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MAIA TYPE FIXED POINT THEOREMS FOR PREŠIĆ TYPE OPERATORS

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Abstract. In this paper, we extend to the case of product spaces, two generalizations of Maia fixed point theorem [Maia, Maria Grazia. Un'osservazione sulle contrazioni metriche. (Italian) Rend. Sem. Mat. Univ. Padova 40 1968 139–143] given by Rus I. A. in [Rus, Ioan A. Generalized contractions. Seminar on Fixed Point Theory, Babeş Bolyai Univ., Cluj-Napoca, 1983, Preprint nr. 3, pp. 1-130, 35] and [Rus, Ioan A. Basic problem for Maia's theorem. Seminar on Fixed Point Theory, Babeş Bolyai Univ., Cluj-Napoca, 1981, Preprint nr. 3, pp. 112-115]. Following the results in [Petruşel, A., Fredholm-Volterra integral equations and Maia's theorem, Seminar on Fixed Point Theory, Babeş Bolyai Univ., Cluj-Napoca, (1988), Preprint nr. 3, pp. 79–82], a theorem on the existence and uniqueness of solutions of Fredholm-Volterra integral equations, using a theorem of Maia type in product metric spaces, is proved.
Key Words and Phrases: Fixed point, Maia, Prešić type contraction, two metrics.
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1. Introduction

In 1968, Maia M. G. [4] generalized the Banach contraction mapping principle for sets endowed with two comparable metrics and is connected with Bielecki's method of changing the norm in the theory of differential equations. Maia type fixed point results for singlevalued or multivalued operators have been studied in [9], [14], [13], [15], [16], [12].

In 1977, Rus I. A. [12] published a result about Maia fixed point theorem, where condition (i) and (iv) are changed. Two years later, the basic problem of the metrical fixed point theory, in the case of Maia type fixed point theorems, was improved by the same author in [11].

Following the above results of Rus and the Bielecki norms technique, in 1988, Petruşel A. [8], proved a theorem on the existence and uniqueness of solutions of Fredholm-Volterra integral equations.

Prešić S. B. [10] extended the famous Banach contraction principle [2] to the case of product spaces in 1965. Recently, in 2007, Ćirić and Prešić [3], generalized the Prešić's theorem introducing Ćirić-Prešić contraction condition. Other important

Prešić fixed point theorem generalizations and some related results can be found in Păcurar's papers [5], [7].

In an extended version, Prešić's result may be stated as follows:

Theorem 1.1. [6] *Let (X,d) be a complete metric space, k a positive integer and $f : X^k \to X$ a Prešić operator, it is, a mapping for which there exists*

$$\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+,\ \sum_{i=1}^{k} \alpha_i = \alpha < 1$$

such that:

$$d\big(f(x_0,\ldots,x_{k-1}), f(x_1,\ldots,x_k)\big) \le \sum_{i=1}^{k} \alpha_i d(x_{i-1},x_i), \tag{1.1}$$

for all $x_0,\ldots,x_k \in X$.

Then:

1) *f has a unique fixed point x^*;*
2) *the sequence $\{y_n\}_{n\ge 0}$, $y_{n+1} = f(y_n,y_n,\ldots,y_n), n \ge 0$, converges to x^*;*
3) *the sequence $\{x_n\}_{n\ge 0}$ with $x_0,\ldots,x_{k-1} \in X$ and*

$$x_n = f(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}), n \ge k,$$

also converges to x^*, with a rate estimated by

$$d(x_{n+1}, x^*) \le \alpha d(x_n, x^*) + M \cdot \theta^n, n \ge 0,$$

where $M > 0$ is constant.

The following lemma was given by Prešić [10] and we shall use it in the proof of our results:

Lemma 1.1. [10] *Let $k \in \mathbb{N}, k \ne 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$ such that $\sum_{i=1}^{k} \alpha_i = \alpha < 1$. If $\{\Delta_n\}_{n\ge 1}$ is a sequence of positive numbers satisfying*

$$\Delta_{n+k} \le \alpha_1 \Delta_n + \alpha_2 \Delta_{n+1} + \cdots + \alpha_k \Delta_{n+k-1}, n \ge 1, \tag{1.2}$$

then there exist $L > 0$ and $\theta \in (0,1)$ such that

$$\Delta_n \le L \cdot \theta^n,\ for\ all\ n \ge 1. \tag{1.3}$$

Starting from these results, the aim of this paper is to extend the results obtained by Rus in [12] and [11], Petruşel A. in [8], to the case of product spaces using Prešić contraction condition.

2. Main results

The first result is the extension of the Theorem in [12], to the case of product metric spaces. For this to be held, the condition (i) is created taking a particular case of Prešić's operator, i.e., $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \alpha$ and using the triangle inequality.

Theorem 2.1. *Let X be a nonempty set, d and ρ two metrics on X and $f : X^k \to X$ a mapping satisfying the condition:*

$$\rho(f(x_0,x_1,\ldots,x_{k-1}), f(x_1,x_2,\ldots,x_k)) \le \alpha_1 \cdot \rho(x_0,x_1) + \alpha_2 \cdot \rho(x_1,x_2) + \cdots + \alpha_k \cdot \rho(x_{k-1},x_k), \tag{2.1}$$

for any $x_0, x_1, \ldots, x_k \in X$, *where* α_i, $i = \overline{1,n}$ *are nonnegative constants such that*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k < 1.$$

If

(1) *there exists* $C > 0$ *such that*

$$d(f(x_0, x_1, \ldots, x_{k-1}), f(x_1, x_2, \ldots, x_k)) \leq C \cdot \rho(x_0, x_k), \tag{2.2}$$

for all $x_0, x_1, \ldots, x_k \in X$;

(2) (X, d) *is a complete metric space;*

(3) $f : (X^k, \bar{d}) \to (X, d)$ *is continuous.*

Then

(*a*) *f has a unique fixed point* x^*, $x^* = f(x^*, x^*, \ldots, x^*)$;

(*b*) *the sequence* $\{x_n\}_{n\geq 0}$ *with* $x_0, \ldots, x_{k-1} \in X$ *and*

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k+1}),\ n \geq k-1,$$

converges to x^* *with respect to d;*

(*c*) *the sequence* $\{y_n\}_{n\geq 0}$, $y_{n+1} = f(y_n, y_n, \ldots, y_n), n \geq 0$, *converges to* x^* *with respect to* ρ.

Proof. Let $\{x_n\}_{n\geq 0}$, $x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k+1})$, $n \geq k-1$,

$$\begin{aligned} \rho(x_n, x_{n+1}) &= \rho\big(f(x_{n-1}, x_{n-2}, \ldots, x_{n-k}), f(x_n, x_{n-1}, \ldots, x_{n-k+1})\big) \\ &\leq \alpha_1 \rho(x_{n-1}, x_n) + \alpha_2 \rho(x_{n-2}, x_{n-1}) + \cdots + \alpha_k \rho(x_{n-k}, x_{n-k+1}). \end{aligned}$$

By Lemma 1.1, we have

$$\rho(x_{n-1}, x_n) \leq L \cdot \theta^n, n \geq 1, \tag{2.3}$$

For $n \geq 1, m \geq 1$, by (2.3) we obtain:

$$\begin{aligned} \rho(x_n, x_{n+m}) &\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{n+m-1}, x_{n+m}) \\ &\leq L \cdot \theta^{n+1} + L \cdot \theta^{n+2} + \cdots + L \cdot \theta^{n+m} \\ &= L \cdot \theta^{n+1}(1 + \theta + \theta^2 + \cdots + \theta^{m-1}) \end{aligned}$$

so

$$\rho(x_n, x_{n+m}) \leq L \cdot \theta^{n+1} \cdot \frac{1-\theta^m}{1-\theta}, n \geq 1, m \geq 1.$$

Since $\theta \in (0,1)$, it follows that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in (X, ρ).

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m})$$

From relation (2.2) we obtain:

$$d(x_n, x_{n+1}) = d(x_{n+1}, x_n)$$

$$= d(f(x_n, x_{n-1}, \ldots, x_{n-k+1}), f(x_{n-1}, x_{n-2} \ldots, x_{n-k+1}, x_{n-k})) \leq C \cdot \rho(x_n, x_{n-k})$$

and

$$\rho(x_n, x_{n-k}) \leq L \cdot \theta^{n-k+1} \cdot \frac{1-\theta^k}{1-\theta}.$$

Since $\frac{1-\theta^k}{1-\theta} < 1$, we have

$$d(x_{n+1}, x_n) \leq C \cdot L \cdot \theta^{n-k+1}.$$

Similarly

$$d(x_{n+2}, x_{n+1}) \leq C \cdot L \cdot \theta^{n-k+2}, \dots, d(x_{n+m}, x_{n+m-1}) \leq C \cdot L \cdot \theta^{n-k+m}.$$

So,

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+m-1}, x_{n+m}) \\ &\leq C \cdot L \cdot \theta^{n-k+1} + C \cdot L \cdot \theta^{n-k+2} + \dots + C \cdot L \cdot \theta^{n-k+m} \\ &= C \cdot L \cdot \theta^{n-k} \cdot (\theta + \theta^2 + \dots + \theta^m) = C \cdot L \cdot \theta^{n-k+1} \cdot \frac{1-\theta^m}{1-\theta}. \end{aligned}$$

Since $\theta \in (0,1)$, it follows that $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in the complete metric space (X, d) so $\{x_n\}_{n \geq 0}$ is also convergent: there exists $x^* \in X$ such that

$$\lim_{n \to \infty} d(x_n, x^*) = 0.$$

By continuity of f and considering the associate operator $F : X \to X$,

$$F(x) = f(x, x, \dots, x),$$

for any $x \in X$ we have:

$$\begin{aligned} d(F(x^*), x^*) &= d(f(x^*, x^*, \dots, x^*), x^*) = d(f(\lim_{n \to \infty} x_n, \dots, \lim_{n \to \infty} x_{n-k+1}), x^*) \\ &= \lim_{n \to \infty} d(f(x_n, \dots, x_{n-k+1}), x^*) = \lim_{n \to \infty} d(x_{n+1}, x^*) = 0. \end{aligned}$$

Therefore $x^* = f(x^*, x^*, \dots, x^*) = F(x^*)$ is a fixed point of f.
We suppose that f has another fixed point $y^* = f(y^*, y^*, \dots, y^*)$.

$$\begin{aligned} \rho(x^*, y^*) &= \rho(f(x^*, x^*, \dots, x^*), f(y^*, y^*, \dots, y^*)) \\ &\leq \rho(f(x^*, x^*, \dots, x^*), f(x^*, x^*, \dots, y^*)) \\ &+ \rho(f(x^*, x^*, \dots, y^*), f(x^*, \dots, x^*, y^*, y^*)) \\ &+ \dots + \rho(f(x^*, y^*, \dots, y^*), f(y^*, y^*, \dots, y^*)) \\ &\leq \alpha_k \cdot \rho(x^*, y^*) + \alpha_{k-1} \cdot \rho(x^*, y^*) + \dots + \alpha_1 \cdot \rho(x^*, y^*) \\ &= \rho(x^*, y^*) \cdot \sum_{i=1}^{k} \alpha_i. \end{aligned}$$

As $\sum_{i=1}^{k} \alpha_i < 1$, we obtain $\rho(x^*, y^*) = 0$, so $x^* = y^*$. The uniqueness of fixed point is proved.
From $\{y_n\}_{n \geq 0}$, $y_{n+1} = f(y_n, y_n, \dots, y_n)$, $n \geq 0$ and considering the associate operator $F : X \to X$, $F(x) = f(x, x, \dots, x)$, for any $x, y \in X$ we have:

$$\begin{aligned} \rho\big(F(x), F(y))\big) &= \rho\big(f(x, x, \dots, x), f(y, y, \dots, y)\big) \\ \leq \rho\big(f(x, x, \dots, x), f(x, x, \dots, y)\big) &+ \rho\big(f(x, x, \dots, y), f(x, \dots, x, y, y)\big) \\ &+ \dots + \rho\big(f(x, y, \dots, y), f(y, y, \dots, y)\big). \end{aligned}$$

By (2.1) we obtain:

$$\begin{aligned} \rho\big(F(x), F(y))\big) \leq & [\alpha_1 \cdot \rho(x, x) + \alpha_2 \cdot \rho(x, x) + \dots + \alpha_{k-1} \cdot \rho(x, x) + \alpha_k \cdot \rho(x, y)] \\ & + [\alpha_1 \cdot \rho(x, x) + \alpha_2 \cdot \rho(x, x) + \dots + \alpha_{k-1} \cdot \rho(x, y) + \alpha_k \cdot \rho(y, y)] \end{aligned}$$

$$+\cdots+[\alpha_1\cdot\rho(x,y)+\alpha_2\cdot\rho(y,y)+\cdots+\alpha_{k-1}\cdot\rho(y,y)+\alpha_k\cdot\rho(y,y)].$$

so

$$\rho\big(F(x),F(y))\big)\leq\sum_{i=1}^{k}\alpha_i\cdot\rho(x,y)=\alpha\cdot\rho(x,y),$$

for any $x,y\in X$, that is, F is a Banach contraction with constant $\alpha\in[0,1)$.
By the Contraction Mapping Principle of Banach, F has a unique fixed point $x^*=f(x^*,x^*,\ldots,x^*)$,

$$\lim_{n\to\infty}F(y_n)=x^*.$$

It follows that $y_{n+1}=f(y_n,y_n,\ldots,y_n)=F(y_n)$ converges to $x^*=f(x^*,x^*,\ldots,x^*)$ in (X,ρ).

Remark 2.1. We have the following particular cases of Theorem 2.1:

1. If $k=1$, $C=1$, by Theorem 2.1 we get Maia's fixed point theorem, see [4].
2. If $k=1$, by Theorem 2.1 we get a Maia type fixed point theorem, given by Rus I. A. in [12].

The following result is the extension of Petruşel's theorem in [8] regarding the existence and uniqueness of solutions of Fredholm-Volterra integral equations, to the case of product spaces.

Theorem 2.2. *Let* $[a,b]\subset\mathbb{R}$, $X=C([a,b])$. *On* X^k *we define the following two metrics:*

$$d(x,y)=\max_{t\in[a,b]}[|x(t)-y(t)|\cdot e^{-\tau(t-a)}] \tag{2.4}$$

$$\rho(x,y)=\left(\int_a^b|x(t)-y(t)|^2\cdot e^{-\tau(t-a)}dt\right)^{\frac{1}{2}} \tag{2.5}$$

where $x=(x_1,x_2,\ldots,x_k),\ y=(y_1,y_2,\ldots,y_k)\in X^k$.

Let the following Fredholm-Volterra equation be:

$$x(t)=\int_a^b K(t,s,x(s))ds+\int_a^t H(t,s,x(s))ds;\ t\in[a,b]; \tag{2.6}$$

We suppose that:

(i) $K,H\in C([a,b]\times[a,b]\times X^k)$;

(ii) there exists $L_{K_1},L_{K_2},\ldots,L_{K_k}\geq 0$, $L_{K_1}+L_{K_2}+\cdots+L_{K_k}=L_K\in(0,1)$ *such that*

$$|K(t,s,x(s))-K(t,s,y(s))|$$
$$\leq L_{K_1}\cdot|x_1(s)-y_1(s)|+L_{K_2}\cdot|x_2(s)-y_2(s)|+\cdots+L_{K_k}\cdot|x_k(s)-y_k(s)|$$

for all $t,s\in[a,b],\ x=(x_1,x_2,\ldots,x_k),y=(y_1,y_2,\ldots,y_k)\in X^k$;

(iii) there exists $L_{H_1},L_{H_2},\ldots,L_{H_k}\geq 0$, $L_{H_1}+L_{H_2}+\cdots+L_{H_k}=L_H\in(0,1)$ *such that*

$$|H(t,s,x(s))-H(t,s,y(s))|$$
$$\leq L_{H_1}\cdot|x_1(s)-y_1(s)|+L_{H_2}\cdot|x_2(s)-y_2(s)|+\cdots+L_{H_k}\cdot|x_k(s)-y_k(s)|$$

for all $t,s\in[a,b],x=(x_1,x_2,\ldots,x_k),y=(y_1,y_2,\ldots,y_k)\in X^k$;

(iv) *there exists* $\tau > 0$ *such that*

$$\frac{L_K \cdot e^{\frac{\tau}{2}(b-t)} + L_H}{\sqrt{\tau}} < 1$$

Then the equation (2.6) *has in* X^k *a unique solution.*
Proof. Let $f : X^k \to X$, where

$$f(x)(t) = \int_a^b K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds;\ t \in [a, b]$$

We consider on X^k the two metrics given in (2.4), (2.5). We have:

$$\rho(f(x), f(y)) = \left(\int_a^b |f(x)(t) - f(y)(t)|^2 \cdot e^{-\tau(t-a)}dt\right)^{\frac{1}{2}}$$
$$= \left(\int_a^b \left|\int_a^b (K(t, s, x(s)) - K(t, s, y(s)))ds\right.\right.$$
$$\left.\left. + \int_a^t (H(t, s, x(s)) - H(t, s, y(s)))ds\right|^2 \cdot e^{-\tau(t-a)}dt\right)^{\frac{1}{2}}$$

Using the Minkovski inequality we have:

$$\rho(f(x), f(y)) \leq \left(\int_a^b \left[\int_a^b |K(t, s, x(s)) - K(t, s, y(s))|ds\right]^2 \cdot e^{-\tau(t-a)}ds\right)^{\frac{1}{2}}$$
$$+ \left(\int_a^b \left[\int_a^t |H(t, s, x(s)) - H(t, s, y(s))|ds\right]^2 \cdot e^{-\tau(t-a)}ds\right)^{\frac{1}{2}}$$

We use now the Hölder inequality:

$$\int_a^b |K(t, s, x(s)) - K(t, s, y(s))|ds = \int_a^b |K(t, s, x(s)) - K(t, s, y(s))| \cdot e^{-\frac{\tau}{2}(s-a)} \cdot e^{\frac{\tau}{2}(s-a)}ds$$
$$\leq \left(\int_a^b |K(t, s, x(s)) - K(t, s, y(s))|^2 \cdot e^{-\tau(s-a)}ds\right)^{\frac{1}{2}} \left(\int_a^b e^{\tau(s-a)}\right)^{\frac{1}{2}}$$
$$= \left(\int_a^b |K(t, s, (x_1, x_2, \ldots, x_k)(s)) - K(t, s, (y_1, y_2, \ldots, y_k)(s))|^2 \cdot e^{-\tau(s-a)}ds\right)^{\frac{1}{2}}$$
$$\cdot \left(\int_a^b e^{\tau(s-a)}\right)^{\frac{1}{2}}$$
$$\leq \Big[\int_a^b (L_{K_1}|x_1(s) - y_1(s)| + L_{K_2}|x_2(s) - y_2(s)| + \ldots$$
$$+ L_{K_k}|x_k(s) - y_k(s)|)^2 \cdot e^{-\tau(s-a)}ds\Big]^{\frac{1}{2}} \left(\int_a^b e^{\tau(s-a)}\right)^{\frac{1}{2}}$$

$$\leq \Big[\int_a^b (L_{K_1}^2|x_1(s)-y_1(s)|^2 + L_{K_2}^2|x_2(s)-y_2(s)|^2 + \dots$$

$$+L_{K_k}^2|x_k(s)-y_k(s)|^2)\cdot e^{-\tau(s-a)}ds\Big]^{\frac{1}{2}}\left(\int_a^b e^{\tau(s-a)}\right)^{\frac{1}{2}}$$

$$\leq \left[\left(\int_a^b L_{K_1}^2|x_1(s)-y_1(s)|^2\cdot e^{-\frac{\tau}{2}(s-a)}ds\right)^{\frac{1}{2}}\right.$$

$$+\left(\int_a^b L_{K_2}^2|x_2(s)-y_2(s)|^2\cdot e^{-\frac{\tau}{2}(s-a)}ds\right)^{\frac{1}{2}}+\dots$$

$$\left.+\left(\int_a^b L_{K_k}^2|x_k(s)-y_k(s)|^2\cdot e^{-\frac{\tau}{2}(s-a)}ds\right)^{\frac{1}{2}}\right]\left(\int_a^b e^{\tau(s-a)}\right)^{\frac{1}{2}}$$

$$\leq \left[L_{K_1}\left(\int_a^b |x_1(s)-y_1(s)|^2\cdot e^{-\frac{\tau}{2}(s-a)}ds\right)^{\frac{1}{2}}\right.$$

$$+L_{K_2}\left(\int_a^b |x_2(s)-y_2(s)|^2\cdot e^{-\frac{\tau}{2}(s-a)}ds\right)^{\frac{1}{2}}+\dots$$

$$\left.+L_{K_k}\left(\int_a^b |x_k(s)-y_k(s)|^2\cdot e^{-\frac{\tau}{2}(s-a)}ds\right)^{\frac{1}{2}}\right]\left(\frac{1}{\tau}\cdot e^{\tau(b-a)}\right)^{\frac{1}{2}}$$

$$= L_{K_1}\cdot\left(\frac{1}{\tau}\cdot e^{\tau(b-a)}\right)^{\frac{1}{2}}\cdot\rho(x_1(s),y_1(s)) + L_{K_2}\cdot\left(\frac{1}{\tau}\cdot e^{\tau(b-a)}\right)^{\frac{1}{2}}\cdot\rho(x_2(s),y_2(s))$$

$$+\dots+L_{K_k}\cdot\left(\frac{1}{\tau}\cdot e^{\tau(b-a)}\right)^{\frac{1}{2}}\cdot\rho(x_k(s),y_k(s)).$$

By a similar approach we have:

$$\int_a^b |H(t,s,x(s))-H(t,s,y(s))|$$

$$\leq L_{H_1}\cdot\left(\frac{1}{\tau}\cdot e^{\tau(t-a)}\right)^{\frac{1}{2}}\cdot\rho(x_1(s),y_1(s)) + L_{H_2}\cdot\left(\frac{1}{\tau}\cdot e^{\tau(t-a)}\right)^{\frac{1}{2}}\cdot\rho(x_2(s),y_2(s))$$

$$+\dots+L_{H_k}\cdot\left(\frac{1}{\tau}\cdot e^{\tau(t-a)}\right)^{\frac{1}{2}}\cdot\rho(x_k(s),y_k(s))$$

and so it follows:

$$\rho(f(x),f(y)) \leq \frac{L_K\cdot e^{\frac{\tau}{2}(b-t)}+L_H}{\sqrt{\tau}}\cdot(\rho(x_1,y_1)+\rho(x_2,y_2)+\dots+\rho(x_k,y_k)).$$

The operator f satisfies the conditions of the Theorem 2.1, and the theorem is proved.

The following result is the extension of Theorem in [11] extended to the case of product metric spaces.

Theorem 2.3. *Let X be a nonempty set, d and ρ two metrics on X and $f : X^k \to X$ a Prešić operator w.r.t. ρ. We suppose that:*

(i) there exists $c > 0$ such that

$$d(f(x_0, x_1, \ldots, x_{k-1}), f(x_1, x_2, \ldots, x_k)) \leq c \cdot \rho(x_0, x_k),$$

for all $x_0, x_1, \ldots, x_k \in X$;

(ii) there exists $c_1 > 0$ such that

$$\rho(x_0, x_1) \leq c_1 \cdot d(x_0, x_1),$$

for all $x_0, x_1 \in X$;

(iii) (X, d) is a complete metric space;

(iv) $f : (X^k, \bar{d}) \to (X, d)$ is continuous.

Then:

(a) f has a unique fixed point $x^* = f(x^*, \ldots, x^*)$;

(b) the sequence $\{x_n\}_{n\geq 0}$ with $x_0, \ldots, x_{k-1} \in X$ and

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k+1}),\ n \geq k-1,$$

converges to x^* w.r.t. d, with a rate estimated by

$$d(x_{n+1}, x^*) \leq c \cdot c_1 \cdot \alpha d(x_{n-k}, x^*) + c^2 \cdot c_1 \cdot M \cdot \theta^{n-k-1}, \tag{2.7}$$

where $M = L[\alpha_1 + (\alpha_1 + \alpha_2)\theta^{-1} + \cdots + (\alpha_1 + \cdots + \alpha_{k-1})\theta^{-k}] > 0$.

(c) let $g : X^k \to X$ be such that there exists $\eta > 0$, for all $x \in X$

$$d(f(x, x, \ldots, x), g(x, x, \ldots, x)) \leq \eta,$$

If $x_g^* \in F_g$, $x_g^* = g(x_g^*, x_g^*, \ldots, x_g^*)$, then

$$d(f(x_f^*, x_f^*, \ldots, x_f^*), g(x_g^*, x_g^*, \ldots, x_g^*)) \leq \frac{\eta}{1 - c \cdot c_1 \cdot \alpha}. \tag{2.8}$$

(d) If $(X^k, \|\cdot\|_1, \|\cdot\|_2)$ is a linear space with two norms and

$$d(x, y) = \|x - y\|_1 = \max_{t\in X}|x(t) - y(t)|,$$

and

$$\rho(x, y) = \|x - y\|_2 = \left(\int_X |x(t) - y(t)|^2 dt\right)^{\frac{1}{2}},$$

where $x = (x_1, x_2, \ldots, x_k), y = (y_1, y_2, \ldots, y_k) \in X^k$, then

$$1_{X^k} - f : (X^k, \|\cdot\|_1) \to (X, \|\cdot\|_2)$$

is a homeomorphism.

Proof. (a) and (b):

Let $\{x_n\}_{n\geq 0}$, $x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k+1}), n \geq k-1$,

$$\begin{aligned}\rho(x_n, x_{n+1}) &= \rho\big(f(x_{n-1}, x_{n-2}, \ldots, x_{n-k}), f(x_n, x_{n-1}, \ldots, x_{n-k+1})\big) \\ &\leq \alpha_1\rho(x_{n-1}, x_n) + \alpha_2\rho(x_{n-2}, x_{n-1}) + \cdots + \alpha_k\rho(x_{n-k}, x_{n-k+1}).\end{aligned}$$

By Lemma 1.1, we have

$$\rho(x_{n-1}, x_n) \leq L \cdot \theta^n, n \geq 1, \tag{2.9}$$

For $n \geq 1, m \geq 1$, by (2.9) we obtain:

$$\begin{aligned}\rho(x_n, x_{n+m}) &\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{n+m-1}, x_{n+m})\\ &\leq L \cdot \theta^{n+1} + L \cdot \theta^{n+2} + \cdots + L \cdot \theta^{n+m}\\ &= L \cdot \theta^{n+1}(1 + \theta + \theta^2 + \cdots + \theta^{m-1})\end{aligned}$$

so

$$\rho(x_n, x_{n+m}) \leq L \cdot \theta^{n+1} \cdot \frac{1-\theta^m}{1-\theta},\ n \geq 1,\ m \geq 1.$$

Since $\theta \in (0,1)$, it follows that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in (X, ρ).

For $n \leq m$,

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m})$$

From relation (i) we obtain:

$$d(x_n, x_{n+1}) = d(x_{n+1}, x_n)$$

$$= d(f(x_n, x_{n-1}, \ldots, x_{n-k+1}), f(x_{n-1}, x_{n-2} \ldots, x_{n-k+1}, x_{n-k})) \leq c \cdot \rho(x_n, x_{n-k})$$

and

$$\rho(x_n, x_{n-k}) \leq L \cdot \theta^{n-k+1} \cdot \frac{1-\theta^k}{1-\theta}.$$

Since $\frac{1-\theta^k}{1-\theta} < 1$, we have

$$d(x_{n+1}, x_n) \leq c \cdot L \cdot \theta^{n-k+1}. \tag{2.10}$$

Similarly

$$d(x_{n+2}, x_{n+1}) \leq c \cdot L \cdot \theta^{n-k+2}, \ldots, d(x_{n+m}, x_{n+m-1}) \leq c \cdot L \cdot \theta^{n-k+m}.$$

So,

$$\begin{aligned}d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m})\\ &\leq c \cdot L \cdot \theta^{n-k+1} + c \cdot L \cdot \theta^{n-k+2} + \cdots + c \cdot L \cdot \theta^{n-k+m}\\ &= c \cdot L \cdot \theta^{n-k} \cdot (\theta + \theta^2 + \cdots + \theta^m) = c \cdot L \cdot \theta^{n-k+1} \cdot \frac{1-\theta^m}{1-\theta}.\end{aligned}$$

Since $\theta \in (0,1)$, it follows that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in the complete metric space (X, d) so $\{x_n\}_{n\geq 0}$ is also convergent: there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

By continuity of f and considering the associate operator $F : X \to X$,

$$F(x) = f(x, x, \ldots, x),$$

for any $x \in X$ we have:

$$\begin{aligned}d(F(x^*), x^*) &= d(f(x^*, x^*, \ldots, x^*), x^*) = d(f(\lim_{n\to\infty} x_n, \ldots, \lim_{n\to\infty} x_{n-k+1}), x^*)\\ &= \lim_{n\to\infty} d(f(x_n, \ldots, x_{n-k+1}), x^*) = \lim_{n\to\infty} d(x_{n+1}, x^*) = 0.\end{aligned}$$

Therefore $x^* = f(x^*, x^*, \ldots, x^*) = F(x^*)$ is a fixed point of f.

We suppose that f has another fixed point $y^* = f(y^*, y^*, \dots, y^*)$.

$$\begin{aligned}\rho(x^*, y^*) &= \rho(f(x^*, x^*, \dots, x^*), f(y^*, y^*, \dots, y^*))\\ &\leq \rho(f(x^*, x^*, \dots, x^*), f(x^*, x^*, \dots, y^*))\\ &+\rho(f(x^*, x^*, \dots, y^*), f(x^*, \dots, x^*, y^*, y^*))\\ &+ \dots + \rho(f(x^*, y^*, \dots, y^*), f(y^*, y^*, \dots, y^*))\\ &\leq \alpha_k \cdot \rho(x^*, y^*) + \alpha_{k-1} \cdot \rho(x^*, y^*) + \dots + \alpha_1 \cdot \rho(x^*, y^*)\\ &= \rho(x^*, y^*) \cdot \sum_{i=1}^{k} \alpha_i.\end{aligned}$$

As $\sum_{i=1}^{k} \alpha_i < 1$, we obtain $\rho(x^*, y^*) = 0$, so $x^* = y^*$. The uniqueness of fixed point is proved.

To obtain the estimation (2.7) we use (i), the Prešić contraction condition and (ii):

$$\begin{aligned}d(x_{n+1}, x^*) &= d(f(x_n, x_{n-1}, \dots, x_{n-k+1}), f(x^*, x^*, \dots, x^*))\\ &\leq c \cdot \rho(x_n, x^*) = c \cdot \rho(f(x_{n-1}, x_{n-2}, \dots, x_{n-k}), f(x^*, x^*, \dots, x^*))\\ &\leq c \cdot [\rho(f(x_{n-1}, x_{n-2}, \dots, x_{n-k}), f(x_{n-2}, \dots, x_{n-k}, x^*))\\ &+\rho(f(x_{n-2}, \dots, x_{n-k}, x^*), f(x_{n-3}, \dots, x_{n-k}, x^*, x^*)) + \dots\\ &+\rho(f(x_{n-k}, x^*, \dots, x^*), f(x^*, x^*, \dots, x^*))]\\ &\leq c \cdot [\alpha_1 \rho(x_{n-1}, x_{n-2}) + \alpha_2 \rho(x_{n-2}, x_{n-3}) + \dots + \alpha_k \rho(x_{n-k}, x^*)\\ &+\alpha_1 \rho(x_{n-2}, x_{n-3}) + \alpha_2 \rho(x_{n-3}, x_{n-4}) + \dots + \alpha_{k-1} \rho(x_{n-k}, x^*) + \alpha_k \rho(x^*, x^*)\\ &+ \dots + \alpha_1 \rho(x_{n-k}, x^*) + \alpha_2 \rho(x^*, x^*) + \dots + \alpha_k \rho(x^*, x^*)]\\ &\leq c \cdot c_1 \cdot [\alpha_1 d(x_{n-1}, x_{n-2}) + \alpha_2 d(x_{n-2}, x_{n-3}) + \dots + \alpha_k d(x_{n-k}, x^*)\\ &+\alpha_1 d(x_{n-2}, x_{n-3}) + \alpha_2 d(x_{n-3}, x_{n-4}) + \dots + \alpha_{k-1} d(x_{n-k}, x^*) + \alpha_k d(x^*, x^*)\\ &+ \dots + \alpha_1 d(x_{n-k}, x^*) + \alpha_2 d(x^*, x^*) + \dots + \alpha_k d(x^*, x^*)]\end{aligned}$$

$$\begin{aligned}d(x_{n+1}, x^*) &\leq c \cdot c_1 \cdot [\alpha_1 d(x_{n-1}, x_{n-2}) + (\alpha_1 + \alpha_2) d(x_{n-2}, x_{n-3})\\ &+(\alpha_1 + \alpha_2 + \alpha_3) d(x_{n-3}, x_{n-4}) + \dots\\ &+(\alpha_1 + \alpha_2 + \dots + \alpha_{k-1}) d(x_{n-k-1}, x_{n-k}) + \alpha d(x_{n-k}, x^*)].\end{aligned}$$

Now using (2.10) it follows that

$$d(x_{n+1}, x^*) \leq c \cdot c_1 \cdot [\alpha_1 \cdot c \cdot L \cdot \theta^{n-k-1} + (\alpha_1 + \alpha_2) \cdot c \cdot L \cdot \theta^{n-k-2}$$
$$+(\alpha_1+\alpha_2+\alpha_3) \cdot c \cdot L \cdot \theta^{n-k-3} + \dots + (\alpha_1+\alpha_2+\dots+\alpha_{k-1}) \cdot c \cdot L \cdot \theta^{n-2k-1} + \alpha d(x_{n-k}, x^*)]$$
$$= c \cdot c_1 \cdot \alpha d(x_{n-k}, x^*) + c^2 \cdot c_1 \cdot L \cdot \theta^{n-k-1} \cdot [\alpha_1 + (\alpha_1+\alpha_2)\theta^{-1} + \dots + (\alpha_1 + \dots + \alpha_{k-1})\theta^{-k}]$$

Denoting $M = L[\alpha_1 + (\alpha_1 + \alpha_2)\theta^{-1} + \dots + (\alpha_1 + \dots + \alpha_{k-1})\theta^{-k}]$, we obtain the estimation (2.7).

(c) :

$$\begin{aligned}d(x_f^*, x_g^*) &= d(f(x_f^*, x_f^*, \dots, x_f^*), g(x_g^*, x_g^*, \dots, x_g^*))\\ &\leq d(f(x_f^*, x_f^*, \dots, x_f^*), f(x_g^*, x_g^*, \dots, x_g^*)) + d(f(x_g^*, x_g^*, \dots, x_g^*), g(x_g^*, x_g^*, \dots, x_g^*))\\ &\leq d(f(x_f^*, x_f^*, \dots, x_f^*), f(x_g^*, x_g^*, \dots, x_g^*)) + \eta\\ &\leq c \cdot \rho(f(x_f^*, x_f^*, \dots, x_f^*), f(x_g^*, x_g^*, \dots, x_g^*)) + \eta\end{aligned}$$

$$\leq c\cdot[\rho(f(x_f^*,x_f^*,\ldots,x_f^*),f(x_f^*,\ldots,x_f^*,x_g^*))+\rho(f(x_f^*,\ldots,x_f^*,x_g^*),f(x_f^*,\ldots,x_f^*,x_g^*,x_g^*))$$
$$+\cdots+\rho(f(x_f^*,x_g^*,\ldots,x_g^*),f(x_g^*,x_g^*,\ldots,x_g^*))]+\eta$$

and further on

$$\leq c\cdot[\alpha_1\rho(x_f^*,x_f^*)+\alpha_2\rho(x_f^*,x_f^*)+\cdots+\alpha_k\rho(x_f^*,x_g^*)$$
$$+\alpha_1\rho(x_f^*,x_f^*)+\cdots+\alpha_{k-1}\rho(x_f^*,x_g^*)+\alpha_k\rho(x_f^*,x_g^*)$$
$$+\cdots+\alpha_1\rho(x_f^*,x_f^*)+\alpha_2\rho(x_g^*,x_g^*)+\cdots+\alpha_k\rho(x_g^*,x_g^*)]+\eta$$
$$=c\cdot\alpha\cdot\rho(x_f^*,x_g^*)+\eta$$
$$d(x_f^*,x_g^*)\leq c\cdot\alpha\cdot c_1\cdot d(x_f^*,x_g^*)+\eta$$
$$d(x_f^*,x_g^*)\leq\frac{\eta}{1-c\cdot c_1\cdot\alpha}.$$

(d) :

Let $x,y\in X^k$, $x=(x_1,x_2,\ldots,x_k)$, $y=(y_1,y_2,\ldots,y_k)$. $1_{X^k}-f:X^k\to X$ is injective if from $1_{X^k}-f(x)=1_{X^k}-f(y)$, we have $x=y$.

In $1_{X^k}-f(x)=1_{X^k}-f(y)$, $1_{X^k}(x,x,\ldots,x)=(x,x,\ldots,x)$, so $f(x)=f(y)$.

$$d(f(x),f(y))=d((x_1,x_2,\ldots,x_k),(y_1,y_2,\ldots,y_k))=|x_1-y_1|+|x_2-y_2|+\cdots+|x_k-y_k|$$

and

$$d(f(x),f(y))=\|f(x)-f(y)\|=0$$

so

$$x_1=y_1,\ x_2=y_2,\ \ldots,\ x_k=y_k,\ \text{that is } x=y.$$

$1_{X^k}-f:X^k\to X$ is surjective if for any $y\in X$, there exists $x=(x_1,x_2,\ldots,x_k)\in X^k$ such that $1_{X^k}-f(x)=y$.

We define the mapping $g:X^k\to X$ by

$$(x_1,x_2,\ldots,x_k)\mapsto f(x_1,x_2,\ldots,x_k)+y.$$

We have

$$\|f(x_1,x_2,\ldots,x_k)-g(x_2,x_3,\ldots,x_{k+1})\|_2=\|f(x_1,x_2,\ldots,x_k)-f(x_2,x_3,\ldots,x_{k+1})\|_2$$
$$\leq\alpha_1\|x_1-x_2\|_2+\alpha_2\|x_2-x_3\|_2+\cdots+\alpha_k\|x_k-x_{k+1}\|_2.$$

By Theorem 2.1, the mapping g has a unique fixed point, $x^*=f(x^*,x^*,\ldots,x^*)$, and $x^*-f(x^*)=y$.

We obtain $1_{X^k}-f:X^k\to X$ is a bijection, from (i), (iii), (iv) and f a Prešić operator.

From (iv), $1_{X^k}-f:(X^k,\bar{d})\to(X,d)$ is continuous.

From (c), $(1_{X^k}-f)^{-1}:(X^k,\bar{d})\to(X,d)$ is continuous.

Remark. We have the following important particular cases of Theorem 2.3:

1. If $k=1$, by Theorem 2.3 we get a Maia type fixed point theorem, given by Rus I. A. in [13].
2. If $d=\rho$ and $k=1$ by Theorem 2.1 we get Banach's fixed point theorem [2].

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