Fixed Point Theory, 20(2019), No. 1, 177-184
DOI: 10.24193/fpt-ro.2019.1.10
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A FIXED POINT THEOREM IN UNIFORM SPACES GENERATED BY A FAMILY OF b-PSEUDOMETRICS

HAMID FARAJI*, KOUROSH NOUROUZI** AND DONAL O'REGAN***

*Department of Mathematics, Science and Research Branch
Islamic Azad University, Tehran, Iran

**Faculty of Mathematics, K. N.Toosi University of Technology
P.O. Box 16315-1618, Tehran, Iran
E-mail: nourouzi@kntu.ac.ir

***School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway, University Road, Galway, Ireland

Abstract. In this paper, we discuss the existence of fixed points of mappings defined on uniform spaces generated by a family of b-pseudometrics. We also give some sufficient conditions under which the fixed point is unique.
Key Words and Phrases: Uniform spaces, b-pseudometrics, fixed points.
2010 Mathematics Subject Classification: 47H10.

1. Introduction

A b-pseudometric d on a nonempty set X is a function $d : X \times X \to [0, \infty)$ which satisfies the following conditions for all $x, y, z \in X$:

(1) $d(x, x) = 0$
(2) $d(x, y) = d(y, x)$
(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ (b-triangular inequality),

where $s \in [1, \infty)$. Then, the pair (X, d) is called a b-pseudometric space with parameter s. If, in addition, $d(x, y) = 0$ implies that $x = y$, for all $x, y \in X$, then (X, d) is called b-metric space.

Bakhtin [3] and Czerwik [4, 5] introduced the notion of a b-metric space and gave some fixed point theorems for single-valued and multi-valued mappings in b-metric spaces. The notion of a b-metric space was reintroduced by Khamsi and Hussain [8] using the name metric type space. For more information on b-metric spaces the reader is referred to [9] and the related references therein. For some recent progress in b-metric spaces we also refer the reader to [10], [11] and [12].

Uniform spaces play a role between metric and topological spaces. Usually they are used to obtain generalizations of some facts in metric spaces. The reader is referred to the books by Angelov [2] and Heinonen [6] and to the references therein where one can obtain more information on fixed point theory in uniform spaces and analysis on metric spaces. For example in [2] the author considered fixed point theory for Φ-contractions, nonexpansive and expansive maps, Φ-densifying maps, and coincidence theory in uniform spaces.

In [1], Acharya considered the existence and uniqueness of fixed points for mappings defined on uniform spaces satisfying a contractive condition. In this paper, we give a generalization of Theorem 3.1 in [1] using uniform spaces generated by a family of b-pseudometrics.

We start with some notions of uniform spaces (see [13]). A nonempty family $\mathcal{U}$ of subsets of $X \times X$ is called a uniformity on X if the following conditions hold:

U1: $U \in \mathcal{U}$ implies $\{(x,x) \in X \times X : x \in X\} \subset U$;

U2: $U_1, U_2 \in \mathcal{U}$ implies $U_1 \cap U_2 \in \mathcal{U}$;

U3: $U \in \mathcal{U}$ implies that $V \circ V \subset U$ for some $V \in \mathcal{U}$;

U4: $U \in \mathcal{U}$ implies that $V^{-1} = \{(x,y) \in X \times X : (y,x) \in V\} \subset U$, for some $V \in \mathcal{U}$;

U5: If $U \in \mathcal{U}$ and $U \subseteq V$ imply $V \in \mathcal{U}$.

Then, the pair $(X, \mathcal{U})$ is called a uniform space. Members of $\mathcal{U}$ are called entourages. If $\cap \mathcal{U} = \{(x,x) : x \in X\}$, then X is called a separated (Hausdorff) uniform space. Let $\{x_n\}$ be a sequence in a uniform space X. Then $\{x_n\}$ in X is convergent to a point $x \in X$, if for each $U \in \mathcal{U}$, there exists a natural number N such that $(x_n, x) \in U$, for all $n \geq N$. Also $\{x_n\}$ is Cauchy if for each $U \in \mathcal{U}$, there is some $N > 0$ such that $m, n \geq N$ implies $(x_m, x_n) \in U$. The uniform space X is called sequentially complete if each Cauchy sequencein X is convergent to some point of X.

Let $\mathcal{F}$ be a nonempty family of b-pseudometrics with the same parameter $s \geq 1$ on X generating the uniformity $\mathcal{U}$ (See Proposition 8.1.14 in [7]). For $p \in \mathcal{F}$ and $r > 0$, define

$$V_{(p,r)} = \{(x,y) \in X : p(x,y) < r\}.$$

Suppose that $\mathcal{V}$ is the family of all sets of the form

$$\bigcap_{i=1}^{k} V_{(p_i, r_i)},$$

where k is a positive integer, $p_i \in \mathcal{F}$ and $r_i > 0$ for $i = 1, \ldots, k$. It is easy to see that $\mathcal{V}$ is a base for the uniformity $\mathcal{U}$. If $V = \bigcap_{i=1}^{k} V_{(p_i, r_i)} \in \mathcal{V}$ and $\alpha > 0$, we have

$$\alpha V = \bigcap_{i=1}^{k} V_{(p_i, \alpha r_i)} \in \mathcal{V}.$$

We need the following lemma.

Lemma 1.1. [1] *Let X be a uniform space. Then*

(i) If $V \in \mathcal{V}$ and α, β are positive numbers, then $\alpha(\beta V) = (\alpha\beta)V$.

(ii) If $V \in \mathcal{V}$ and $0 < \alpha \leq \beta$, then $\alpha V \subseteq \beta V$.

(iii) If p is a b-pseudometric on X with parameter $s \geq 1$ and α, β are any two positive numbers such that

$$(x,y) \in \alpha V_{(p,r_1)} \circ \beta V_{(p,r_2)},$$

then

$$p(x,y) < s(\alpha r_1 + \beta r_2).$$

(iv) If $V \in \mathcal{V}$ and α, β are positive, then

$$\alpha V \circ \beta V \subset s(\alpha+\beta)V.$$

(v) If $x, y \in X$ and $V \in \mathcal{V}$, then there is a positive number λ such that $(x,y) \in \lambda V$.
(vi) If $V \in \mathcal{V}$, then there is a b-pseudometric p on X such that

$$V = V_{(p,1)}.$$

Proof. We just prove (iii) and (iv). To prove (iii), let

$$(x,y) \in \alpha V_{(p,r_1)} \circ \beta V_{(p,r_2)} = V_{(p,\alpha r_1)} \circ V_{(p,\beta r_2)}.$$

There exists $z \in X$ such that $(x,z) \in V_{(p,\beta r_2)}$ and $(z,y) \in V_{(p,\alpha r_1)}$. Then we have

$$p(x,z) < \beta r_2 \quad , \quad p(z,y) < \alpha r_1.$$

Using the b-triangular inequality, we obtain

$$p(x,y) \leq s\big(p(x,z) + p(y,z)\big) < s(\alpha r_1 + \beta r_2).$$

For (iv), let $V = \bigcap_{i=1}^{k} V_{(p_i,r_i)}$ and $(x,y) \in \alpha V \circ \beta V$ and there exists $z \in X$ such that $(x,z) \in \beta V = \bigcap_{i=1}^{k} V_{(p_i,\beta r_i)}$ and $(z,y) \in \alpha V = \bigcap_{i=1}^{k} V_{(p_i,\alpha r_i)}$. For $i = 1, ..., k$, we have

$$p_i(x,y) \leq s\big(p_i(x,z) + p_i(z,y)\big) < s\big(\beta r_i + \alpha r_i\big).$$

Therefore for $i = 1, ..., k$, we have

$$(x,y) \in V_{(p_i, s(\alpha+\beta)r_i)},$$

Hence

$$(x,y) \in s(\alpha+\beta)\left(\bigcap_{i=1}^{k} V_{(p_i,r_i)}\right) = s(\alpha+\beta)V. \qquad \square$$

2. Acharya Type Theorem

Throughout this section we assume that $(X, \mathcal{U})$ is a uniform space whose uniformity $\mathcal{U}$ is generated by the family $\mathcal{F}$ of b-pseudometrics with the same parameter $s \geq 1$ on X. Furthermore $\mathcal{V}$ is the collection of all sets of the form

$$\bigcap_{i=1}^{k} \Big\{(x,y) \in X \times X : p_i(x,y) < r_i\Big\},$$

where k is a positive integer, $p_i \in \mathcal{F}$ and $r_i > 0$ for $i = 1, \ldots, k$.
Acharya Type Theorem. Let $(X, \mathcal{U})$ be a sequentially complete Hausdorff uniform space and $T : X \to X$ satisfy

$$(Tx, Ty) \in \alpha V \qquad if \qquad (x,y) \in V, \tag{2.1}$$

for all $V \in \mathcal{V}$ and $x, y \in X$, where $0 < \alpha < 1$. Then T has a unique fixed point.

Proof. Choose $n \in \mathbb{N}$ such that
$$\alpha^n < \frac{1}{4s^2}. \tag{2.2}$$
Put $g = T^n$ and for each $m \in \mathbb{N}$ set $x_m = g^m(x_0)$, where x_0 is not a fixed point of T. Suppose that $V = \bigcap_{i=1}^{k} V_{(p_i,\varepsilon_i)} \in \mathcal{V}$, where $p_i \in \mathcal{F}$. By Lemma 1.1 there is a positive number $\lambda > 0$ such that
$$(T^n x_0, x_0) \in \lambda V = W.$$
Using Condition (2.1), we obtain
$$(T^{mn}(T^n x_0), T^{mn} x_0) \in \alpha^{mn} W = \alpha^{mn} \lambda V.$$
Choose m so large that $\alpha^{mn}\lambda < \frac{1}{4s^2}$. Then we have
$$(x_{m+1}, x_m) = (T^{mn}(T^n x_0), T^{mn} x_0) \in \frac{1}{4s^2} V = \frac{1}{4s^2} \bigcap_{i=1}^{k} V_{(p_i,\varepsilon_i)} = \frac{1}{2s} \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)}. \tag{2.3}$$
Since $(x_m, x_m) \in \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)}$, by Condition (2.1) for $n \in \mathbb{N}$, we have
$$(T^n x_m, T^n x_m) \in \alpha^n \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)}.$$
Using (2.2), we obtain
$$(g x_m, x_{m+1}) \in \frac{1}{4s^2} \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)}. \tag{2.4}$$
From (2.3) and (2.4), we have
$$(g x_m, x_m) \in s\left(\frac{1}{2s} + \frac{1}{4s^2}\right) \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)} \subset \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)},$$
Using Condition (2.1) and (2.2) we get
$$(T^n g x_m, T^n x_m) \in \alpha^n \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)} \subseteq \frac{1}{4s^2} \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)}. \tag{2.5}$$
From (2.3) and (2.5), we have
$$(g^2 x_m, x_m) \in s(\frac{1}{2s} + \frac{1}{4s^2}) \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)} \subset \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)}.$$
Similarly, for each $k \in \mathbb{N}$, we have
$$(g^k x_m, x_m) \in s\left(\frac{1}{2s} + \frac{1}{4s^2}\right) \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)} \subset \bigcap_{i=1}^{k} V_{\left(p_i,\frac{\varepsilon_i}{2s}\right)}. \tag{2.6}$$

Now, suppose that $j, t \geq m$. Let $t = m + k_1$ and $j = m + k_2$, for some k_1, k_2. Using (2.6), we obtain

$$(x_m, g^{k_1} x_m) \in \bigcap_{i=1}^{k} V_{\left(p_i, \frac{\varepsilon_i}{2s}\right)},$$

and then

$$(x_m, x_t) \in \bigcap_{i=1}^{k} V_{\left(p_i, \frac{\varepsilon_i}{2s}\right)}. \tag{2.7}$$

Similarly, for $j \geq m$, we can show that

$$(x_m, x_j) \in \bigcap_{i=1}^{k} V_{\left(p_i, \frac{\varepsilon_i}{2s}\right)}. \tag{2.8}$$

Then by (2.7) and (2.8) for $t, j \geq m$, we have

$$(x_j, x_t) \in 2s \bigcap_{i=1}^{k} V_{\left(p_i, \frac{\varepsilon_i}{2s}\right)} = \bigcap_{i=1}^{k} V_{(p_i, \varepsilon_i)} = V. \tag{2.9}$$

This shows that $\{x_n\}$ is a Cauchy sequence. Therefore, there exists x^* in X such that

$$\lim_{n \to \infty} x_n = x^*. \tag{2.10}$$

The continuity of T implies the continuity of g and so

$$x^* = \lim_{m \to \infty} x_m = \lim_{m \to \infty} x_{m+1} = \lim_{n \to \infty} g(x_m) = g(x^*). \tag{2.11}$$

That is, x^* is fixed point of g. Now, let $x \in X$, $V \in \mathcal{V}$ and choose $\lambda > 0$ such that

$$(x^*, x) \in \lambda V.$$

Using Condition (2.1), we get

$$(g^m x^*, g^m x) \in \alpha^{mn} \lambda V.$$

Choose m so large such that $\alpha^{mn}\lambda < 1$. Then, we have

$$(g^m x^*, g^m x) \in V.$$

Since x^* is fixed point g, we have

$$(x^*, g^m x) \in V. \tag{2.12}$$

Since $V \in \mathcal{V}$ and $x \in X$ were arbitrary, by (2.12) for all $x \in X$ we obtain

$$g^m x \to x^*. \tag{2.13}$$

However, by the continuity of T and using (2.10) and (2.13) we have

$$T(x^*) = \lim_{m \to \infty} T(x_m) = \lim_{m \to \infty} T(g^m(x_0)) = \lim_{m \to \infty} g^m(T(x_0)) = x^*. \tag{2.14}$$

Therefore x^* is the fixed point of T. We claim that x^* is the unique fixed point of T. Let $Ty^* = y^*$. Take any $V \in \mathcal{V}$ and choose $\lambda > 0$ such that

$$(x^*, y^*) \in \lambda V.$$

Then

$$(x^*, y^*) = (Tx^*, Ty^*) \in \alpha \lambda V.$$

Therefore after n steps, we get

$$(x^*, y^*) = (T^n x^*, T^n y^*) \in \alpha^n \lambda V.$$

Choose n so large that $\alpha^n \lambda < 1$. Then

$$(x^*, y^*) \in V.$$

Since V was arbitrary, it follows that $x^* = y^*$. □

Corollary 2.1. *Let $(X, \mathcal{U})$ be a sequentially complete Hausdorff uniform space and $T : X \to X$ satisfy*

$$(T^n x, T^n y) \in \alpha_n V \qquad if \qquad (x, y) \in V \tag{2.15}$$

for all $V \in \mathcal{V}, n \geq 1$ and $x, y \in X$, where $\alpha_n \to 0$. Then T has a unique fixed point.

Proof. Let $0 < \alpha < \frac{1}{s}$. Since $\alpha_n \to 0$, choose $n \in \mathbb{N}$ so that $\alpha_n < \alpha$. If $V \in \mathcal{V}$ and $(x, y) \in V$, we have $(T^n x, T^n y) \in \alpha_n V$ and thus $(T^n x, T^n y) \in \alpha V$. Therefore by our Acharya Type Theorem T^n has a unique fixed point z. That is $T^n(z) = z$ and $T^n(T(z)) = T(z)$. This implies that $T(z) = z$. □

In the following corollary by $F(T)$ we mean the set of all fixed points of T.

Corollary 2.2. *Let $(X, \mathcal{U})$ be a sequentially complete Hausdorff uniform space and $T : X \to X$ satisfy (2.1). Then, $F(T) = F(T^n)$ for each $n \in N$.*

Proof. By our Acharya Type Theorem, $F(T^n)$ is nonempty. Assume that $z \in F(T^n)$ for some $n > 1$. Suppose that $z \neq Tz$ and $V \in \mathcal{V}$ be arbitrary. By Lemma 1.1, there exists a positive number $\lambda > 0$ such that $(z, Tz) \in \lambda V = W$. Using Condition (2.1), we have $(T^n z, T^{n+1} z) \in \alpha^n W$. Since $z \in F(T^n)$, we have

$$(z, Tz) \in \alpha^n W = \alpha^n \lambda V.$$

Therefore after m steps, we get

$$(z, Tz) \in \alpha^{mn} \lambda V.$$

Choose m so large that $\alpha^{mn} \lambda < 1$. Then $(z, Tz) \in V$. Since V was arbitrary, it follows that $z = Tz$ which is a contraction. Therefore $z \in F(T)$. □

References

[1] S.P. Acharya, *Some results on fixed points in uniform spaces*, Yokohama Math. J., **22**(1974), 105-116.

[2] V. Angelov, *Fixed Points in Uniform Spaces and Applications*, Cluj University Press, 2009.

[3] I.A. Bakhtin, *The contraction mapping principle in almost metric space, Functional analysis*, (Russian), Ulýanovsk. Gos. Ped. Inst., Ulýanovsk, (1989), 26-37.

[4] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1**(1993), 5-11.

[5] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena, **46**(1998), 263-276.

[6] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Universitext, Springer-Verlag, New York, 2001.

[7] K.D. Joshi, *Introduction to General Topology*, John Wiley Sons, Inc., 1983.

[8] M.A. Khamsi, N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal., **73**(2010), no. 9, 3123-3129.
[9] W. Kirk, N. Shahzad, *Fixed Point Theory in Distance Spaces*, Springer, Cham, 2014.
[10] A. Petruşel, G. Petruşel, B. Samet, J.-C. Yao, *Coupled fixed point theorems for symmetric multi-valued contractions in b-metric space with applications to systems of integral inclusions*, J. Nonlinear Convex Anal., **17**(2016), no. 7, 1265-1282.
[11] A. Petruşel, G. Petruşel, B. Samet, J.-C. Yao, *Coupled fixed point theorems for symmetric contractions in b-metric spaces with applications to operator equation systems*, Fixed Point Theory, **17**(2016), no. 2, 457-475.
[12] W. Sintunavarat, *Nonlinear integral equations with new admissibility types in b-metric spaces*, J. Fixed Point Theory Appl., **18** (2016), no. 2, 397-416.
[13] S. Willard, *General Topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.

Received: September 6, 2016; Accepted: November 11, 2016.