\beta)}$ converges to $\frac{1}{5}\sqrt[4]{1000}\approx 1.1246$ and

$$\frac{1}{2}\left(1+\sqrt{1+4\cdot N(E_\beta)\cdot(\kappa_0(E_\beta)-1)}\right)$$

converges to 1 as $\beta\nearrow\frac{1}{2}\sqrt{5}\approx 1.118$. Therefore, for $\beta$ close to $\frac{1}{2}\sqrt{5}$, more precisely $\beta>1.0556$, the constant which appears in Corollary 3.3 is strictly bigger than the constant

$$\frac{1}{2}\left(1+\sqrt{1+4\cdot N(E_\beta)\cdot(\kappa_0(E_\beta)-1)}\right),$$

which appears in the fixed point theorem for uniformly Lipschitzian mappings due to Domínguez Benavides [6] and its multi-valued version [10].

**B.** Metric hyperconvexity was introduced in 1956. A metric space $(E,d)$ is called *hyperconvex* if

$$\bigcap_{\alpha\in\Gamma}B(x_\alpha,r_\alpha)\neq\emptyset$$

for any collection of closed balls $\{B(x_\alpha,r_\alpha)\}_{\alpha\in\Gamma}$ such that

$$d(x_\alpha,x_\beta)\leqslant r_\alpha+r_\beta,\ \alpha,\beta\in\Gamma.$$

The space $C(S)$ of continuous real function on a Stonian space with "supremum" norm, the classical spaces $l_\infty$, $L_\infty$ are examples of hyperconvex spaces. Two facts are pertinent to what follows: any hyperconvex metric space is complete and $N(E)=2$ if $E$ is hyperconvex, [8].

Thus, Theorem 3.2 implies the following corollary for hyperconvex spaces.

**Corollary 4.1.** *Let $(E,d)$ be a bounded hyperconvex metric space with property $(P)$ and $T:E\to\mathcal{C}(E)$, i.e. $T(x)$ is a nonempty closed convex subset of $E$, for any $x\in E$. If $T$ is uniformly $k-$Lipschitzian with $k<\sqrt{2}$, then $T$ has a fixed point, i.e. there exists $x\in E$ such that $x\in T(x)$.*

**Remark 4.2.** In the classical Kirk's fixed point theorem, having a bounded orbit implies the existence of a fixed point. The following example of Prus [8] shows that boundedness of orbits does not imply the existence of a fixed point even for nonexpansive mappings. Indeed, consider the classical Banach space $l_\infty$ and the map $T : l_\infty \to l_\infty$ defined by

$$T((x_n)) = (1 + LIM x_n, x_1, x_2, \ldots),$$

where $LIM$ denotes a Banach limit. Then $\|T^n x\| \leqslant 1 + \|x\|$ and

$$\|T^n x - T^n y\| = \|x - y\|$$

for every $x, y \in l_\infty$. If $z = (z_1, z_2, \ldots) \in l_\infty$ satisfies $Tz = z$, then

$$z_1 = 1 + LIM z_n, z_2 = z_1, z_3 = z_2, \ldots,$$

and hence $LIM z_n = 1 + LIM z_n$, a contradiction, which shows that $Fix(T) = \emptyset$. On the other hand, we have $T^n(0) = (1, 1, \ldots, 1, 0, 0, \ldots)$ where the first block of length $n$ has all entries equal 1 and then 0 after that. So $T$ has bounded orbits.

**Question 4.3.** Does every hyperconvex space have the property $(P)$?

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