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APPROXIMATING COINCIDENCE POINTS BY α-DENSE CURVES

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Abstract. The purpose of this paper is to show, under suitable conditions, an iterative procedure which if converges, the limit point is a coincidence point of two given itself mappings defined in a subset of a metric space. Also, under additional conditions, the convergence of proposed iterative procedure holds. Our main tool will be the so called α-dense curves, which allow us to construct such procedure in a stable way, in the specified sense, providing also a bound for the error approximation at each iteration. To justify our result, we will analyze certain integral equations of Volterra type.
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1. Introduction

Many nonlinear problems arising from economics, physics and others applied sciences can be expressed as an equation of the form:

$$S(x) = T(x), \tag{1.1}$$

where $S, T : B \subset E \longrightarrow E$ are known mappings and (E, d) a metric space. A point $x \in B$ satisfying (1.1) is said to be a *coincidence point of S and T*. Of course, for the particular case that S be the identity mapping, (1.1) is a fixed point problem. Moreover, under suitable conditions, a solution for (1.1) is equivalent to find a fixed point of certain mapping. In fact, in [13, 21] to cite a pair of examples, the Banach contraction principle is used to solve such problem.

Usually, under suitable conditions, a constructive way to approximate a solution for (1.1) is, fixed an initial $x_1 \in B$, by an iterative procedure (often called iteration) of the form $S(x_{n+1}) = f(T, x_n)$. The simplest of such iterations, essentially introduced by Jungck [18, 19], is the following:

$$S(x_{n+1}) = T(x_n), \tag{1.2}$$

although, as expected, such iteration has been modified to obtain convergence under less restrictive conditions; see [1, 2, 3, 4, 7, 9, 10, 12, 17, 20, 32] and references therein.

Note that, fixed $x_1 \in B$ in the iterative procedure (1.2), once calculated $a_1 := T(x_1)$, we need to solve $S(x_2) = a_1$, or equivalently, to find $x_2 \in S^{-1}(a_1)$. The same can be said for others more general iterations of the form $S(x_{n+1}) = f(T, x_n)$. So, due to discretization or rounding off (or other reasons), the value of $S(x_2)$ might not be exactly equal to a_1. Then, in general, we will obtain a sequence $(S(z_n))_{n\geq 1}$ which is approximately equal to $(S(x_n))_{n\geq 1}$. This means that the sequence $(S(z_n))_{n\geq 1}$ can be converging to a point which is not a solution of (1.1). Therefore, in actual numerical environment, the *stability* of an iterative procedure is crucial.

The concept of stability for the Picard iteration $x_{n+1} = T(x_n)$ (that is, the iteration (1.2) when S is the identity mapping), was introduced by Ostrowski [28]. However, Singh et al. [31] (see also [14, 15, 32]) generalized and studied this concept for others iterations:

Definition 1.1. Let $S, T : B \longrightarrow B$, with $T(B) \subset S(B)$, $f(T, x_n)$ an iterative procedure and $x_0 \in B$ a coincidence point of S and T, put $p := S(x_0) = T(x_0)$. Given any sequence $(S(z_n))_{n\geq 1}$, define $\varepsilon_n := d(S(z_{n+1}), f(T, x_n))$. Then, the iterative procedure $f(T, x_n)$ is said to be (S, T)-stable, or simply stable if there is no ambiguity, if $\lim_n \varepsilon_n = 0$ implies that $\lim_n S(z_{n+1}) = p$.

Henceforth, by a stable iterative procedure (or stable iteration) we will mean in the above sense.

On the other hand, in this paper we will introduce a new iteration (see Theorem 3.1) to approximate, if such iteration converges, a solution for (1.1) when the mappings S and T satisfy certain conditions. Our main tool to define such iterative procedure will be the so called α-dense curves, which are detailed in Section 2 and provide us the following (*desirable* for any iteration) properties:

(a) The constructed iteration is stable.
(b) A bound for the error approximation, in the specified sense, in each iteration.

Also, we show in Corollary 3.1 sufficient conditions to guarantee the convergence of the proposed iterative procedure in Theorem 3.1. As the mentioned procedure is based on α-curves, a compactness condition for the mapping T is required.

Although the compactness condition mentioned above, required in our iteration, may seem very *strong*, the reality is that integral operators with sufficiently regular kernels provide the most important examples of compact operators on infinite dimensional Banach spaces; see, for instance, [22, Chapter 5]. Thus, as application of our results, in Section 4 we will analyze the existence of solutions, by showing a sequence converging to some of them, of certain Volterra integral equations.

2. α-DENSE CURVES AND FIXED POINTS

In what follows $\mathcal{B}(E)$ will be the class of non-empty and bounded subsets of the metric space (E, d), and $I := [0, 1]$. The concept of α-dense curve and densifiable set were introduced in 1997 by Mora and Cherruault [24]:

Definition 2.1. Given $\alpha \geq 0$ and $B \in \mathcal{B}(E)$, a continuous mapping $\gamma : I \longrightarrow (E, d)$ is said to be an α-dense curve in B if the following conditions hold:

(i) $\gamma(I) \subset B$.

(ii) For any $x \in B$, there is $y \in \gamma(I)$ such that $d(x,y) \leq \alpha$.

If for every $\alpha > 0$ there is an α-dense curve in B, then B is said to be densifiable.

Remark 2.1. If $B := I^N$, $N > 1$, and γ is a 0-dense curve in B, then γ is precisely, a *space-filling curve* in B, i.e. $\gamma(I) = B$ (see [30]). So, we can say that the α-dense curves are a generalization of the so called space-filling curves.

Example 2.1. The cosines curve. For each integer $n \geq 1$, let $\gamma_n : I \longrightarrow \mathbb{R}^N$, $N > 1$, given by

$$\gamma_n(t) := \Big(t, \frac{1}{2}(1 - \cos(n\pi t)), \dots, \frac{1}{2}(1 - \cos(n^{N-1}\pi t))\Big) \quad \text{for all } t \in I,$$

Then γ_n is a $\frac{\sqrt{N-1}}{n}$-dense curve in I^N, see [8, Proposition 9.5.4, p. 144].

Other examples of α-dense curves and its applications can be found in [8, 11, 23, 25, 26] and references therein. As expected, not every bounded set of a Banach space, even compact and connected, is densifiable:

Example 2.2. In the Euclidean plane consider the set

$$B := \big\{(x, \sin(x^{-1})) : x \in [-1,0) \cup (0,1]\big\} \cup \big\{(0,y) : y \in [-1,1]\big\}.$$

Then, given $\alpha > 0$ and any continuous mapping $\gamma : I \longrightarrow \mathbb{R}^2$, if $\gamma(I) \subset B$ then it is contained in some arc-connected component of B. So, taking $0 < \alpha < 1$, it is clear that there is no an α-dense curve in B, and therefore B is not densifiable.

So, the class of densifiable sets is strictly between the class of Peano Continua (i.e. those sets which are the continuous image of I) and the class of connected and precompacts sets. However, we have the following result (see [27]):

Proposition 2.1. *The following properties hold for any $B \in \mathcal{B}(E)$:*

(1) If B is densifiable, then is precompact (i.e., its closure is compact).

(2) If B is precompact and arc-connected, then is densifiable.

3. Main result

The following result holds:

Theorem 3.1. *Let $B \in \mathcal{B}(X)$ and $S, T : B \longrightarrow B$ two continuous mappings satisfying the following conditions:*

(1) $T(B) \subset S(B)$.

(2) $T(B)$ is densifiable and $S^{-1}(T(B))$ is precompact.

Fixed $x_1 \in B$, define the iterative procedure

$$S(x_{n+1}) = y_n, \tag{3.1}$$

where $y_n \in \gamma_n(I)$ is such that $d(y_n, T(x_n)) \leq \alpha_n$, γ_n being, for each $n \geq 1$, an α_n-dense curve in $T(B)$ with $\alpha_n \to 0$. Then, if the sequence $(x_n)_{n\geq 1}$ converges to some $x_0 \in B$, x_0 is a coincidence point of S and T and it is stable. Moreover, we have:

$$|d(S(x_{n+1}), S(x_0)) - d(T(x_n), T(x_0))| \leq \alpha_n \quad \textit{for all } n \geq 1. \tag{3.2}$$

Proof. First at all, note that for each $n \geq 1$ the α_n-dense curves γ_n of the statement and the sequence (3.1) are well defined by conditions (1)-(2). By the continuity of S

$$S(x_0) = \lim_n S(x_{n+1}) = \lim_n y_n. \tag{3.3}$$

Noticing that the continuity of T and (3.3) we find

$$\begin{aligned} &\lim_n d(y_n, T(x_0)) \leq \lim_n [d(y_n, T(x_n)) + d(T(x_n), T(x_0))] \\ &\leq \lim_n [\alpha_n + d(T(x_n), T(x_0))] = 0, \end{aligned} \tag{3.4}$$

So, joining (3.3) and (3.4), we conclude that x_0 is a coincidence point of T and S. Now, let $(S(z_n))_{n\geq 1}$ be any sequence, and assume that $\varepsilon_n := d(S(z_{n+1}), y_n) \to 0$. Putting $p := S(x_0) = T(x_0)$, we have

$$\begin{aligned} &d(S(z_{n+1}), p) \leq d(S(z_{n+1}), y_n) + d(y_n, T(x_n)) + d(T(x_n), T(x_0)) \\ &\leq \varepsilon_n + \alpha_n + d(T(x_n), T(x_0)) \longrightarrow 0, \end{aligned}$$

and therefore, the iterative procedure (3.1) is stable.
On the other hand, fixed $n \geq 1$, assume that $d(S(x_{n+1}), p) \geq d(T(x_n), p)$. Then, we have

$$d(S(x_{n+1}), p) = d(y_n, p) \leq d(y_n, T(x_n)) + d(T(x_n), p) \leq \alpha_n + d(T(x_n), p),$$

and so

$$0 \leq d(S(x_{n+1}), p) - d(T(x_n), p) \leq \alpha_n. \tag{3.5}$$

If $d(T(x_n), p) > d(S(x_{n+1}), p)$, as

$$d(T(x_n), p) \leq d(T(x_n), y_n) + d(y_n, p) \leq \alpha_n + d(S(x_n), p),$$

we deduce that

$$0 < d(T(x_n), p) - d(S(x_{n+1}), p) \leq \alpha_n. \tag{3.6}$$

Then, joining (3.5) and (3.6), the inequality (3.2) holds and the proof is now complete. □

If S^{-1} is continuous, B is closed and $(y_n)_{n\geq 1}$ is converging, then the sequence $(x_n)_{n\geq 1}$ is also converging. For instance, assume that B is closed and $y_n := \gamma_n(t_n)$ for certain sequence $(t_n)_{n\geq 1} \subset I$, and $(t_n)_{n\geq 1}$ is monotone. Then, $(t_n)_{n\geq 1}$ is converging, and therefore $y_n \to y_0$, for some $y_0 \in B$. So, under these conditions, $x_{n+1} = S^{-1}(y_n)$ converges to $S^{-1}(y_0)$.

On the other hand, under some additional conditions, the procedure of Theorem 3.1 is converging, as we show in the next result.

Corollary 3.1. *Let $B \in \mathcal{B}(X)$ and $S, T : B \longrightarrow B$ as in Theorem 3.1. Assume that*

(1) $S(B)$ is complete and S^{-1} is continuous.

(2) There is $r \in (0,1)$ such that $d(T(x), T(y)) \leq r d(S(x), S(y))$ for each $x, y \in B$.

Then, taking $(\alpha_n)_{n\geq 1}$ such that $\sum_{n\geq 1} \alpha_n < \infty$, the iterative procedure of Theorem 3.1 converges to a coincidence point of S and T.

Proof. For each $n > 1$, we have

$$d(S(x_{n+1}), S(x_n)) \leq \alpha_n + d(T(x_n), S(x_n)) \leq \alpha_n + \alpha_{n-1} + d(T(x_n), T(x_{n-1}))$$
$$\leq \alpha_n + \alpha_{n-1} + rd(S(x_n), S(x_{n-1}). \tag{3.7}$$

It is a well known fact (see, for instance, [6]) that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequence of non-negative numbers such that $a_{n+1} \leq b_n + a_n$ and $\sum_{n\geq 1} b_n < \infty$, then the sequence $(a_n)_{n\geq 1}$ converges to some number a. Therefore, putting

$$a_n := d(S(x_n), S(x_{n-1})) \text{ and } b_n := \alpha_n + \alpha_{n-1},$$

from (3.7) we conclude from the inequality

$$0 \leq a = \lim_n a_{n+1} \leq r \lim_n a_n = ra$$

that $\lim_n d(S(x_{n+1}), S(x_n)) = 0$.

Thus, $(S(x_n))_{n\geq 1}$ is a Cauchy sequence in the complete metric space $S(B)$ and, consequently, converges. Of course, as $\lim_n y_n = \lim_n S(x_{n+1})$ we infer that the sequence y_n is also converging and by the continuity of S^{-1}, the sequence $x_{n+1} = S^{-1}(y_n)$ converges. The results follows then by Theorem 3.1. $\square$

It is important to stress that the contraction condition (2) of the above result, or even others more general, are often used to prove the existence of a coincidence point of S and T; see, for instance,[32] and references therein. However, usually, an explicit iterative procedure to approximate such coincidence point is not provided.

4. Application to certain Volterra integral equations

Results on existence and approximation of coincidence points play an important role in the analysis of integral equations, see for instance [3, 5, 16, 29] and references therein. In this section, we will consider the following Volterra integral equation:

$$f(x(t)) = g(t) + \int_0^t K\big(t, s, x(s)\big)ds \quad \text{for all } t \in I, \tag{4.1}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$, $g : I \longrightarrow \mathbb{R}$ and $K : I \times I \times \mathbb{R} \longrightarrow \mathbb{R}$ are known and obey certain conditions that we will describe below. In what follows, E will be the Banach space of continuous functions $x : I \longrightarrow \mathbb{R}$, endowed the usual supremum norm $\|\cdot\|_\infty$.

Let the following conditions:

(C1) The functions f, g and K are continuous.

(C2) There are two numbers $a_K < b_K$ such that $K(t, s, x) \in [a_K, b_K]$ for each $s, t \in I$ and $x \in \mathbb{R}$.

(C3) There is a closed and bounded interval $J := [a_f, b_f] \subset \mathbb{R}$ such that

$$[a_K + \min\{g(t) : t \in I\}, b_K + \max\{g(t) : t \in I\}] \subset f(J) \subset J,$$

and $f : J \longrightarrow f(J)$ is a homeomorphism.

(C4) There is an integrable function $\varphi : I^2 \longrightarrow [0, +\infty)$ such that for each $s, t \in I$ and $x, y \in \mathbb{R}$, $|K(t, s, x) - K(t, s, y)| \leq \varphi(t, s)|f(x) - f(y)|$, and

$$\sup \left\{ \int_0^t \varphi(t, s) ds : t \in I \right\} \in (0, 1).$$

Under the above conditions, we can show that the equation (4.1) has some solution. More specifically:

Proposition 4.1. *Assume that conditions (C1)-(C4) hold and let*

$$C := \{x \in E : a_f \leq x(t) \leq b_f, \forall t \in I\}.$$

Then, there is for each integer $n \geq 1$ an α_n-dense curve in $T(C)$, put γ_n, with $\sum_{n \geq 1} \alpha_n < \infty$, such that fixed $x_1 \in C$ the iteration

$$f(x_{n+1}) = y_n, \tag{4.2}$$

where $y_n \in \gamma_n(I)$ satisfies

$$\left| y_n(t) - g(t) - \int_0^t K(t, s, x_n(s)) ds \right| \leq \alpha_n \quad \textit{for all } t \in I,$$

for each $n \geq 1$, converges to

$$f(x_0(t)) = g(t) + \int_0^t K(t, s, x_0(s)) ds,$$

for some $x_0 \in C$, and is stable. Moreover, for each $n \geq 1$, the following inequality holds for each $t \in I$:

$$\left| |f(x_{n+1}(t)) - f(x_0(t))| - \left| \int_0^t [K(t, s, x_n(s)) - K(t, s, x_0(s))] ds \right| \right| \leq \alpha_n. \tag{4.3}$$

Proof. Let $S, T : E \longrightarrow E$ be the mappings given by:

$$S(x)(t) := f(x(t)), \quad T(x)(t) := g(t) + \int_0^t K(t, s, x(s)) ds,$$

for each $x \in E$, that clearly are well defined and are continuous by condition (C1). Also, a solution for (4.1) is equivalent to the existence of a coincidence point of S and T. Therefore, we will apply Corollary 3.1 to show the existence of such point.

By conditions (C2)-(C3) we find that $T(C) \subset C$, $S(C) \subset C$ and

$$T(C) \subset \{x \in E : a_K + m_1 \leq x(t) \leq b_K + M_1, \forall t \in I\} \subset S(C) \subset C,$$

where $m_1 := \min\{g(t) : t \in I\}$, $M_1 := \max\{g(t) : t \in I\}$. In addition, as S^{-1} is continuous in $S(C)$, $S^{-1}(T(C))$ is precompact.

A direct application of the well known Arzelá-Ascoli theorem, shows that T is a compact mapping, that is to say, $T(B)$ is precompact for each $B \subset C$ non-empty. Then, as $T(C)$ is arc-connected (by the convexity of C) and precompact, by virtue of Proposition 2.1 is densifiable.

On the other hand, given $x, y \in C$, noticing condition (C4), for each $t \in I$ we have:

$$|T(x)(t) - T(y)(t)| \leq \int_0^t \varphi(t,s)|f(x(s)) - f(y(s))|ds$$

$$\leq \|S(x) - S(y)\|_\infty \int_0^t \varphi(t,s)ds,$$

and so, by the arbitrariness of $t \in I$, we conclude that the inequality

$$\|T(x) - T(y)\|_\infty \leq r\|S(x) - S(y)\|_\infty$$

with

$$r := \sup\Big\{\int_0^t \varphi(t,s)ds : t \in I\Big\} \in (0,1)$$

holds for all $x, y \in C$. Therefore, the conditions of Corollary 3.1 are satisfied, and consequently the iteration (4.2) converges to

$$f(x_0(t)) = g(t) + \int_0^t K(t,s,x_0(s))ds$$

for some $x_0 \in C$, and is stable. Moreover, the inequality (4.3) follows from the inequality (3.2) of Theorem 3.1. □

Example 4.1. Consider the following integral equation:

$$\frac{1}{2}x^2(t) = \ln(1+t) + \frac{1}{2}\int_0^t \frac{e^{-st}}{2+ts^2}\cos^2(x^2(s))ds, \quad \text{for all } t \in I. \tag{4.4}$$

With the notation used so far, we have:

$$f(x) := \frac{1}{2}x^2, \quad g(t) := \ln(1+t), \quad K(s,t,x) := \frac{1}{2}\frac{e^{-st}}{2+ts^2}\cos^2(x^2),$$

for each $t, s \in I$ and $x \in \mathbb{R}$. Condition (C1) is clearly satisfied, and taking

$$[a_K, b_K] := [0, 1/2] \text{ and } J := [0, \sqrt{2(1+\ln(2))}] \simeq [0, 1.8401]$$

we have the inclusions

$$[0, \ln(2) + 1/2] \simeq [0, 1.1931] \subset f(J) = [0, 1 + \ln(2)] \simeq [0, 1.6931] \subset J,$$

and therefore, conditions (C2) and (C3) follow. Also, given any $a, b \geq 0$, by the Mean Value Theorem the inequality $|\cos^2(a) - \cos^2(b)| \leq |a - b|$ holds, and so

$$|K(t,s,x) - K(t,s,y)| \leq \frac{1}{2}\frac{e^{-st}}{2+ts^2}|x^2 - y^2| = \frac{e^{-st}}{2+ts^2}|f(x) - f(y)|.$$

Then, condition (C4) is satisfied with $\varphi(t,s) := e^{-st}/(2+ts^2)$. So, the conditions of Proposition 4.1 hold. Furthermore,

$$C := \{x \in E : 0 \leq x(t) \leq \sqrt{2(1+\ln(2))}, \forall t \in I\}.$$

Now, for each $n \geq 1$, let γ_n be an α_n-curve in the set

$$\Big\{\ln(1+t) + \frac{1}{2}\int_0^t \frac{e^{-st}}{2+ts^2}\cos^2(x^2(s))ds : x \in C\Big\},$$

with $\sum_{n\geq 1} \alpha_n < \infty$. Fixed $x_1 \in C$ the iteration (4.2) is given by

$$\frac{1}{2}x_{n+1}^2 = y_n, \tag{4.5}$$

with $y_n \in \gamma_n(I)$ such that

$$|y_n(t) - \ln(1+t) + \frac{1}{2}\int_0^t \frac{e^{-st}}{2+ts^2}\cos^2(x_n^2(s))ds| \leq \alpha_n \quad \text{for all } t \in I.$$

Then, by Proposition 4.1, the iteration (4.5) converges to

$$\frac{1}{2}x_0^2(t) = \ln(1+t) - \frac{1}{2}\int_0^t \frac{e^{-st}}{2+ts^2}\cos^2(x_0^2(s))ds$$

for some $x_0 \in C$, is stable and the inequality (4.3) of Proposition 4.1 holds.

We end the paper noticing that the iterative procedure given in [16, Theorem 12] can not be applied in the above example, as the function $K(t,s,\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ is not increasing for all $t, s \in I$.

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