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A NOTE ON EXISTENCE AND UNIQUENESS FOR INTEGRAL EQUATIONS WITH SUM OF TWO OPERATORS: PROGRESSIVE CONTRACTIONS

T.A. BURTON

Northwest Research Institute, 732 Caroline St., Port Angeles, WA, 98362 USA
E-mail: taburton@olypen.com

Abstract. In this note we show a simple way to obtain a unique solution on $[0, \infty)$ of a scalar integral equation

$$x(t) = g(t, x(t)) + \int_0^t A(t-s) f(s, x(s)) ds$$

where $x, y \in \Re$ and $t \geq 0$ imply that $|g(t,x) - g(t,y)| \leq \alpha |x-y|, 0 < \alpha < 1$, and for each $E > 0$ there is a $K > 0$ so that $x, y \in \Re$ and $0 \leq t \leq E$ imply $|f(t,x) - f(t,y)| \leq K|x-y|$. We introduce a *progressive contraction*. The constant K is a function of E and, hence, may tend to infinity as $E \to \infty$. The conclusion is that there is a single function $\xi(t)$ satisfying the equation on $[0, \infty)$ without resorting to any of the classical translations and extensions of solutions which, in fact, must invoke Zorn's Lemma and which can encounter difficulties as $K \to \infty$.
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1. Introduction

In an earlier, and largely unrelated project, we had introduced the idea of a *progressive contraction* in the context of showing uniqueness of solutions of an integral equation. That work took place in a Banach space. Some time later we noted that if the setting is a complete metric space with an initial function in the way El'sgol'ts [2, p. 16] taught us fifty years ago then one can obtain a very general global existence theorem in a trivially simple way.

This work had its roots in three things:

1. The ideas of a fixed point theorem of Krasnoselskii on the sum of two operators [5, p. 31].

2. The goal of simplifying the classical existence theory [4, pp. 93-98] in which we prove the existence of a solution on a short interval, then the equation is translated to a new starting time so that a solution on another short interval is fitted onto the first solution, and finally tacitly appealing to Zorn's Lemma [3, p. 42] to conclude that the process can be extended to $[0, \infty)$. But, in fact, the process is so complicated that even in fairly simple cases it is difficult to tell if the solution is entering regions in which

there is difficulty in continuing the process on all of $[0,\infty)$, as well as determining the exact character of the translated functions.

3. The goal of countering the natural inclination of adding the contraction constants when we have the sum of two contractions, and then insisting that the sum of the constants be less than one.

In this elementary and somewhat expository note we show in a clear and simple way exactly how to get a clean solution on $[0,\infty)$, never worrying about the problems arising in the aforementioned classical method.

2. Main results

We consider a scalar integral equation

$$x(t) = g(t, x(t)) + \int_0^t A(t-s) f(s, x(s)) ds \tag{2.1}$$

in which we suppose there is an $\alpha \in (0,1)$ so that $x, y \in \Re$ and $0 \le t < \infty$ imply that

$$|g(t,x) - g(t,y)| \le \alpha |x-y| \tag{2.2}$$

and for each $E > 0$ there is a $K > 0$ so that for $x, y \in \Re$ and $0 \le t \le E$ we have

$$|f(t,x) - f(t,y)| \le K|x-y|. \tag{2.3}$$

We suppose that the kernel $A : (0,\infty) \to \Re$ is continuous, that ϕ continuous on $[0,\infty)$ implies that

$$\int_0^t A(t-s)\phi(s) ds \text{ is continuous,} \tag{2.4}$$

and that

$$\int_0^t |A(s)| ds \text{ is continuous and converges to zero as } t \downarrow 0 \tag{2.5}$$

so that for T small enough we have

$$K \int_0^T |A(s)| ds < \frac{1-\alpha}{2}. \tag{2.6}$$

Moreover, $g : [0,\infty) \times \Re \to \Re$ and $f : [0,\infty) \times \Re \to \Re$ are continuous.

When $T_2 - T_1 \le T$ and $T_1 \le t \le T_2$ a change of variable yields

$$\int_{T_1}^t |A(t-s)| ds < \frac{(1-\alpha)}{2K}. \tag{2.7}$$

There is an important set of kernels satisfying (2.4)-(2.7) in [4, p. 209] with consequences on pp. 212-213. They are defined as follows:

(A1) $A(t) \in C(0,\infty) \cap L^1(0,1)$.

(A2) $A(t)$ is positive and non-increasing for $t > 0$.

(A3) For each $T > 0$ the function $A(t)/A(t+T)$ is non-increasing in t for $0 < t < \infty$.

We turn now to our existence theorem and we name the type of proof *a progressive contraction.* The complete metric space used here is found in El'sgol'ts [2, p. 16] and repeated in Burton [1, p. 177].

Theorem 2.1 *Let (2.2)-(2.6) hold for (2.1). For every* $E > 0$ *there is a unique solution of (2.1) on* $[0, E]$.

Proof. Divide the interval $[0, E]$ into n equal parts, each of length $S < T$, denoting the end points by

$$T_0 = 0, T_1, T_2, ..., T_n = E.$$

Step 1. Let $(\mathcal{M}_1, \|\cdot\|_1)$ be the complete metric space of continuous functions $\phi : [0, T_1] \to \Re$ with the supremum metric. Define a mapping $P_1 : \mathcal{M}_1 \to \mathcal{M}_1$ by $\phi \in \mathcal{M}_1$ implies that

$$(P_1\phi)(t) = g(t, \phi(t)) + \int_0^t A(t-s)f(s, \phi(s))ds.$$

Then for $\phi, \psi \in \mathcal{M}_1$ and $0 \le t \le T_1$ we have

$$\begin{aligned} |(P_1\phi)(t) - (P_1\psi)(t)| &\le \alpha|\phi(t) - \psi(t)| + \int_0^t |A(t-s)||f(s,\phi(s)) - f(s,\psi(s))|ds \\ &\le \alpha\|\phi - \psi\|_1 + K\|\phi - \psi\|_1 \int_0^{T_1} |A(s)|ds \\ &\le \left[\alpha + \frac{(1-\alpha)}{2}\right]\|\phi - \psi\|_1 \\ &= \frac{1+\alpha}{2}\|\phi - \psi\|_1, \end{aligned}$$

a contraction with a unique fixed point ξ_1 on $[0, T_1]$ with

$$(P_1\xi_1)(t) = \xi_1(t) = g(t, \xi_1(t)) + \int_0^t A(t-s)f(s, \xi_1(s))ds \tag{2.8}$$

for $0 \le t \le T_1$.

Before going to Step 2 we note that the supremum of a function ϕ restricted to an interval $[a, b]$ is denoted by $\|\phi\|^{[a,b]}$.

Step 2. Let $(\mathcal{M}_2, \|\cdot\|_2)$ be the complete metric space of continuous functions $\phi : [0, T_2] \to \Re$ with the supremum metric and

$$\phi(t) = \xi_1(t) \text{ on } [0, T_1].$$

Define $P_2 : \mathcal{M}_2 \to \mathcal{M}_2$ by $\phi \in \mathcal{M}_2$ implies

$$(P_2\phi)(t) = g(t, \phi(t)) + \int_0^t A(t-s)f(s, \phi(s))ds.$$

Notice that for $0 \le t \le T_1$ and $\phi \in \mathcal{M}_2$ then $\phi = \xi_1$ which is a fixed point and so from (2.8) we have

$$(P_2\phi)(t) = (P_2\xi_1)(t) = g(t, \xi_1(t)) + \int_0^t A(t-s)f(s, \xi_1(s))ds = \xi_1(t).$$

This means that $P_2 : \mathcal{M}_2 \to \mathcal{M}_2$, as claimed in the above definition of P_2.

For $\phi, \psi \in \mathcal{M}_2$ then

$$|(P_2\phi)(t) - (P_2\psi)(t)| \le |g(t, \phi(t)) - g(t, \psi(t))|$$

$$+\int_0^t |A(t-s)||f(s,\phi(s))-f(s,\psi(s))|ds$$
$$\leq \alpha|\phi(t)-\psi(t)|+\int_0^t |A(t-s)|K|\phi(s)-\psi(s)|ds$$
$$\text{(since } \phi(t)=\psi(t)=\xi_1(t) \text{ on } [0,T_1]\text{, now take } t>T_1)$$
$$\leq \alpha\|\phi-\psi\|^{[T_1,T_2]}+\int_{T_1}^t |A(t-s)|K|\phi(s)-\psi(s)|ds$$
$$\leq \left[\alpha+\int_{T_1}^t K|A(t-s)|ds\right]\|\phi-\psi\|^{[T_1,T_2]} \quad \text{(see (2.7))}$$
$$\leq \frac{(1+\alpha)}{2}\|\phi-\psi\|_2$$

a contraction on $[0,T_2]$ with unique fixed point ξ_2 on that entire interval. It is a unique continuous solution of (2.1) on that interval and it agrees with ξ_1 on $[0,T_1]$ by uniqueness.

Step 3. The next step is essentially the inductive hypothesis. We define the complete metric space $(\mathcal{M}_3,\|\cdot\|_3)$ of continuous functions $\phi:[0,T_3]\to\Re$ with $\phi(t)=\xi_2$ on $[0,T_2]$. But ξ_2 is a fixed point and so P_3 defined as in (2.8) does map $\mathcal{M}_3$ into $\mathcal{M}_3$. Exactly as in Step 2 we obtain a continuous solution ξ_3 on $[0,T_3]$. By induction we obtain a unique continuous solution on $[0,E]$. While we feel this is sufficient for a complete understanding, here are the induction details.

For $2<i<n-1$ let ξ_{i-1} be the unique solution of (2.1) on $[0,T_{i-1}]$. Let $(\mathcal{M}_i,\|\cdot\|_i)$ be the complete metric space of continuous functions $\phi:[0,T_i]\to\Re$ with the supremum metric and $\phi=\xi_{i-1}$ on $[0,T_{i-1}]$. Define $P_i:\mathcal{M}_i\to\mathcal{M}_i$ by $\phi\in\mathcal{M}_i$ imply

$$(P_i\phi)(t)=g(t,\phi(t))+\int_0^t A(t-s)f(s,\phi(s))ds.$$

To see that this is a contraction, let $\phi,\psi\in\mathcal{M}_i$ and $0<t\leq T_i$ so that

$$|(P_i\phi)(t)-(P_i\psi)(t)|\leq |g(t,\phi(t))-g(t,\psi(t))|$$
$$+\int_0^t |A(t-s)||f(s,\phi(s))-f(s,\psi(s)|ds$$
$$\leq \alpha|\phi(t)-\psi(t)|+\int_0^t |A(t-s)|K|\phi(s)-\psi(s)|ds$$
$$(\ \phi=\psi \text{ on } [0,T_{i-1}] \implies T_{i-1} \text{ is the lower limit so now take } T_{i-1}<t)$$
$$\leq \left[\alpha+\int_{T_{i-1}}^t K|A(t-s)|ds\right]\|\phi-\psi\|^{[T_{i-1},T_i]}$$
$$\text{(by a change of variable as in (2.7))}$$
$$\leq \left[\alpha+\int_0^{T_1} K|A(s)ds\right]\|\phi-\psi\|^{[T_{i-1},T_i]}$$
$$\leq \frac{(1+\alpha)}{2}\|\phi-\psi\|_i$$

a contraction with unique fixed point ξ_i on $[0, T_i]$ which completes the proof.

It is to be noted that as $E \to \infty$, the constant K may also tend to infinity. Still, we determine T from the same relation; as K increases, T decreases. The process works for any $E > 0$. This is important for our next result in that we need to see that we can let $E \to \infty$ and always get a solution on $[0, E]$.

We will now show that we can select a well-defined function on $[0, \infty)$ which is a unique solution of (2.1) and it involves no translations or unfinished steps on the road to a solution on $[0, \infty)$.

Theorem 2.2. *Under the conditions of Theorem 2.1 there is a unique solution of (2.1) on* $[0, \infty)$.

Proof. Using Theorem 2.1 we will obtain a sequence of uniformly continuous functions on $[0, \infty)$ which converge uniformly on compact sets to a continuous function which is the unique solution of (2.1). Here are the details.

For each positive integer n use Theorem 2.1 to obtain a solution of (2.1) on $[0, n]$. Then denote by $x_n(t)$ the solution on $[0, n]$ extended to $[0, \infty)$ by $x_n(t) = x_n(n)$ for $t \geq n$. This sequence converges uniformly and to a continuous function, $x(t)$, a solution of (2.1) because at every t the function $x(t)$ agrees with a solution $x_n(t)$ where $n > t$. This completes the proof.

3. What about Krasnoselskii?

We began with a partial motivation from Krasnoselskii's fixed point theorem on the sum of a contraction and compact map. In fact, we can get a compact map out of the integral term and we can satisfy Krasnoselskii's conditions for f continuous and A satisfying (A1)-(A3). But uniqueness has been absolutely crucial at every step of the material offered here. Something like (2.3) seems to be needed for the uniqueness so we see little point in pursuing a different route than the one offered here.

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