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FIXED POINT APPROACH TO THE STABILITY OF GENERALIZED POLYNOMIALS

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Dedicated to Professor Ioan A. Rus

Abstract. Using a new fixed point theorem for linear operators which act on function spaces, we give an iterative method for proving the generalized stability in three essential cases and the hyperstability for polynomial equation $\Delta_y^{n+1} f(x) = 0$ on commutative monoids.The proposed iterative fixed point method leads to final concrete unitary estimates, and also improves and complements the known stability results for generalized polynomials.

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1. Introduction

Over the last twenty-five years, the stability theory of functional equations has developed in two main directions. The first direction concerns some refinements and, on the other hand, generalizations of the original concept of stability (introduced by Hyers in [10] as an answer to a question posed by S.M. Ulam about the stability of the group homomorphisms). The second one involves the stability's proof techniques.

The vast majority of stability theorems require two proof techniques: the *direct method* - firstly applied by Hyers in [10] for proving the stability of Cauchy's functional equation on Banach spaces -, and the *fixed point method* - firstly applied by Baker, who solve a stability problem using a variant of Banach's fixed point theorem (see [4]).

Our paper is included in the second direction and analyses the well-known polynomial equation with differences

$$\Delta_y^{n+1} f(x) = 0, \tag{1.1}$$

equation with various applications in many branches of mathematics and applied sciences.

The first stability result on commutative semigroups for this equation was obtained by Albert and Baker in [1], through the direct method, as an extension of Hyers' result from [10]. The generalized stability for equation (1.1) has been proved in

2014 on commutative $n!-$ divisible groups, via the generalized stability of Fréchet's polynomial equation

$$\Delta_{x_1} \circ \cdots \circ \Delta_{x_{n+1}} f(0) = 0, \tag{1.2}$$

using the direct method, too (see [8] and [9]).

Here we firstly prove a fixed point theorem for linear operators which act on function spaces. We use this result to prove the generalized stability in three essential cases and the hyperstability for equation (1.1) on commutative monoids. Also, in each case, we give consequences using control functions of Aoki-Rassias type. We note that the proposed iterative fixed point method leads to final concrete and unitary estimates.

A new proof of Fréchet's functional characterization of real polynomials, but using equation (1.1) is given in [2]. Applications of this equation to spectral synthesis can be found in [15]. A perspective on the current state of the functional equations stability theory is the handbook [14]. Different types of stability for functional equations as well as the interlinked results are discussed in [5]. Applications of different fixed point theorems to the theory of stability of functional equations can be found in [6]. A fixed point technique for solving the generalized stability of equation (1.2) for $n = 2$ is available in [3]. Finally, an overview of stability results regarding the equations (1.1) and (1.2) - including their equivalence - can be found in [8].

2. Preliminaries

In the following lines A denotes a commutative monoid, n is a positive integer, B is a Banach space, B^A is the linear space of the functions $A \to B$, $\mathbb{R}$ is the set of real numbers, $\mathbb{N}$ is the set of natural numbers $0, 1, 2, ...$ and i, j, k, s denote natural numbers.

We remember that for all $f \in B^A$ and all $x, y \in A$

- $\Delta_y^n f(x) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x + iy);$
- $\Delta_y^n f(x + jy) = \Delta_y^n f(x) + \sum_{i=0}^{n-1} \Delta_y^{n+1} f(x + iy)$ for all $j \in \mathbb{N} \backslash \{0\};$
- $\Delta_{(j+1)y}^n f(x) = \sum_{i=0}^{jn} (j)_i^n \cdot \Delta_y^n f(x + iy)$ for all $j \in \mathbb{N}$, where $(j)_i^n$ denotes the coefficient of α^i from $\left(1 + \alpha + \cdots + \alpha^j\right)^n$ - *Marchoud's formula* (see [12], p. 368), and from this:
- $\sum_{i=0}^{jn} (j)_i^n = (1 + j)^n$ for all $j \in \mathbb{N}$, and
- $\Delta_{2y}^n f(x) - 2^n \Delta_y^n f(x) = \sum_{s=1}^{n} \binom{n}{s} \sum_{i=1}^{s} \Delta_y^{n+1} f(x + (i - 1)y).$

The function $p \in B^A$ is an *n-polynomial* (or, by abuse of language, *Fréchet polynomial* of degree less then or equals n) if and only if $\Delta_y^{n+1} p(x) = 0$ for all $x, y \in A$

or, equivalent,

$$p = \sum_{i=0}^{n} m_i,$$

where $m_i \in B^A$ is an i-*monomial*, i.e.

$$\Delta_y^i m_i(x) = i! \cdot m_i(y)$$

for all $x, y \in A$. We denote $\mathcal{P}_n(A,B)$ the linear space of all n-polynomials from B^A, and $\mathcal{M}_i(A,B)$ the linear space of all i-monomials from B^A, where $i \in \mathbb{N}$.

If $m_n \in \mathcal{M}_n(A,B)$ then

$$m_n(kx) = k^n m_n(x)$$

for all $x \in A$ and all $k \in \mathbb{N}$. Also, if $m \in \mathcal{P}_n(A,B)$, $i \in \{1,...,n\}$ and $m(kx) = k^i m(x)$ for all $x \in A$ and all $k \in \mathbb{N}$, then $m \in \mathcal{M}_i(A,B)$.

For details on (generalized) polynomials and monomials see [8] and the papers referred there.

We shall use the following elementary facts about the control functions.

Lemma 2.1. ([8]) *Let $\varphi' \in \mathbb{R}_+^A$ and $i \in \mathbb{N}$ such that*

$$\varphi_i(y) := \sum_{k=0}^{\infty} 2^{-i(k+1)} \varphi'\left(2^k y\right) < \infty$$

for all $y \in A$. Then, for all $y \in A$

$$\sum_{k=0}^{\infty} 2^{-(i+1)(k+1)} \varphi_i\left(2^k y\right) \le 2^{-i} \varphi_i(y) \quad \text{and} \quad \lim_{k\to\infty} 2^{-ik} \varphi_i\left(2^k y\right) = 0.$$

Lemma 2.2. ([9]) *Let A be a 2-divisible Abelian monoid, $\varphi' \in \mathbb{R}_+^A$ and $i \in \mathbb{N}$ such that $\widetilde{\varphi}_i(y) := \sum_{k=0}^{\infty} 2^{ik} \varphi'\left(2^{-k-1} y\right) < \infty$ for all $y \in A$. Then*

$$\sum_{k=0}^{\infty} 2^{(i-1)k} \widetilde{\varphi}_i\left(2^{-k-1} y\right) \le 2\widetilde{\varphi}_i\left(2^{-1} y\right) \quad \text{and} \quad \lim_{k\to\infty} 2^{ik} \widetilde{\varphi}_i\left(2^{-k} y\right) = 0$$

for all $y \in A$.

3. Fixed point theorem for $J-$contractive linear operators

In the following lines we give a fixed point theorem for linear J-contractive operators in the spirit of [7].

We remember that, if $L : B^A \to B^A$ and $J : \mathbb{R}^A \to \mathbb{R}^A$ are linear operators, then L is J-*contractive* iff

$$[h \in B^A, \delta \in \mathbb{R}^A \text{ and } \|h(y)\| \le \delta(y) \text{ for all } y \in A]$$
$$\Rightarrow [\|(Lh)(y)\| \le (J\delta)(y) \text{ for all } y \in A].$$

Theorem 3.1. *Let $J : \mathbb{R}^A \to \mathbb{R}^A$ be a linear operator, $L : B^A \to B^A$ be a linear J-contractive operator, $\alpha \in \mathbb{R}_+^A$ and $g \in B^A$ such that*

(*i*) $\overline{\alpha}(y) := \sum_{k=0}^{\infty} \left(J^k \alpha\right)(y) < \infty$ *and* $(J\overline{\alpha})(y) = \sum_{k=0}^{\infty} \left(J^{k+1} \alpha\right)(y)$,

(ii) $\|g(y) - (Lg)(y)\| \leq \alpha(y)$
for all $y \in A$. *Then*

$$m(y) := \lim_{k \to \infty} \left(L^k g\right)(y) \tag{3.1}$$

defines the unique fixed point of the operator L *for which*

$$\|g(y) - m(y)\| \leq \overline{\alpha}(y) \text{ for all } y \in A. \tag{3.2}$$

Proof. 1. Let $y \in A$. Since L is J-contractive linear operator, from (ii) we have

$$\left\|\left(L^k g\right)(y) - \left(L^{k+1} g\right)(y)\right\| \leq \left(J^k \alpha\right)(y) \text{ for all } k \in \mathbb{N} \text{ and } y \in A.$$

Using (i) we obtain

$$\left\|\left(L^k g\right)(y) - \left(L^{k+s+1} g\right)(y)\right\| \leq \sum_{i=k}^{k+s} \left(J^i \alpha\right)(y) \leq \sum_{i=k}^{\infty} \left(J^i \alpha\right)(y) \tag{3.3}$$

for all $k, s \in \mathbb{N}$. Therefore $\left(\left(L^k g\right)(y)\right)_{k \geq 0}$ is a Cauchy sequence in the Banach space B. Let $m : A \to B$ be the function defined by (3.1). Letting $k = 0$ and $s \to \infty$ in (3.3) we obtain immediately (3.2).

2. Let $y \in A$ and $s \to \infty$ in (3.3). Then

$$\left\|\left(L^k g\right)(y) - m(y)\right\| \leq \sum_{i=k}^{\infty} \left(J^i \alpha\right)(y).$$

Since L is a J-contractive linear operator and J is linear, using (i) we obtain

$$\left\|\left(L^{k+1} g\right)(y) - (Lm)(y)\right\| \leq \sum_{i=k}^{\infty} \left(J^i \alpha\right)(y) \to 0 \text{ when } k \to \infty.$$

Therefore $Lm = m$, i.e. m is a fixed point of L which verifies (3.2).

3. It remains to show that m is the only fixed point of L which satisfies (3.2). First, we remark that, from (i) it follows

$$\overline{\alpha} = \alpha + J\overline{\alpha} \text{ and } J^i \overline{\alpha} = J^i \alpha + J^{i+1} \overline{\alpha}$$

for all $i \in \mathbb{N}$. Therefore $\overline{\alpha} = J^{k+1}\overline{\alpha} + \sum_{i=0}^{k} J^i \alpha$ for all $k \in \mathbb{N}$ and

$$\lim_{k \to \infty} \left(J^k \overline{\alpha}\right)(y) = 0 \text{ for all } y \in A. \tag{3.4}$$

Let m' be a fixed point of L which verify (3.2). Then $\|m'(y) - m(y)\| \leq 2\overline{\alpha}(y)$ for all $y \in A$. Since m and m' are fixed points of the J-contractive operator L, from (3.4) we have

$$\|m'(y) - m(y)\| = \left\|\left(L^k m'\right)(y) - \left(L^k m\right)(y)\right\| \leq 2\left(J^k \overline{\alpha}\right)(y) \to 0$$

when $k \to \infty$. Therefore $m' = m$.

4. The iterative fixed point method

The fundamental problem of generalized stability for an equation is to find control functions φ for which the equation is φ-stable (see, for instance, [5] or [8]). In our case, equation (1.1) is φ-stable, where $\varphi : A \times A \to [0,\infty)$ is a function, iff there exists a function $\Psi : A \to [0,\infty)$ such that for every function $f : A \to B$ for which $\left\|\Delta_y^{n+1} f(x)\right\| \leq \varphi(x,y)$ for all $x, y \in A$ there exists a polynomial $p \in \mathcal{P}_n(A,B)$ such that $\|f(y) - p(y)\| \leq \Psi(y)$ for all $y \in A$.

The central idea of our method is to prove that, starting with a function $f : A \to B$ which verify approximately equation (1.1), the procedure

$$f_n := f,\ \ m_0(y) := f(0)\,, \text{ and for } i = n, n-1, ..., 1:$$

$$m_i(y) := \lim_{k\to\infty} 2^{-ik}\Delta^i_{2^k y} f_i(0)\,,\ \ f_{i-1} := f_i - \frac{1}{i!}m_i \tag{4.1}$$

defines the monomials $m_i \in \mathcal{M}_i(A,B)$ such that the $n-$polynomial $p := \sum_{i=0}^{n} \frac{1}{i!} m_i$ to approximate the function f.

In the following lines we use the coefficients

$$c'_{n+1} = \pi_{n+1} = 1,\ \ \pi_i = (3i-2)\,\pi_{i+1},\ \ c_i = i2^{i-1}\pi_{i+1},\ \ \text{and}\ \ c'_i = 2^{i-1}\pi_i$$

for $i = n, n-1, ..., 1$. Also, if $\varphi : A \times A \to [0,\infty)$ is a function, we define $\varphi' : A \to [0,\infty)$ by

$$\varphi'(y) := \max\left\{\varphi(jy, y)\,|\,j \in \{0, 1, ..., n-1\}\right\}.$$

Theorem 4.1. *Let $i_0 \in \{1, ..., n\}$, $\varphi : A \times A \to [0,\infty)$, and $f : A \to B$ such that*

$$\varphi_{i_0}(y) := \sum_{k=0}^{\infty} 2^{-i_0(k+1)}\varphi'\left(2^k y\right) < \infty, \tag{4.2}$$

$$\lim_{k\to\infty} 2^{-i_0 k}\varphi\left(2^k x, 2^k y\right) = 0 \tag{4.3}$$

$$\left\|\Delta_y^{n+1} f(x)\right\| \leq \varphi(x,y) \tag{4.4}$$

for all $x, y \in A$. Then procedure (4.1) defines $m_i \in \mathcal{M}_i(A,B)$ for $i = n, n-1, ..., i_0$, and

$$\left\|\Delta_y^{i_0} f_{i_0-1}(0)\right\| \leq c_{i_0}\varphi_{i_0}(y) \tag{4.5}$$

for all $y \in A$. Moreover, if $i_0 = 1$ then $p := \sum_{i=0}^{n} \frac{1}{i!} m_i$ is the unique $n-$polynomial for which $p(0) = f(0)$ and

$$\|f(y) - p(y)\| \leq \varphi_1(y) \prod_{i=1}^{n} (3i-2) \tag{4.6}$$

for all $y \in A$.

Proof. Let $\varphi_{n+1} := \varphi'$ and, if $i \in \{i_0, i_0+1, ..., n\}$

$$\varphi_i(y) := \sum_{k=0}^{\infty} 2^{-i(k+1)}\varphi'\left(2^k y\right)$$

for all $y \in A$. We remark that $\varphi_i(y) = 2^{-i}\varphi'(y) + 2^{-i}\varphi_i(2y)$, hence

$$\varphi'(y) \leq 2^i \varphi_i(y) \text{ and } \varphi_i(2y) \leq 2^i \varphi_i(y) \tag{4.7}$$

for all $y \in A$. Also, if $n > 1$ and $i \geq i_0 + 1$, from Lemma 2.1 we have

$$\sum_{k=0}^{\infty} 2^{-(i-1)(k+1)} \varphi_i\left(2^k y\right) = \sum_{k=0}^{\infty} 2^{-i(k+1)} \varphi_{i-1}\left(2^k y\right) \leq 2^{-(i-1)} \varphi_{i-1}(y). \tag{4.8}$$

1. We prove, by reverse induction on $i = n, n-1, ..., i_0$, the following three assertions:

$$\left\|\Delta_y^{i+1} f_i(jy)\right\| \leq c'_{i+1} \varphi_{i+1}(y) \text{ for } j \in \{0, 1, ..., i-1\}, \tag{4.9}$$

$$m_i(y) := \lim_{k\to\infty} 2^{-ik} \Delta_{2^k y}^i f_i(0) \text{ defines } m_i \in \mathcal{M}_{\rangle}(A, B), \tag{4.10}$$

$$\left\|\Delta_y^i f_{i-1}(0)\right\| \leq c_i \varphi_i(y) \tag{4.11}$$

for all $y \in A$, where $f_n := f$, and $f_{i-1} := f_i - \frac{1}{i!} m_i$ (as in procedure (4.1)).

1.1. Let $i = n$ and $y \in A$. From (4.4) we have

$$\left\|\Delta_y^{n+1} f_n(jy)\right\| \leq \varphi(jy, y) \leq \varphi'(y) = c'_{n+1} \varphi_{n+1}(y) \tag{4.12}$$

for $j \in \{0, 1, ..., n-1\}$ and (4.9) is proved.

For proving (4.10) let first remark that

$$\Delta_{2y}^n f_n(0) - 2^n \Delta_y^n f_n(0) = \sum_{s=1}^{n} \binom{n}{s} \sum_{i=1}^{s} \Delta_y^{n+1} f_n((i-1)y);$$

from (4.12), and because $\sum_{s=1}^{n} s \binom{n}{s} = n2^{n-1}$ we have

$$\left\|\Delta_y^n f_n(0) - 2^{-n} \Delta_{2y}^n f_n(0)\right\| \leq n2^{-1} \varphi'(y). \tag{4.13}$$

Now, we are able to introduce our fixed point method, using Theorem 3.1 for

$$\begin{aligned} (Lh)(y) :&= 2^{-n} h(2y), \quad (J\delta)(y) := 2^{-n} \delta(2y), \\ \alpha(y) :&= n2^{-1} \varphi'(y), \quad \text{and} \quad g(y) := \Delta_y^n f_n(0). \end{aligned}$$

Of course, J is a linear operator and L is a linear $J-$contractive operator. From (4.2) it follows that $\overline{\alpha}$ is well defined, because

$$\overline{\alpha}(y) = \sum_{k=0}^{\infty} 2^{-nk} \alpha\left(2^k y\right) = n2^{-1} \sum_{k=0}^{\infty} 2^{-nk} \varphi'\left(2^k y\right) \leq n2^{-1} \varphi_{i_0}(y),$$

and, since

$$(J\overline{\alpha})(y) = 2^{-n} \overline{\alpha}(2y) = \sum_{k=0}^{\infty} 2^{-n(k+1)} \alpha\left(2^{k+1} y\right) = \sum_{k=0}^{\infty} \left(J^{k+1} \alpha\right)(y),$$

condition (i) is verified. Moreover, (4.13) is exactly (ii). Therefore, from Theorem 3.1, it follows that

$$m_n(y) := \lim_{k\to\infty} 2^{-nk} \Delta_{2^k y}^n f_n(0) \tag{4.14}$$

defines $m_n \in B^A$, and

$$\left\|\Delta_y^n f_n(0) - m_n(y)\right\| \le c_n \varphi_n(y) \tag{4.15}$$

for all $y \in A$.

For proving that m_n is an $n-$monomial we show first that

$$m_n(y) = \lim_{k\to\infty} 2^{-nk} \Delta_{2^k y}^n f_n\left(2^k s y\right) \tag{4.16}$$

for all $s \in \mathbb{N}$. For $s = 0$ relation (4.16) is exactly (4.14). Let $s > 0$. From (4.3) and (4.4) it follows that

$$\lim_{k\to\infty} 2^{-nk} \Delta_{2^k y}^{n+1} f_n\left(2^k j y\right) = 0$$

for $j \in \mathbb{N}$. Therefore, from (4.14) and

$$\Delta_y^n f_n(sy) = \Delta_y^n f_n(0) + \sum_{j=0}^{s-1} \Delta_y^{n+1} f_n(jy)$$

it follows (4.16).

Secondly, we prove that

$$m_n(jy) = j^n m_n(y)\,, \text{ for } j \in \mathbb{N} \tag{4.17}$$

Since $\Delta_0^n = 0$ we have $m_n(0) = 0$. Applying (4.16), Marchaud's formula, and taking into account the relation $\sum_{i=0}^{jn} (j)_i^n = (j+1)^n$ we have for $j \in \mathbb{N}:$

$$\begin{aligned} m_n((j+1)y) &= \lim_{k\to\infty} 2^{-nk} \Delta_{2^k(j+1)y}^n f_n(0) \\ = \lim_{k\to\infty} 2^{-nk} \sum_{i=0}^{jn} (j)_i^n \, \Delta_{2^k y}^n f_n\left(i 2^k y\right) &= \sum_{i=0}^{jn} (j)_i^n \, m_n(y) = (j+1)^n \, m_n(y)\,, \end{aligned}$$

and (4.17) is proved.

Taking into account (4.17), for proving that $m_n \in \mathcal{M}_n(A,B)$ it is sufficient to show that m_n is an $n-$polynomial. Let $x, y \in A$. Then

$$\Delta_y^{n+1} m_n(x) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} m_n(x+jy)$$

$$= \lim_{k\to\infty} 2^{-nk} \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \Delta_{2^k(x+jy)}^n f_n(0)$$

$$= \lim_{k\to\infty} 2^{-nk} \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} f_n\left(2^k s x + j 2^k s y\right)$$

$$= \lim_{k\to\infty} 2^{-nk} \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} \Delta_{2^k s y}^{n+1} f_n\left(2^k s x\right);$$

but, from (4.3) and (4.4) we have

$$0 \le \lim_{k\to\infty} 2^{-nk} \left\|\Delta_{2^k s y}^{n+1} f_n\left(2^k s x\right)\right\| \le \lim_{k\to\infty} 2^{-nk} \varphi\left(2^k s x, 2^k s y\right) = 0,$$

hence $\Delta_y^{n+1} m_n(x) = 0$ for all $x, y \in A$., i.e. $m_n \in \mathcal{M}_n(A,B)$ and (4.10) is proved.

It remains to prove (4.11). Since

$$f_{n-1} := f_n - \frac{1}{n!} m_n,$$

we have $\Delta_y^n f_{n-1}(0) = \Delta_y^n f_n(0) - m_n(y)$ and (4.15) becomes (4.11).

1.2. Suppose that $n > 1$ and (4.9), (4.10), and (4.11) are true for $i \in \{i_0+1, ..., n\}$. For proving the three assertions for $i-1$, we work similarly as in the first part.

a. First we show that for all $y \in A$

$$\left\| \Delta_y^i f_{i-1}(jy) \right\| \le c_i' \varphi_i(y) \text{ for all } j \in \{0, ..., i-2\}. \tag{4.18}$$

Since $c_i \le c_i'$ from (4.11) it follows (4.18) for $j = 0$. Since $m_i \in \mathcal{M}_i(A,B)$, we have $\Delta^{i+1} f_{i-1} = \Delta^{i+1} f_i$, and, for $j \ge 1$ and $y \in A$ we have

$$\Delta_y^i f_{i-1}(jy) = \Delta_y^i f_{i-1}(0) + \sum_{s=0}^{j-1} \Delta_y^{i+1} f_i(sy).$$

Therefore, using (4.11), (4.9), and the inequality

$$\varphi_{i+1}(y) \le \varphi_i(y),$$

we have for $j \le i-2$

$$\left\| \Delta_y^i f_{i-1}(jy) \right\| \le c_i \varphi_i(y) + (i-1) c_{i+1}' \varphi_{i+1}(y) \le 2^{i-1}(3i-2) \pi_{i+1} \varphi_i(y),$$

and (4.18) is proved.

b. As in the first part for proving that

$$m_{i-1}(y) := \lim_{k \to \infty} 2^{-(i-1)k} \Delta_{2^k y}^{i-1} f_{i-1}(0) \text{ defines } m_{i-1} \in \mathcal{M}_{i-1}(A,B) \tag{4.19}$$

we use the formula

$$\Delta_{2y}^{i-1} f_{i-1}(0) - 2^{i-1} \Delta_y^{i-1} f_{i-1}(0) = \sum_{s=1}^{i-1} \binom{i-1}{s} \sum_{j=1}^{s} \Delta_y^i f_{i-1}((j-1)y),$$

and (4.18). We obtain immediately

$$\left\| \Delta_y^{i-1} f_{i-1}(0) - 2^{-(i-1)} \Delta_{2y}^{i-1} f_{i-1}(0) \right\| \le c_{i-1} \varphi_i(y) \tag{4.20}$$

for all $y \in A$. Taking in Theorem 3.1

$$\begin{aligned} (Lh)(y) : &= 2^{-(i-1)} h(2y), \quad (J\delta)(y) := 2^{-(i-1)} \delta(2y), \\ \alpha(y) : &= c_{i-1} \varphi_i(y), \quad \text{and} \quad g(y) := \Delta_y^{i-1} f_{i-1}(0), \end{aligned}$$

from (4.20) it follows that the limit from (4.19) defines a function $m_{i-1} : A \to B$ and

$$\left\| \Delta_y^{i-1} f_{i-1}(0) - m_{i-1}(y) \right\| \le \overline{\alpha}(y);$$

but

$$\overline{\alpha}(y) = c_{i-1} \sum_{k=0}^{\infty} 2^{-(i-1)(k+1)} \varphi_i\left(2^k y\right),$$

and from (4.8) it follows that $\overline{\alpha}(y) \le 2^{1-i} c_{i-1} \varphi_{i-1}(y)$, hence

$$\left\| \Delta_y^{i-1} f_{i-1}(0) - m_{i-1}(y) \right\| \le c_{i-1} \varphi_{i-1}(y) \tag{4.21}$$

for all $y \in A$. Following the technique used in the first part we obtain successively:

$$m_{i-1}(jy) := \lim_{k\to\infty} 2^{-(i-1)k} \Delta_{2^k y}^{i-1} f_{i-1}\left(2^k sy\right) \text{ for all } s \in \mathbb{N},$$

$$m_{i-1}(jy) = j^{i-1} m_{i-1}(y), \text{ for all } j \in \mathbb{N},$$
$$\Delta_y^{n+1} m_{i-1}(x) = 0 \text{ for all } x, y \in A.$$

Therefore m_{i-1} is an $(i-1)-$monomial and (4.21) becomes exactly (4.11) for $i-1$.

2. For $i = i_0$ from (4.11) we obtain immediately (4.5).

3. Let $i_0 = 1$. Since

$$\Delta_y^1 f_0(0) = f_0(y) - f_0(0), c_1 = \prod_{i=1}^{n} (3i-2), \text{ and } f_0 = f - \sum_{j=0}^{n} \frac{1}{j!} m_j,$$

where $m_0(y) := f(0) = f_0(0)$, from (4.5) it follows that $p := \sum_{j=0}^{n} \frac{1}{j!} m_j$ verifies (4.6).

4. Uniqueness. Suppose that $\sum_{j=0}^{n} m'_j$ is an $n-$ polynomial that verifies (4.6), where $m'_j \in \mathcal{M}_j(A,B)$ for $j \in \{0,...,n\}$, $m'_0(y) := f(0)$. Then for all $y \in A$

$$\left\| f(y) - \sum_{j=0}^{n} m'_j(y) \right\| \le c_1 \varphi_1(y)$$

and from (4.6) we obtain

$$\left\| \sum_{j=1}^{n} m'_j(y) - \frac{1}{j!} m_j(y) \right\| \le 2c_1 \varphi_1(y). \tag{4.22}$$

From Lemma 2.1 we have $\lim_{k\to\infty} 2^{-k} \varphi_1\left(2^k y\right) = 0$, and from (4.21) (replacing y with $2^k y$) it follows

$$\lim_{k\to\infty} 2^{-nk} \left\| \sum_{j=1}^{n} 2^{jk} \left(m'_j(y) - \frac{1}{j!} m_j(y) \right) \right\| = 0.$$

Therefore $m'_n = \frac{1}{n!} m_n$ and (4.21) becomes

$$\left\| \sum_{j=1}^{n-1} m'_j(y) - \frac{1}{j!} m_j(y) \right\| \le 2c_1 \varphi_1(y).$$

By reverse induction we finally obtain $m'_i = \frac{1}{i!} m_i$ for all $i \in \{0,...,n\}$.

An alternative procedure to find the unique $p \in \mathcal{P}_n(A,B)$ which verifies (4.6) is the following.

Corollary 4.2. *Let φ and f as in Theorem 4.1 and $i_0 = 1$. Then the unique $p \in \mathcal{P}_n(A,B)$ which verifies (4.6) and $p(0) = f(0)$ has the form $p := \sum_{i=0}^{n} m'_i(y)$, where*

$m_i' \in \mathcal{M}_i(A,B)$ can be find with the procedure

$$f_0' :\, = f,\ m_0'(y) :\, = f(0),\ \text{ and for } i = 1,2,...,n\text{:}$$
$$f_i' :\, = f_{i-1}' - m_{i-1}',\ \text{ and } m_i'(y) :\, = \lim_{k\to\infty} 2^{-ik} f_i'\left(2^k y\right). \tag{4.23}$$

Proof. We prove by induction on $i \in \{0,1,...,n\}$ that $m_i' = \frac{1}{i!}m_i$. By hypotesis $m_0' = m_0$. Suppose that $i \in \{0,1,...,n-1\}$ and $m_j' = \frac{1}{j!}m_j$ for $j \in \{0,...,i\}$. Then

$$f_{i+1}' = f - \sum_{j=0}^{i} m_j'$$

and (4.6) becomes

$$\left\| f_{i+1}'(y) - \sum_{j=i+1}^{n} \frac{1}{j!} m_j(y) \right\| \le c_1 \varphi_1(y)$$

for all $y \in A$. Then, for all $y \in A$ and $k \in \mathbb{N}$:

$$\left\| 2^{-(i+1)k} f_{i+1}'\left(2^k y\right) - \sum_{j=i+1}^{n} \frac{1}{j!} 2^{(j-i-1)k} m_j(y) \right\| \le 2^{-(i+1)k} c_1 \varphi_1\left(2^k y\right).$$

But, from Lemma 2.1 we have $\lim_{k\to\infty} 2^{-k}\varphi_1\left(2^k y\right) = 0$. Therefore

$$\lim_{k\to\infty} 2^{-(i+1)k} f_{i+1}'\left(2^k y\right) = \frac{1}{(i+1)!} m_{i+1}(y)$$

for all $y \in A$ and $m_{i+1}' = \dfrac{1}{(i+1)!} m_{i+1}$.

For bounded control functions we obtain an extension of the main result from [1].

Corollary 4.3. *Let $\epsilon > 0$ and $f : A \to B$ such that $\left\|\Delta_y^{n+1} f(x)\right\| \le \epsilon$ for all $x, y \in A$. Then procedure (4.1) (or procedure (4.23)) defines the unique $n-$polynomial p for which $p(0) = f(0)$ and $\|f(y) - p(y)\| \le \epsilon \prod_{i=1}^{n} (3i-2)$ for all $y \in A$.*

We conclude this section with a stability result -which generalizes and improves Corollary 5.5 from [9]- for control functions of Aoki-Rassias type.

Corollary 4.4. *Let A be a normed linear space, $\epsilon > 0$, $r_1, r_2 \in (0,1)$ and $f : A \to B$ such that*

$$\left\|\Delta_y^{n+1} f(x)\right\| \le \epsilon\left(\|x\|^{r_1} + \|y\|^{r_2}\right)$$

for all $x, y \in A$. Then procedure (4.1) (or procedure (4.23)) defines the unique $n-$polynomial p for which

$$\|f(y) - p(y)\| \le \epsilon\left[\frac{(n-1)^{r_1}}{2-2^{r_1}}\|y\|^{r_1} + \frac{1}{2-2^{r_2}}\|y\|^{r_2}\right]\prod_{i=1}^{n}(3i-2) \tag{4.24}$$

for all $y \in A$.

Proof. We apply Theorem 4.1. Since $\varphi'(y) := \epsilon\left[(n-1)^{r_1}\|y\|^{r_1} + \|y\|^{r_2}\right]$ and

$$\varphi_1(y) := \epsilon\left[\frac{(n-1)^{r_1}}{2-2^{r_1}}\|y\|^{r_1} + \frac{1}{2-2^{r_1}}\|y\|^{r_2}\right]$$

we immediately obtain (4.24).

5. Hyperstability for Fréchet polynomials

We remember that, according to [5], equation (1.1) is φ-hyperstable (where $\varphi \in \mathbb{R}_+^{A\times A}$) if from $f \in B^A$ and $\left\|\Delta_y^{n+1} f(x)\right\| \le \varphi(x,y)$ for all $x, y \in A$ it follows that f is an n-polynomial.

Theorem 5.1. *Let A be an Abelian group, $\varphi \in \mathbb{R}_+^{A\times A}$ and $f \in B^A$ such that*

$$\lim_{i,j\to\infty} \varphi(ix, jy) = 0 \tag{5.1}$$

$$\left\|\Delta_y^{n+1} f(x)\right\| \le \varphi(x,y) \tag{5.2}$$

for all $x, y \in A$ (where the double limit is taken in Pringsheim's sense (see [13]*)). Then f is an n-polynomial.*

Proof. We use Theorem 4.1 and the notation introduced there. Let $y \in A$. First, we remark that the sequence $\left(\varphi'\left(2^k y\right)\right)_{k\ge 0}$ converges to 0 (from (5.1)). Therefore there exists an $i_y \in \mathbb{N}$ such that $\varphi'\left(2^k y\right) \le \varphi'\left(2^{i_y} y\right)$ for all $k \in \mathbb{N}$, hence

$$\varphi_1(y) \le \sum_{k=0}^{\infty} 2^{-k-1}\varphi'\left(2^{i_y} y\right) = \varphi'\left(2^{i_y} y\right). \tag{5.3}$$

If $k \to \infty$ then $k 2^{i_{ky}} \ge k \to \infty$, and from (5.1) we have $\lim_{k\to\infty} \varphi'\left(2^{i_{ky}} ky\right) = 0$; therefore, from (5.3) it follows that

$$\lim_{k\to\infty} \varphi_1(ky) = 0. \tag{5.4}$$

Let $F := f - \sum_{j=0}^{n} \frac{1}{j!} m_j$, where the j-monomials m_j are defined by procedure (4.1). From Theorem 4.1 (for $i_0 = 1$) we have

$$\|F(y)\| \le c_1 \varphi_1(y) \tag{5.5}$$

for all $y \in A$. Let $k \in \mathbb{N}\setminus\{0\}$. Then

$$\Delta_{-ky}^{n+1} F((k+1)y) = (-1)^n (n+1) F(y)$$
$$+ \sum_{i=0, i\ne 1}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} F((1+k-ik)y) \tag{5.6}$$

But $\Delta_y^{n+1} F = \Delta_y^{n+1} f$. Therefore, from (5.6) we have

$$\|F(y)\| \le \frac{1}{n+1}[\|\Delta_{-ky}^{n+1} f((k+1)y)\|$$
$$+ \sum_{i=0, i\ne 1}^{n+1} \binom{n+1}{i} \|F((1+k-ik)y)\|]. \tag{5.7}$$

From (5.7), (5.2), (5.5) and (5.1) it follows that

$$\|F(y)\| \le \lim_{k\to\infty} \frac{1}{n+1}[\varphi((k+1)y, -ky) + c_1 \sum_{i=0, i\ne 1}^{n+1} \binom{n+1}{i} \varphi_1((1+k-ik)y)] = 0$$

for all $y \in A$. Consequently $f = \sum_{j=0}^{n} \frac{1}{j!} m_j \in \mathcal{P}_n(A, B)$.

We remark that, since Δ_0 is the null operator, in the above theorem we can define, without any loss of generality, $\varphi(x, 0) = 0$ for all $x \in A$. Consequently, from the previous theorem we obtain the following hyperstability result (see [11] and the papers referred there for similar results obtained by the direct method for $n \leq 5$).

Corollary 5.2. *Let A be a normed linear space, $r_1, r_2, r_3 < 0$, $\epsilon_1, \epsilon_2 \geq 0$, and $f : A \to B$ be a function which verify*

$$\left\| \Delta_y^{n+1} f(x) \right\| \leq \epsilon_1 \left(\|x\|^{r_1} + \|y\|^{r_2} \right), \text{ and } \left\| \Delta_y^{n+1} f(0) \right\| \leq \epsilon_2 \|y\|^{r_3}$$

for all $x, y \in A \setminus \{0\}$. Then $\Delta_y^{n+1} f(x) = 0$ for all $x, y \in A$.

6. A new class of control functions

In this section A denotes a 2−divisible Abelian monoid. The coefficients π_i, c_i, and the function φ' are defined in Section 4.

We remark that Theorem 3.2 from [9] combined with Lemma 5.1 from [8] provides a stability result for difference equation (1.1) (realized with the direct method) on $n!$−divisible commutative groups. Here, using our fixed point theorem, and the fixed point method described in Section 4, we generalize this result in the following theorem.

Theorem 6.1. *Let $\varphi : A \times A \to \mathbb{R}_+$ and $f : A \to B$ such that*

$$\overline{\varphi}(y) := \sum_{k=0}^{\infty} 2^{nk} \varphi'\left(2^{-k-1} y\right) < \infty, \tag{6.1}$$

$$\lim_{k \to \infty} 2^{nk} \varphi\left(2^{-k} x, 2^{-k} y\right) = 0, \tag{6.2}$$

$$\left\| \Delta_y^{n+1} f(x) \right\| \leq \varphi(x, y) \tag{6.3}$$

for all $x, y \in A$. Then there exists a unique n−polynomial p such that

$$\|f(y) - p(y)\| \leq \overline{\varphi}(y) \prod_{i=1}^{n} (3i - 2) \tag{6.4}$$

for all $y \in A$. Moreover, $p := \sum_{i=0}^{n} \frac{1}{i!} m_i$, and $m_i \in \mathcal{M}_i(A, B)$ can be calculated with the procedure

$$f_n := f,\ m_i(y) := \lim_{k \to \infty} 2^{ik} \Delta_{2^{-k} y}^{i} f_i(0),\ f_{i-1} := f_i - \frac{1}{i!} m_i \text{ for}$$
$$i = 1, 2, ..., n, \text{ and } m_0(y) := f(0). \tag{6.5}$$

Proof. 1. We remark that $\overline{\varphi}(2y) = \varphi'(y) + 2^n \overline{\varphi}(y)$, hence

$$\varphi'(y) \leq \overline{\varphi}(2y) \tag{6.6}$$

$$\overline{\varphi}(y) \leq 2^{-n} \overline{\varphi}(2y) \tag{6.7}$$

for all $y \in A$.

2. We prove, by reverse induction on $i \leq n$ the following three assertions:

$$\left\| \Delta_y^{i+1} f_i(jy) \right\| \leq 2^{i-n} \pi_{i+1} \overline{\varphi}(2y) \text{ for } j \in \{0, ..., i-1\}, \tag{6.8}$$

$$m_i(y) := \lim_{k\to\infty} 2^{ik}\Delta^i_{2^{-k}y} f_i(0) \text{ defines } m_i \in \mathcal{M}_i(A,B), \tag{6.9}$$

$$\left\|\Delta^i_y f_{i-1}(0)\right\| \le c_i\overline{\varphi}(y) \tag{6.10}$$

for all $y \in A$ in a similarly manner as in Theorem 4.1.

2.1. Let $i = n$.

2.1.1. Using (6.3) and (6.6) we obtain

$$\left\|\Delta^{n+1}_y f(jy)\right\| \le \varphi'(y) \le \pi_{n+1}\overline{\varphi}(2y) \text{ for } j \in \{0,...,n-1\}. \tag{6.11}$$

2.1.2. Since $\sum_{s=1}^{n} s\binom{n}{s} = n2^{n-1}$ and

$$\Delta^n_{2y} f_n(0) - 2^n\Delta^n_y f_n(0) = \sum_{s=1}^{n}\binom{n}{s}\sum_{j=1}^{s}\Delta^{n+1}_y f_n((j-1)y),$$

using (6.11) we obtain

$$\left\|\Delta^n_y f_n(0) - 2^n\Delta^n_{2^{-1}y} f_n(0)\right\| \le n2^{n-1}\varphi'\left(2^{-1}y\right). \tag{6.12}$$

We apply Theorem 3.1 for

$$\begin{aligned} (Lh)(y) &:= 2^n h\left(2^{-1}y\right), \qquad (J\delta)(y) := 2^n\delta\left(2^{-1}y\right), \\ \alpha(y) &:= n2^{n-1}\varphi'\left(2^{-1}y\right), \text{ and } g(y) := \Delta^n_y f_n(0). \end{aligned}$$

We have:

- if $h \in B^A$, $\delta \in \mathbb{R}^A$ and $\|h(x)\| \le \|\delta(x)\|$ for all $x \in A$, then $\left\|2^n h\left(2^{-1}x\right)\right\| \le \left\|2^n\delta\left(2^{-1}x\right)\right\|$, hence $\|(Lh)(x)\| \le \|(J\delta)(x)\|$ for all $x \in A$;
- $\overline{\alpha}(y) = \sum_{k=0}^{\infty}\left(J^k\alpha\right)(y) = \sum_{k=0}^{\infty} 2^{nk}\alpha\left(2^{-k}y\right) = n2^{n-1}\sum_{k=0}^{\infty} 2^{nk}\varphi'\left(2^{-k-1}y\right)$ $= n2^{n-1}\overline{\varphi}(y)$;
- $(J\overline{\alpha})(y) = 2^n\overline{\alpha}\left(2^{-1}y\right) = \sum_{k=0}^{\infty} 2^{n(k+1)}\alpha\left(2^{-k-1}y\right) = \sum_{k=0}^{\infty}\left(J^{k+1}\alpha\right)(y)$;
- (6.12) is equivalent with $\|g(y) - (Lg)(y)\| \le \alpha(y)$,

and all the requirements of Theorem 3.1 are verified.
Since $\left(L^k g\right)(y) = 2^{nk}\Delta^n_{2^{-k}y} f_n(0)$, from Theorem 3.1 it follows that

$$m_n(y) := \lim_{k\to\infty} 2^{nk}\Delta^n_{2^{-k}y} f_n(0)$$

defines $m_n \in B^A$ and

$$\left\|\Delta^n_y f_n(0) - m_n(y)\right\| \le c_n\overline{\varphi}(y) \tag{6.13}$$

for all $y \in A$.

2.1.3. For proving that $m_n \in \mathcal{M}_n(A,B)$ we follow the technique used in Theorem 4.1. Let $y \in A$. From (6.3) and (6.2) it follows that

$$\lim_{k\to\infty} 2^{nk}\Delta^{n+1}_{2^{-k}y} f_n\left(2^{-k}sy\right) \text{ for all } s \in \mathbb{N}.$$

Therefore, using the formula

$$\Delta_y^n f_n(jy) = \Delta_y^n f_n(0) + \sum_{s=0}^{j-1} \Delta_y^{n+1} f_n(sy), \; j \geq 1,$$

it follows immediately that

$$m_n(y) := \lim_{k\to\infty} 2^{nk} \Delta_{2^{-k}y}^n f_n\left(2^{-k} jy\right) \tag{6.14}$$

for all $j \in \mathbb{N}$. Using this, we prove that

$$m_n(jy) = j^n m_n(y) \tag{6.15}$$

for all $j \in \mathbb{N}$ and $y \in A$. Of course $m_n(0) = 0$. From Marchaud's formula and (6.14) we have $\sum_{i=0}^{jn} (j)_i^n = (j+1)^n$ for all $j \in \mathbb{N}$, and

$$m_n((j+1)y) = \lim_{k\to\infty} 2^{nk} \Delta_{2^{-k}(j+1)y}^n f_n(0) = \lim_{k\to\infty} 2^{nk} \sum_{i=0}^{jn} (j)_i^n \Delta_{2^{-k}y}^n f_n\left(2^k iy\right)$$

$$= \sum_{i=0}^{jn} (j)_i^n m_n(y) = (j+1)^n m_n(y);$$

and (6.15) is completely proved.

Taking into account (6.15), for proving that $m_n \in \mathcal{M}_n(A,B)$ it is sufficient to show that m_n is an $n-$polynomial. Let $x, y \in A$. Then

$$\Delta_y^{n+1} m_n(x) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} m_n(x+jy)$$

$$= \lim_{k\to\infty} 2^{nk} \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \Delta_{2^{-k}(x+jy)}^n f_n(0)$$

$$= \lim_{k\to\infty} 2^{nk} \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} f_n\left(2^{-k} sx + j2^{-k} sy\right)$$

$$= \lim_{k\to\infty} 2^{nk} \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} \Delta_{2^{-k}sy}^{n+1} f_n\left(2^{-k} sx\right);$$

but, from (6.3) and (6.2) we have

$$0 \leq \lim_{k\to\infty} 2^{nk} \left\| \Delta_{2^{-k}sy}^{n+1} f_n\left(2^{-k} sx\right) \right\| \leq \lim_{k\to\infty} 2^{nk} \varphi\left(2^{-k} sx, 2^{-k} sy\right) = 0,$$

hence $\Delta_y^{n+1} m_n(x) = 0$ for all $x, y \in A$., i.e. $m_n \in \mathcal{M}_n(A,B)$. Moreover, since

$$f_{i-1} := f_i - \frac{1}{i!} m_i,$$

then

$$\Delta_y^n f_{n-1}(0) = \Delta_y^n f_n(0) - m_n(y),$$

hence (6.13) becomes exactly (6.10) (for $i = n$).

2.2. Let $n>1$ and suppose that (6.8), (6.9), and (6.10) are true for $i\in\{2,...,n\}$. For proving (6.8), (6.9), and (6.10) for $i-1$ we proceed similarly as in case 2.1.

2.2.1. For proving

$$\left\|\Delta_y^i f_{i-1}(jy)\right\| \le 2^{i-n-1}\pi_i\overline{\varphi}(2y) \text{ for all } j\in\{0,...,i-2\}. \tag{6.16}$$

we remark first that, from (6.7) and from $i\le 3i-2$ we have

$$c_i\overline{\varphi}(y)\le i2^{i-1}\pi_{i+1}2^{-n}\overline{\varphi}(2y)\le 2^{i-n-1}\pi_i\overline{\varphi}(2y),$$

hence, from (6.10) it follows (6.16) for $j=0$. Since $\Delta^{i+1}m_i=0$, for $j\in\{1,...,i-1\}$ we have

$$\Delta_y^i f_{i-1}(jy)=\Delta_y^i f_{i-1}(0)+\sum_{s=0}^{j-1}\Delta_y^{i+1}f_i(sy).$$

Therefore, from (6.10), (6.8), and (6.7) it follows

$$\left\|\Delta_y^i f_{i-1}(jy)\right\|\le\left[2^{-n}c_i+(i-1)\,2^{i-n}\pi_{i+1}\right]\overline{\varphi}(2y).$$

But

$$2^{-n}c_i+(i-1)\,2^{i-n}\pi_{i+1}=\left(2^{-n}i2^{i-1}+(i-1)\,2^{i-n}\right)\pi_{i+1}=2^{i-n-1}\pi_i,$$

and (6.16) is completely proved.

2.2.2. As in the case 2.1.2, using (6.16) we have

$$\begin{aligned}\left\|\Delta_{2y}^{i-1}f_{i-1}(0)-2^{i-1}\Delta_y^{i-1}f_{i-1}(0)\right\| &\le \sum_{s=1}^{i-1}\binom{i-1}{s}\sum_{j=1}^{s}\left\|\Delta_y^i f_i((j-1)\,y)\right\|\\ &\le (i-1)\,2^{2i-n-3}\overline{\varphi}(2y),\end{aligned}$$

hence

$$\left\|\Delta_y^{i-1}f_{i-1}(0)-2^{i-1}\Delta_{2^{-1}y}^{i-1}f_{i-1}(0)\right\|\le(i-1)\,2^{2i-n-3}\overline{\varphi}(y) \tag{6.17}$$

We apply Theorem 3.1 for

$$(Lh)(y):=2^{i-1}h\left(2^{-1}y\right),\quad (J\delta)(y):=2^{i-1}\delta\left(2^{-1}y\right),$$

$$\alpha(y)=:(i-1)\,2^{2i-n-3}\overline{\varphi}(y),\text{ and } g(y):=\Delta_y^{i-1}f_{i-1}(0).$$

From (6.17) and Theorem 3.1 it follows that

$$\left\|\Delta_y^{i-1}f_{i-1}(0)-m_{i-1}(y)\right\|\le\overline{\alpha}(y) \tag{6.18}$$

where

$$m_{i-1}(y):=\lim_{k\to\infty}2^{(i-1)k}\Delta_{2^{-k}y}^{i-1}f_{i-1}(0).$$

We remark that, using Lemma 2.2,

$$\begin{aligned}\overline{\alpha}(y)=\sum_{k=0}^{\infty}2^{(i-1)k}\alpha\left(2^{-k}y\right)&\le(i-1)\,2^{2i-n-3}\pi_i\sum_{k=0}^{\infty}2^{(n-1)k}\overline{\varphi}\left(2^{-k-1}(2y)\right)\\ &\le(i-1)\,2^{2i-n-2}\pi_i\overline{\varphi}(y).\end{aligned}$$

Since $i\le n$, we have $(i-1)\,2^{2i-n-2}\pi_i\le c_{i-1}$, and, from (6.18),

$$\left\|\Delta_y^{i-1}f_{i-1}(0)-m_{i-1}(y)\right\|\le c_{i-1}\overline{\varphi}(y). \tag{6.19}$$

Using the same technique as at 2.1.3 it follows that m_{i-1} is an $(i-1)$ −monomial. Denoting

$$f_{i-2} := f_{i-1} - \frac{1}{(i-1)!} m_{i-1}$$

relation (6.19) becomes

$$\left\| \Delta_y^{i-1} f_{i-2}(0) \right\| \le c_{i-1} \overline{\varphi}(y),$$

and our reverse induction is complete.

3. From (6.10) (for $i = 1$) we have

$$\| f_0(y) - f_0(0) \| \le c_1 \overline{\varphi}(y) \text{ for all } y \in A.$$

But $f_0 = f - \sum_{i=1}^{n} \frac{1}{i!} m_i$, $f_0(0) = f(0) = m_0(y)$ and $c_1 = \prod_{i=1}^{n} (3i-2)$. Therefore

$$\left\| f(y) - \sum_{i=0}^{n} \frac{1}{i!} m_i(y) \right\| \le \overline{\varphi}(y) \prod_{i=1}^{n} (3i-2) \text{ for all } y \in A.$$

4. Uniqueness. Finally, suppose that $m_i' \in \mathcal{M}_i(A, B)$ and

$$\left\| f(y) - \sum_{i=0}^{n} m_i'(y) \right\| \le c_1 \overline{\varphi}(y) \text{ for all } y \in A. \tag{6.20}$$

From (6.4) and (6.20) we have

$$\left\| \sum_{i=0}^{n} m_i'(y) - \frac{1}{i!} m_i(y) \right\| \le 2c_1 \overline{\varphi}(y) \text{ for all } y \in A. \tag{6.21}$$

We prove by induction on $j \in \{1, ..., n\}$ that

$$m_{j-1}' = \frac{1}{(j-1)!} m_{j-1} \text{ and } \left\| \sum_{i=j}^{n} \left[m_i'(y) - \frac{1}{i!} m_i(y) \right] \right\| \le 2c_1 \overline{\varphi}(y) \tag{6.22}$$

for all $y \in A$. From (6.1) it follows that $\overline{\varphi}(0) = 0$ and from (6.21) (for $y = 0$) we obtain $m_0' = m_0$. Therefore, from (6.21) it follows (6.22) for $j = 1$.

Suppose that $n > 1$ and (6.22) is true for $j \in \{1, ..., n-1\}$. From Lemma 2.2 we have

$$\lim_{k\to\infty} 2^{jk} \overline{\varphi}\left(2^{-k} y\right) = 0 \text{ for all } y \in A. \tag{6.23}$$

From (6.22) and (6.23) we have

$$\lim_{k\to\infty} 2^{jk} \left\| \sum_{i=j}^{n} 2^{-ik} \left[m_i'(y) - \frac{1}{i!} m_i(y) \right] \right\| \le \lim_{k\to\infty} 2^{jk} \cdot 2c_1 \overline{\varphi}\left(2^{-k} y\right) = 0$$

for all $y \in A$, hence $m_j' = \frac{1}{j!} m_j$. Moreover, if $j < n$, from (6.22) we have

$$\left\| \sum_{i=j+1}^{n} m_i'(y) - \frac{1}{i!} m_i(y) \right\| \le 2c_1 \overline{\varphi}(y) \text{ for all } y \in A.$$

Therefore $p=\sum_{i=0}^{n}\frac{1}{i!}m_i$ is the unique $n-$monomial that verifies (6.4).

An Aoki-Rassias type stability result is the following consequence (see [11] and the papers referred there for similar results obtained by the direct method for $n\leq 5$).

Corollary 6.4. *Let A be a normed linear space, $\epsilon>0$, $r>n$ and $f:A\to B$ such that*

$$\left\|\Delta_y^{n+1}f\left(x\right)\right\|\leq\epsilon\left(\left\|x\right\|^r+\left\|y\right\|^r\right)\text{ for all }x,y\in A.$$

Then procedure (6.5) defines the unique $n-$polynomial p for which

$$\left\|f\left(x\right)-p\left(x\right)\right\|\leq\epsilon\frac{\left(n-1\right)^r+1}{2^r-2^n}\left\|x\right\|^r\prod_{i=1}^{n}\left(3i-2\right)$$

for all $x\in A$.

7. A hybrid stability theorem

Combining Theorem 4.1 and Theorem 6.1 we can give the following result. We use the coefficients π_i,c_i,c_i' and the functions φ',φ_i defined in Section 4.

Theorem 7.1. *Let $n>1$, A be a $2-$divisible commutative monoid, $i_0\in\{2,...,n\}$, $\varphi:A\times A\to[0,\infty)$, and $f:A\to B$ which verify conditions (4.2), (4.3), (4.4) and*

$$\overline{\varphi}\left(y\right):=\sum_{k=0}^{\infty}2^{(i_0-1)k}\varphi_{i_o}\left(2^{-k}y\right)<\infty,\tag{7.1}$$

$$\lim_{k\to\infty}2^{(i_0-1)k}\varphi\left(2^{-k}x,2^{-k}y\right)=0\tag{7.2}$$

for all $x,y\in A$. Then there exists $p\in\mathcal{P}_n\left(A,B\right)$ such that

$$\left\|f\left(y\right)-p\left(y\right)\right\|\leq\overline{\varphi}\left(y\right)\prod_{i=1}^{n}\left(3i-2\right)\tag{7.3}$$

for all $y\in A$. Moreover, $p=\sum_{i=0}^{n}\frac{1}{i!}m_i$, where the monomials $m_n,m_{n-1},...,m_{i_0}$ are defined by procedure (4.1), and the monomials $m_{i_0-1},m_{i_0-2},...,m_0$ are defined by procedure (6.5). If, in addition

$$\lim_{k\to\infty}2^{-i_0k}\overline{\varphi}\left(2^ky\right)=0\text{ for all }y\in A,\tag{7.4}$$

then p is the unique $n-$polynomial which fulfills (7.3).

Proof. From Theorem 4.1 it follows that procedure (4.1) defines the functions f_i and the monomials m_i such that

$$\left\|\Delta_y^{i_0}f_{i_0-1}\left(0\right)\right\|\leq c_{i_0}\varphi_{i_0}\left(y\right)\text{ and }\left\|\Delta_y^{i_0+1}f_{i_0}\left(jy\right)\right\|\leq c_{i_0+1}'\varphi_{i_0+1}\left(y\right)\tag{7.5}$$

for all $y\in A$ and $j\in\{0,...,i_0-1\}$ (see part 1 from the proof of Theorem 4.1). From (7.1) we have $\overline{\varphi}\left(y\right)=\varphi_{i_0}\left(y\right)+2^{i_0-1}\overline{\varphi}\left(2^{-1}y\right)$, hence

$$\varphi_{i_0}\left(y\right)\leq\overline{\varphi}\left(y\right)\leq 2^{1-i_0}\overline{\varphi}\left(2y\right),y\in A.\tag{7.6}$$

1. We shall prove, by reverse induction on $i \in \{1, ..., i_0 - 1\}$, the following three assertions:

$$\left\| \Delta_y^{i+1} f_i (jy) \right\| \leq c'_{i+1} \overline{\varphi} (y) \text{ for } j \in \{0, ..., i-1\}, \tag{7.7}$$

$$m_i (y) := \lim_{k \to \infty} 2^{ik} \Delta^i_{2^{-k} y} f_i (0) \text{ defines } m_i \in \mathcal{M}_i (A, B), \tag{7.8}$$

$$\left\| \Delta_y^i f_{i-1} (0) \right\| \leq c_i \overline{\varphi} (y) \tag{7.9}$$

for all $y \in A$.

1.1. Let $i = i_0 - 1$. First we prove (7.7), i.e.

$$\left\| \Delta_y^{i_0} f_{i_0 - 1} (jy) \right\| \leq c'_{i_0} \overline{\varphi} (y) \text{ for } j \in \{0, ..., i_0 - 2\} \text{ and } y \in A. \tag{7.10}$$

Since $c_{i_0} \leq c'_{i_0}$, from (7.6) and (7.5) if follows (7.10) for $j = 0$. Let $j \geq 1$. Since $\varphi_{i_0+1} (y) \leq \varphi_{i_0} (y)$ and

$$\Delta_y^{i_0} f_{i_0-1} (jy) = \Delta_y^{i_0} f_{i_0-1} (0) + \sum_{s=0}^{j-1} \Delta_y^{i_0+1} f_{i_0} (sy),$$

using (7.5) we obtain

$$\left\| \Delta_y^{i_0} f_{i_0-1} (jy) \right\| \leq \left[c_{i_0} + (i_0 - 1) c'_{i_0+1} \right] \varphi_{i_0} (y) = c'_{i_0} \varphi_{i_0} (y), \tag{7.11}$$

and from (7.6) we obtain immediately (7.10).

For proving (7.8) and (7.9) we proceed as in the proof of Theorem 6.1. Using the formula

$$\Delta_{2y}^{i_0-1} f_{i_0-1} (0) - 2^{i_0-1} \Delta_y^{i_0-1} f_{i_0-1} (0) = \sum_{s=1}^{i_0-1} \binom{i_0 - 1}{s} \sum_{j=1}^{s} \Delta_y^{i_0} f_{i_0} ((j-1) y)$$

and (7.11) we obtain immediately

$$\left\| \Delta_y^{i_0-1} f_{i_0-1} (0) - 2^{i_0-1} \Delta_{2^{-1}y}^{i_0-1} f_{i_0-1} (0) \right\| \leq (i_0 - 1) 2^{i_0-2} c'_{i_0} \varphi_{i_0} \left(2^{-1} y \right) \tag{7.12}$$

for all $y \in A$. We apply again Theorem 3.1 for

$$(Lh)(y) : = 2^{i_0-1} h \left(2^{-1} y \right), \qquad (J\delta)(y) := 2^{i_0-1} \delta \left(2^{-1} y \right),$$
$$\alpha (y) : = (i_0 - 1) 2^{i_0-2} c'_{i_0} \varphi_{i_0} \left(2^{-1} y \right), \text{ and } g(y) := \Delta_y^{i_0-1} f_{i_0-1} (0);$$

and (7.12); if follows that

$$m_{i_0-1} (y) := \lim_{k \to \infty} 2^{(i_0-1)k} \Delta_{2^{-k} y}^{i_0-1} f_{i_0-1} (0)$$

defines $m_{i_0-1} : A \to B$ such that

$$\left\| \Delta_y^{i_0-1} f_{i_0-1} (0) - m_{i_0-1} (y) \right\| \leq \overline{\alpha} (y) \tag{7.13}$$

for all $y \in A$. But

$$\overline{\alpha} (y) = (i_0 - 1) 2^{i_0-2} c'_{i_0} \sum_{k=0}^{\infty} 2^{(i_0-1)k} \varphi_{i_0} \left(2^{-k-1} y \right) = (i_0 - 1) 2^{i_0-2} c'_{i_0} \overline{\varphi} \left(2^{-1} y \right);$$

using (7.6) we have $\overline{\alpha} (y) \leq (i_0 - 1) 2^{i_0-2} \pi_{i_0} \overline{\varphi} (y)$, and from (7.13) we obtain

$$\left\| \Delta_y^{i_0-1} f_{i_0-1} (0) - m_{i_0-1} (y) \right\| \leq c_{i_0-1} \overline{\varphi} (y). \tag{7.14}$$

Since $m_{i_0-1}(jy) = j^{i_0-1} m_{i_0-1}(y)$ for all $j \in \mathbb{N}$ and $y \in A$ (see part 2.1.3 from the proof of Theorem 6.1), for proving that $m_{i_0-1} \in \mathcal{M}_{i_0-1}(A, B)$ it is sufficient to show that $m_{i_0-1} \in \mathcal{P}_n(A, B)$. Let $x, y \in A$. Then

$$\Delta_y^{n+1} m_{i_0-1}(x) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} m_{i_0-1}(x + jy)$$

$$= \lim_{k\to\infty} 2^{(i_0-1)k} \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \Delta_{2^{-k}(x+jy)}^{i_0-1} f_{i_0-1}(0)$$

$$= \lim_{k\to\infty} 2^{(i_0-1)k} \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \sum_{s=0}^{i_0-1} (-1)^{i_0-1-s} \binom{i_0-1}{s} f_{i_0-1}\left(2^{-k} s (x + jy)\right)$$

$$= \lim_{k\to\infty} 2^{(i_0-1)k} \sum_{s=0}^{i_0-1} (-1)^{i_0-1-s} \binom{i_0-1}{s} \Delta_{2^{-k}sy}^{n+1} f_{i_0-1}\left(2^{-k} s x\right).$$

But $\Delta^{n+1} f_{i_0-1} = \Delta^{n+1} f$ (since $m_j \in \mathcal{M}_j(A, B)$ for $j \in \{i_0, i_0+1, ..., n\}$), and from (4.4) and (7.2) we have

$$0 \le \lim_{k\to\infty} 2^{(i_0-1)k} \left\| \Delta_{2^{-k}sy}^{n+1} f_{i_0-1}\left(2^{-k} s x\right) \right\| \le \lim_{k\to\infty} 2^{(i_0-1)k} \varphi\left(2^{-k} s x, 2^{-k} s y\right) = 0;$$

hence $\Delta_y^{n+1} m_n(x) = 0$ for all $x, y \in A$. Therefore $m_n \in \mathcal{M}_n(A, B)$. Moreover, since $\Delta^{i_0-1} f_{i_0-2} = \Delta^{i_0-1} f_{i_0-1} - m_{i_0-1}$ where $f_{i_0-1} := f_{i_0} - \frac{1}{i_0!} m_{i_0}$, then (7.14) is exactly (7.9) (for $i = i_0 - 1$).

1.2. Suppose now that (7.7), (7.8) and (7.9) are true for $i \in \{2, 3, ..., i_0 - 1\}$. Following the ideas used in part 2.1 from the proof of Theorem 6.1, we prove the three assertions for $i - 1$.

- Let $j \ge 1$. Since $\Delta_y^i f_{i-1}(jy) = \Delta_y^i f_{i-1}(0) + \sum_{s=0}^{j-1} \Delta_y^{i+1} f_i(sy)$, from (7.9) and (7.7) it follows that

$$\left\| \Delta_y^i f_{i-1}(jy) \right\| \le \left(c_i + (i-1) c'_{i+1}\right) \overline{\varphi}(y) = c'_i \overline{\varphi}(y); \tag{7.15}$$

  but $c_i \le c'_i$; therefore, from (7.15) and (7.9) we have

$$\left\| \Delta_y^i f_{i-1}(jy) \right\| \le c'_i \overline{\varphi}(y) \text{ for } j \in \{0, 1, ..., i-1\} \tag{7.16}$$

  for all $y \in A$.

- Since $\Delta_{2y}^{i-1} f_{i-1}(0) - 2^{i-1} \Delta_y^{i-1} f_{i-1}(0) = \sum_{s=1}^{i-1} \binom{i-1}{s} \sum_{j=1}^{s} \Delta_y^i f_i((j-1)y)$, using (7.16) we obtain

$$\left\| \Delta_{2y}^{i-1} f_{i-1}(0) - 2^{i-1} \Delta_y^{i-1} f_{i-1}(0) \right\| \le (i-1) 2^{2i-3} \pi_i \overline{\varphi}(y) \tag{7.17}$$

for all $y \in A$. Now, from (7.17) and Theorem 3.1 applied for

$$(Lh)(y) := 2^{i-1} h\left(2^{-1} y\right), \quad (J\delta)(y) := 2^{i-1} \delta\left(2^{-1} y\right),$$

$$\alpha(y) =: (i-1) 2^{2i-3} \pi_i \overline{\varphi}(y), \text{ and } g(y) := \Delta_y^{i-1} f_{i-1}(0)$$

it follows that $m_{i-1}(y) := \lim_{k\to\infty} 2^{(i-1)k}\Delta^{i-1}_{2^{-k}y} f_{i-1}(0)$ defines $m_{i-1} : A \to B$ such that

$$\left\|\Delta^{i-1}_y f_{i-1}(0) - m_{i-1}(y)\right\| \le \overline{\alpha}(y) \tag{7.18}$$

for all $y \in A$. Since

$$\overline{\alpha}(y) = \sum_{k=0}^{\infty} 2^{(i-1)k}\alpha\left(2^{-k}y\right) \le (i-1)\,2^{2i-3}\pi_i \sum_{k=0}^{\infty} 2^{(i-1)k}\overline{\varphi}\left(2^{-k-1}y\right),$$

using Lemma 2.2 and (7.4), since $i \le n$, we have

$$\overline{\alpha}(y) \le (i-1)\,2^{2i-2}\pi_i\overline{\varphi}\left(2^{-1}y\right) \le (i-1)\,2^{2i-2}\pi_i 2^{1-i_0}\overline{\varphi}(y) \le (i-1)\,2^{2i-2}\pi_i\overline{\varphi}(y)$$

and, from (7.18) it follows that

$$\left\|\Delta^{i-1}_y f_{i-1}(0) - m_{i-1}(y)\right\| \le c_{i-1}\overline{\varphi}(y) \tag{7.19}$$

for all $y \in A$. As in the first part we show that m_{i-1} is an $(i-1)-$monomial. Hence (7.19) becomes

$$\left\|\Delta^{i-1}_y f_{i-2}(0)\right\| \le c_{i-1}\overline{\varphi}(y)$$

for all $y \in A$, and the reverse induction is complete.

2. From $i = 1$ in (7.9) we obtain

$$\|f_0(y) - f_0(0)\| \le c_1\overline{\varphi}(y),$$

or, equivalent

$$\left\|f(y) - \sum_{i=0}^{n} \frac{1}{i!} m_i(y)\right\| \le c_1\overline{\varphi}(y)$$

and (7.3) is proved.

3. Uniqueness. Suppose that $\sum_{i=0}^{n} m'_i$ is an other $n-$polynomial which verify (7.3) (where $m'_i \in \mathcal{M}_i(A, B)$). Then

$$\left\|\sum_{i=0}^{n} m'_i(y) - \frac{1}{i!} m_i(y)\right\| \le 2c_1\overline{\varphi}(y) \tag{7.20}$$

for all $y \in A$.

As in the proof of Theorem 6.1 it follows that $m'_i = \frac{1}{i!} m_i$ for $i \in \{0, 1, ..., i_0 - 1\}$. Then (7.20) becomes

$$\left\|\sum_{i=i_0}^{n} m'_i(y) - \frac{1}{i!} m_i(y)\right\| \le 2c_1\overline{\varphi}(y) \tag{7.21}$$

for all $y \in A$. Applying the technique from part 4, Theorem 4.1 to inequality (7.21) and using (7.4), it follows that $m'_i = \frac{1}{i!} m_i$ for $i \in \{i_0, i_0 + 1, ..., n\}$.

Corollary 4.4 and Corollary 6.4 can be completed with the following consequence of the above theorem.

Corollary 7.2. *Let $n \geq 2$, $\epsilon > 0$, $f : A \to B$, $r \in \mathbb{R}$, and $i_0 \in \{2, ..., n\}$ such that $i_0 - 1 < r < i_0$ and*

$$\left\| \Delta_y^{n+1} f(x) \right\| \leq \epsilon \left(\|x\|^r + \|y\|^r \right) \text{ for all } x, y \in A.$$

Then there exists a unique $n-$ polynomial p such that

$$\|f(x) - p(x)\| \leq \epsilon \frac{2^r \left[(n-1)^r + 1 \right]}{\left(2^{i_0} - 2^r\right)\left(2^r - 2^{i_0 - 1}\right)} \|x\|^r \prod_{i=1}^{n} (3i - 2)$$

for all $x \in A$.

Proof. In the above theorem, for $\varphi(x, y) := \epsilon \left(\|x\|^r + \|y\|^r \right)$ we have

$$\varphi'(x) := \epsilon \left[(n-1)^r + 1 \right] \|x\|^r, \ \varphi_{i_0}(x) := \epsilon \frac{(n-1)^r + 1}{\left(2^{i_0} - 2^r\right)} \|x\|^r,$$

and

$$\overline{\varphi}(y) := \epsilon \frac{2^r \left[(n-1)^r + 1 \right]}{\left(2^{i_0} - 2^r\right)\left(2^r - 2^{i_0 - 1}\right)} \|x\|^r.$$

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