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CARISTI-LIKE CONDITION. EXISTENCE OF SOLUTIONS TO EQUATIONS AND MINIMA OF FUNCTIONS IN METRIC SPACES

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Abstract. In the paper, there were studied Caristi-like conditions that guaranteed existence of a minimum of a function on a metric space. For functions dependent on a parameter, there were obtained conditions for existence of a minimum for each value of the parameter. These results were applied to derive conditions for fixed point and coincidence point existence for mappings in metric spaces. For mappings dependent on a parameter, there were obtained conditions of coincidence point existence for each value of the parameter.
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Introduction

In the paper [5] by J.Caristi, conditions for the existence of a fixed point for a mapping $g : X \to X$ acting in a complete metric space X were proposed. This result can be interpreted as sufficient conditions for the function $x \mapsto \rho_X(x, g(x))$ to attain its minimum and the minimal value to coincide with zero. This approach was used in [1]. In this paper, it is proved that under a Caristi-like condition (see Definition 1.1 below) a bounded from below lower semi-continuous function attains its minimum and an estimate of the distance from an arbitrary point to the minimum holds (see

Theorem 3 from [1] and Theorem 1.2 below). Some results from [1] developed results from [3].

It is important to note that these results on the existence of a minimum of a function are not an end in itself and are not related to the theory of extremal problems but provide a powerful research tool for studying non-linear equations. The main purpose of these theorems is to be applied to obtain coincidence points and fixed points existence conditions. The subsequent possible application of these results lies in the areas of optimal control (see, for example, [11]), theory of ordinary differential equations (see, for example, [13]) and control systems (see, for example, [12]).

Let us briefly describe the structure of this paper. It consists of four sections. In the first section, we obtain global and local theorems (Theorems 1.2 and 1.3, respectively) on the existence of a minimum for functions acting in metric spaces and satisfying Caristi-like conditions. The proofs are based on Ekeland's variational principle, which is a powerful tool for research (e.g., [6, 4]). In addition, we study a stability of both functions satisfying Caristi-like conditions and their minima (Propositions 1.5 – 1.8). For a function acting in a Banach space, we obtain conditions in the terms of upper Dini derivative for the function to satisfy Caristi-like condition (Lemma 1.9).

In the second section, we investigate the minimum existence problem for a function dependent on a parameter. We obtain a theorem on the existence of a minimum for all values of the parameter (Theorem 2.1), we obtain conditions for continuous dependence of the set of minimum points on a parameter (Propositions 2.6 - 2.9), and provide examples showing the essentiality of the assumptions of Theorem 2.1 (Example 2.10) and Propositions 2.7, 2.9 (Example 2.11).

In the third section, the obtained results on the minimum existence are applied to obtain solvability conditions for equations in metric spaces. A local analogue of Caristi fixed point theorem (Theorem 3.1) and coincidence point theorem for two set-valued mappings in metric spaces (Theorem 3.2) as well as its corollary on fixed point of a continuous mapping (Theorem 3.2′) are derived. An analogue of the coincidence point theorem is obtained for mappings, dependent on a parameter (Theorem 3.7). This result, in particular, generalizes the elementary implicit function theorem for contractive mappings from [8, 7].

The fourth section contains discussions and comparison of Caristi fixed point theorem and Theorem 3.2′. Examples of mappings for which only one out of these two theorems can be meaningfully applied are considered (Examples 4.1 – 4.4). The problem of the fixed point uniqueness is studied.

1. The existence of a minimum of lower semi-continuous functions and Caristi-like conditions

Let (X, ρ_X) be a complete metric space with the metric ρ_X. Given an arbitrary point $\bar{x} \in X$ and a number $\delta \geq 0$, denote a closed ball centered at $\bar{x}$ with the radius δ by $B_X(\bar{x}, \delta)$, i.e.

$$B_X(\bar{x}, \delta) := \{x \in X : \rho_X(\bar{x}, x) \leq \delta\}.$$

For non-empty subsets $M_1, M_2 \subset X$, denote the distance between them by

$$\operatorname{dist}_X(M_1, M_2) := \inf_{x_1 \in M_1,\, x_2 \in M_2} \rho_X(x_1, x_2),$$

the excess from M_1 to M_2 by

$$h_X^+(M_1, M_2) := \sup_{x \in M_1} \operatorname{dist}_X(x, M_2),$$

and the Hausdorff distance between M_1 and M_2 by

$$h_X(M_1, M_2) := \max\{h_X^+(M_1, M_2),\ h_X^+(M_2, M_1)\}.$$

Let $U : X \to \mathbb{R} \cup \{+\infty\}$ be a bounded from below function. Standardly denote by $\operatorname{dom} U$ the domain of U, i.e.

$$\operatorname{dom} U := \{x \in X : U(x) < +\infty\}.$$

Our goal is to find conditions that guarantee that the function U attains its minimum.

Definition 1.1. Given a subset $\widetilde{X} \subset X$, we say that the function U satisfies the Caristi-like condition on the set $\widetilde{X}$ for some $k > 0$ if

$$\forall\, x \in \widetilde{X} \quad \exists\, x' \in X : \quad x' \neq x, \quad U(x') + k\rho_X(x', x) \leq U(x). \tag{1.1}$$

We say that U satisfies the Caristi-like condition with $k > 0$ (without reference to the set $\widetilde{X}$) if it satisfies the Caristi-like condition on the set $\{x \in X : U(x) > \inf\limits_{\xi \in X} U(\xi)\}$ with $k > 0$.

Let us discuss this definition. Let X be a Banach space. Assume that the function U attains its minimum at the point $x_0 \in X$, and U is continuously differentiable in the neighbourhood of x_0. It is a straightforward task to ensure that U does not satisfy the Caristi-like condition on any punctured neighbourhood of the point x_0 for any $k > 0$, since $U'(x_0) = 0$. At the same time, for a function $U(x) := \|x - x_0\|$ that is non-differentiable at point x_0, the Caristi-like condition holds for $k = 1$ on any open set that does not contain the point x_0.

Consider the following property of the Caristi-like condition. Let the function $U : X \to \mathbb{R} \cup \{+\infty\}$ satisfy the Caristi-like condition for some $k > 0$, let a subset $\overline{X} \subset X$ be closed, and $\overline{U}$ be the restriction of U to $\overline{X}$. Then the Caristi-like condition does not necessarily hold for the function $\overline{U}$ for any $k > 0$. This can be illustrated by the following example. Set $X = [-1, 1]$, $U(x) = x^2\ \forall\, x \in [-1, 1/2)$, and $U(x) = (-5x + 3)/2\ \forall\, x \in [1/2, 1]$. The Caristi-like condition holds for $k = 1/2$ (condition (1.1) is satisfied if one takes $x' = 1$). At the same time, Caristi-like conditions does not hold for the restriction $\overline{U}$ of the function U on any ball $\overline{X} = B_X(0, \delta)$, $\delta \leq 1/2$, for any $k > 0$, since $\overline{U}(x) \equiv x^2$.

Theorem 1.2. *Assume that the function U is lower semi-continuous and $U(x) \geq \gamma$ for any $x \in X$. Given $\bar{x} \in \operatorname{dom} U$, $k > 0$, and $\delta \geq (U(\bar{x}) - \gamma)/k$, assume that U satisfies the Caristi-like condition (1.1) on the set*

$$\widetilde{X} = \{x \in B_X(\bar{x}, \delta) :\ \gamma < U(x) \leq U(\bar{x})\}.$$

Then

$$\exists\, \xi \in B_X(\bar{x}, \delta) : \quad \rho_X(\xi, \bar{x}) \leq \frac{U(\bar{x}) - \gamma}{k}, \quad U(\xi) = \min_{x \in X} U(x) = \gamma.$$

Proof. We will carry out the proof in analogy with Theorem 3 from [1] using Ekeland's variational principle (see [4], Chapter 5, §3).

The function $x \mapsto U(x) - \gamma$ satisfies the assumptions of the theorem. Therefore, we can assume without loss of generality that $\gamma = 0$ and, thus, $U(x) \geq 0$ for each $x \in X$. Set

$$\overline{X} = \{x \in X : U(x) \leq U(\bar{x})\}.$$

The set $\overline{X}$ is closed, since U is lower semi-continuous. Set $\varepsilon := U(\bar{x})$, $\lambda := \varepsilon/k$. In virtue of Ekeland's variational principle (see, for example, [4], Chapter 5, §3), applied in the complete metric space $(\overline{X}, \rho_X)$, there exists $\xi \in X$, such that

$$U(\xi) \leq U(\bar{x}), \qquad \rho_X(\bar{x}, \xi) \leq \lambda,$$

$$U(x) + \frac{\varepsilon}{\lambda}\rho_X(x, \xi) > U(\xi) \quad \forall\, x \in X: \quad U(x) \leq U(\bar{x}),\ x \neq \xi. \tag{1.2}$$

Let's prove that ξ is the desired point. Since $\lambda \leq \delta$ we have $\xi \in B(\bar{x}, \delta)$. So, it is enough to prove that $U(\xi) = 0$. Assume the contrary, i.e. $U(\xi) > 0$. In virtue of Caristi-like condition (1.1) there exists $x' \neq \xi$ such that

$$U(x') + k\rho_X(x', \xi) \leq U(\xi). \tag{1.3}$$

This contradicts the strict inequality in (1.2), since $\varepsilon/\lambda = k$. The contradiction proves that $U(\xi) = 0$. □

In [10], it is proved that if U satisfies the Caristi-like condition (1.1) on the whole X and the rest of the assumptions of Theorem 1.2 hold then U attains its minimum.

Note that Theorem 1.2 can be proved using a partial ordering of $X \times \mathbb{R}$ proposed by E. Bishop and R. Phelps (for more details see [1], [13]).

Assume that $\delta = +\infty$. Then Theorem 1.2 coincide with Theorem 3 from [1]. In addition, under the assumptions of Theorem 1.2, condition (1.1) holds if for each $x \in X$ one takes x' equal to the minimum point ξ such that $\rho_X(\xi, x) \leq (U(x) - \gamma)/k$. Moreover,

$$\operatorname{dist}_X(x, \Xi) \leq \frac{U(x) - \min\limits_{\chi \in X} U(\chi)}{k} \quad \forall\, x \in X,$$

where Ξ is the set of minima of the function U.

Consider the local analogue of Theorem 1.2. Let a point $\bar{x} \in X$, numbers $\gamma \in \mathbb{R}$, $k > 0$, $\delta \in [0, +\infty]$, and a function $U : B_X(\bar{x}, \delta) \to \mathbb{R} \cup \{+\infty\}$ be given.

Theorem 1.3. *Assume that*

- **(i)** *U is lower semi-continuous, bounded from below and $U(x) \geq \gamma$ for all $x \in B_X(\bar{x}, \delta)$;*
- **(ii)** *$\bar{x} \in \operatorname{dom} U$, $U(\bar{x}) \leq \gamma + \delta k$;*
- **(iii)** *for each $x \in B_X(\bar{x}, \delta)$, if*

$$\gamma < U(x) \leq U(\bar{x}) - k\rho_X(x, \bar{x}) \tag{1.4}$$

then there exists $x' \in B_X(\bar{x}, \delta)$ such that $x' \neq x$ and

$$U(x') + k\rho_X(x, x') \leq U(x). \tag{1.5}$$

Then there exists $\xi \in B_X(\bar{x}, \delta)$ *such that*

$$U(\xi) = \min_{x \in B_X(\bar{x},\delta)} U(x) = \gamma, \qquad \rho_X(\bar{x}, \xi) \leq \frac{U(\bar{x}) - \gamma}{k}.$$

Proof. As before, we can assume without loss of generality that $\gamma = 0$ and, therefore, $U(x) \geq 0 \ \forall x \in B_X(\bar{x}, \delta)$. Set

$$\overline{X} := \{x \in B_X(\bar{x}, \delta) : U(x) \leq U(\bar{x}) - k\rho_X(x, \bar{x})\}.$$

This set is closed, since the function U is lower semi-continuous, and non-empty, since it contains $\bar{x}$. Therefore, the metric space $(\overline{X}, \rho_X)$ is complete. Let us show that the assumptions of Theorem 1.2 hold for the function U on the space $\overline{X}$.

Take $x \in \overline{X}$ such that $U(x) > 0$. It follows from (iii) that there exists $x' \in B_X(\bar{x}, \delta)$ such that (1.5) holds. Moreover, $x' \in \overline{X}$, since

$$U(x') \leq U(x) - k\rho_X(x, x') \leq U(\bar{x}) - k\rho_X(\bar{x}, x) - k\rho_X(x, x') \leq U(\bar{x}) - k\rho_X(\bar{x}, x').$$

Here the first inequality follows from (1.5), the second one follows from the fact that $x \in \overline{X}$, and the third one from the triangle inequality for the metric ρ_X. Thus, the metric space $\overline{X}$ and the restriction of the function U to $\overline{X}$ satisfy all the assumptions of Theorem 1.2. So, the existence of the desired point ξ follows from Theorem 1.2. □

In condition (iii) of Theorem 1.3, the set of points x, for which condition (1.5) have to be verified, is defined by inequality (1.4). It is important that this set is smaller than the one in Theorem 1.2, where it was defined by the inequality $U(x) \leq U(\bar{x})$. In order to illustrate this fact consider the following simple example in which all the assumptions of Theorem 1.3 hold. However, if we replace the second inequality in (1.4) by the inequality $U(x) \leq U(\bar{x})$ then condition (1.5) does not hold.

Example 1.4. Let $X = \mathbb{R}$, $\bar{x} = 0$. Fix arbitrary $a > 0$, $b \in \mathbb{R}$, and set $U(x) = b + ax$ for $x \leq 0$, $U(x) = b$ for $x > 0$.

Take an arbitrary $\delta > 0$ and set $\gamma = \min\{U(x) : x \in B(0, \delta)\} = b - a\delta$. Condition (iii) of Theorem 1.3 holds for $k = a$, since (1.4) implies that inequality (1.5) should be verified only for $x \leq 0$. At the same time, inequality (1.5) fails for each $x \in (0, \delta]$.

Let us discuss the dependence of the Caristi-like conditions and minimum points on perturbations of the function U. Let a function $\widetilde{U} : X \times X \to \mathbb{R}$, a subset $\widetilde{X} \subset X$, and a number $k > 0$ be given.

Proposition 1.5. *Assume that for each* $x_2 \in \widetilde{X}$ *the function* $\widetilde{U}(\cdot, x_2) : X \to \mathbb{R}$ *satisfies Caristi-like condition (1.1) on the set* $\widetilde{X}$ *with the chosen* k*, and for each* $x_1 \in \widetilde{X}$*, the function* $\widetilde{U}(x_1, \cdot) : X \to \mathbb{R}$ *is* l*-Lipschitz on the set* $\widetilde{X}$*, i.e.*

$$|\widetilde{U}(x_1, x) - \widetilde{U}(x_1, x')| \leq l\rho_X(x, x') \ \ \forall x, x' \in \widetilde{X}.$$

If $l < k$ *then the function*

$$V : X \to \mathbb{R}, \quad V(x) = \widetilde{U}(x, x) \ \ \forall x \in X,$$

satisfies the Caristi-like condition on the set $\widetilde{X}$ *with* $k - l$*.*

Proof. Take arbitrary $x \in \widetilde{X}$. Since (1.1) holds for the function $\widetilde{U}(\cdot, x)$, we have

$$\exists\, x' \in X : \quad x' \neq x, \quad \widetilde{U}(x', x) + k\rho_X(x', x) \leq \widetilde{U}(x, x).$$

Since $\widetilde{U}(x', \cdot)$ is l-Lipschitz, we have $|\widetilde{U}(x', x') - \widetilde{U}(x', x)| \leq l\rho_X(x', x)$. Thus,

$$\widetilde{U}(x, x) \geq \widetilde{U}(x', x) + k\rho_X(x', x) = \widetilde{U}(x', x) + l\rho_X(x', x) + (k - l)\rho_X(x', x)$$
$$\geq \widetilde{U}(x', x) + |\widetilde{U}(x', x') - \widetilde{U}(x', x)| + (k - l)\rho_X(x', x) \geq \widetilde{U}(x', x') + (k - l)\rho_X(x', x).$$

Therefore, the function V satisfies Caristi-like condition on $\widetilde{X}$ with $k - l$. □

Given functions $U, \Delta : X \to \mathbb{R}$, consider the special case, when

$$\widetilde{U}(x_1, x_2) \equiv U(x_1) + \Delta(x_2).$$

In this case, Proposition 1.5 implies the following assertion.

Corollary 1.6. *Assume that the function U satisfies the Caristi-like condition on the set $\widetilde{X}$ with $k > 0$, and the function Δ is l-Lipschitz on the set $\widetilde{X}$. If $l < k$ then the function $U_\Delta : X \to \mathbb{R}$, $U_\Delta(x) = U(x) + \Delta(x)$, satisfies the Caristi-like condition on the set $\widetilde{X}$ with $k - l$.*

Corollary 1.6 and Theorem 1.2 imply the following assertion concerning minima of the function U_Δ. Let the assumptions of Theorem 1.2 hold and $\delta = +\infty$. Assume that U has the only point of minimum (existence of this point follows from Theorem 1.2). Then Theorem 1.2 implies

$$\exists\, \xi \in X : \quad U(\xi) = \min_{x \in X} U(x), \quad U(x) \geq U(\xi) + k\rho_X(\xi, x) \;\; \forall x \in X.$$

Let the function Δ be l-Lipscitz on the set X, $l < k$. Then $\Delta(x) \geq \Delta(\xi) - l\rho_X(\xi, x)$ for each $x \in X$. Therefore,

$$U(x) + \Delta(x) \geq U(\xi) + \Delta(\xi) + (k - l)\rho_X(\xi, x) \quad \forall x \in X.$$

Summarizing the above, *under the above mentioned assumptions the function U_Δ attains its minimum at the only point ξ, which is the point of minimum of the function U, and*

$$U_\Delta(x) \geq U_\Delta(\xi) + (k - l)\rho_X(\xi, x) \;\; \forall x \in X.$$

Note that in these reasonings the assumption of the function U minimum point uniqueness cannot be omitted. Namely, there exists a nonnegative function U with multiple minimum points, satisfying Caristi-like condition with $k = 1$, such that for every $l \in (0, 1)$ there exists a bounded l-Lipschitz function Δ such that the function U_Δ does not attain its minimum.

Let us consider now arbitrarily uniformly small perturbations of the function U. Let $U : X \to \mathbb{R} \cup \{+\infty\}$ be a bounded from below lower-semicontinuous function, $\{U_n\}$ be a sequence of lower semi-continuous functions $U_n : X \to \mathbb{R} \cup \{+\infty\}$ that converge uniformly to U, $k > 0$ be given.

Proposition 1.7. *Let each function U_n satisfy Caristi type condition with given k. Then the functions U satisfies Caristi-like conditions with arbitrary positive $\bar{k} < k$.*

Proof. We can assume without loss of generality that

$$\inf_{x\in X} U(x) = \inf_{x\in X} U_n(x) = 0.$$

Take $\bar{k} < k$ and prove that U satisfies Caristi-like condition. Fix an arbitrary point $x \in X$ such that $U(x) > 0$. Then $U_n(x) > 0$ for sufficiently large n. Theorem 1.2 implies that for each n there exists $x'_n \in X$ such that

$$x'_n \neq x, \qquad U_n(x'_n) = 0, \qquad k\rho_X(x'_n, x) \leq U_n(x). \tag{1.6}$$

So, each sequence $\{x'_n\}$ does not converge to x, since otherwise the uniform convergence of $\{U_n\}$ to U and (1.6) imply that $U(x) = 0$, that contradicts the choice of x. Therefore, there exists $\varepsilon > 0$ such that $\rho_X(x'_n, x) \geq \varepsilon$ for sufficiently large n. Since, $U_n(x'_n) = 0$ and $U_n(x'_n) - U(x'_n) \to 0$ as $n \to \infty$, for each n we have $U(x'_n) + |U(x) - U_n(x)| \leq (k - \bar{k})\varepsilon$ for sufficiently large n. Substituting $x' = x'_n \neq x$ into (1.6) we obtain $U(x') + \bar{k}\rho_X(x', x) \leq U(x)$. □

Note that assumption of uniform convergence in Proposition 1.7 is essential and cannot be replaced by pointwise convergence. For example, consider the function $U(x) = x^2$, $x \in X = [-1, 1]$. For this function the Caristi-like condition does not hold for any $k > 0$, since $U'(0) = 0$. Define a sequence of continuous functions $\{U_n\}$ as follows: $U_n(x) := U(x)$ for each $x \notin [1/(n+1), 1/n]$, the function U_n linearly decreases until $U_n(x_n) = -2$ on the segment $[1/(n+1), x_n]$, and linearly increases on the segment $[x_n, 1/n]$. Here x_n is the midpoint of the segment $[1/(n+1), 1/n]$. Then $U_n(x) \to U(x)$ for each x and for every function U_n, Caristi-like condition holds with $k = 1$ (in order to verify this it is enough to set $x' := x_n$ for each $x \neq x_n$, for each n).

Note also that if the function U satisfies the Caristi-like condition with $\bar{k} > 0$, then there may exist a uniformly convergent to U sequence of functions that do not satisfy the Caristi-like condition with any $k > 0$. For example, consider the function $U(x) := |x|$, $x \in \mathbb{R}$. For each n, let U_n be a continuously differentiable function such that $U_n(x) = U(x)$ for each x such that $|x| > 1/n$, and U_n is a quadratic function on the segment $[-1/n, 1/n]$ that attains its minimum at zero.

The following proposition shows the stability of the minimum point to perturbations of the function U. Let $k > 0$ be given.

Proposition 1.8. *Assume that the function $U : X \to \mathbb{R} \cup \{+\infty\}$ attains its minimum at the point $\bar{x}$, the functions U_n are bounded from below, lower semi-continuous, and satisfy the Caristi-like condition with the same k for each n. Let $U_n(\bar{x}) \to U(\bar{x})$ and*

$$\gamma_n := \inf_{x\in X} U_n(x) \to U(\bar{x}).$$

Then there exits a sequence $\{x_n\} \subset X$ such that $x_n \to \bar{x}$, for each n the point x_n is a minimum point of the function U_n and

$$\rho_X(\bar{x}, x_n) \leq \frac{U_n(\bar{x}) - \gamma_n}{k}. \tag{1.7}$$

Proof. It follows from Theorem 1.2 that for each n there exists a minimum point $x_n \in X$ of the function U_n such that (1.7) holds. The assumptions of the proposition imply that $\{x_n\}$ is the desired sequence. □

In conclusion, let us state the sufficient conditions for the Caristi-like condition (1.1) to hold on an open subset of a normed space. Let X be a normed space with the norm $\|\cdot\|$, a function $U : X \to \mathbb{R}$ be given. Given $x,\ e \in X$, $\|e\| = 1$, denote the upper Dini derivative of the function U at the point x along the direction e by $U'_+(x; e)$, i.e.

$$U'_+(x;e) = \overline{\lim_{\varepsilon \to 0+}} \frac{U(x+\varepsilon e) - U(x)}{\varepsilon}.$$

It is obvious that for two functions $U, V : X \to \mathbb{R}$ the inequality

$$(U+V)'_+(x;e) \le U'_+(x;e) + V'_+(x;e)$$

holds if the values in the right-hand side of the inequality are not infinities with opposite signs.

Lemma 1.9. *Given a number $k > 0$ and an open subset $\widetilde{X} \subset X$, assume that for each $x \in \widetilde{X}$ we have*

$$\exists\, e \in X: \ \ \|e\| = 1, \quad U'_+(x;e) < -k. \tag{1.8}$$

Then the function U satisfies the Caristi-like condition (1.1) on the set $\widetilde{X}$ with the given k.

Proof. Take an arbitrary $x \in \widetilde{X}$, a vector e satisfying (1.8), and a number $\varepsilon > 0$ such that $U'_+(x;e) < -(k+\varepsilon)$. For $s > 0$ we have $U(x+se) - U(x) \le -(k+\varepsilon)s + o(s)$. For a sufficiently small $s > 0$ such that $o(s) - \varepsilon s < 0$ and $x + se \in \widetilde{X}$ set $x' := x + se$. We have $x' \ne x$, $U(x') + ks \le U(x)$. Since $\|x' - x\| = s$, the Caristi-like condition holds on $\widetilde{X}$ with k. □

Remark 1.10. If the function U is differentiable on the set $\widetilde{X}$ and $|U'(x)| > k$ for each $x \in \widetilde{X}$ then (1.8) holds.

2. The existence of a minimum of functions depending on a parameter

As before, we assume that X is a complete metric space with a metric ρ_X. Let the following be given: a metric space T with a metric ρ_T, a closed subset $A \subset X$ with the boundary ∂A and the non-empty interior $\operatorname{int} A$, a function $U : A \times T \to \mathbb{R} \cup \{+\infty\}$. Elements t of the space T are considered as a parameter.

We say that the function U is continuous in t at a point $t_0 \in T$ uniformly in x, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $t \in B_T(t_0, \delta)$ inequality $|U(x,t) - U(x,t_0)| \le \varepsilon$ holds for each $x \in A$. The function U is continuous in t uniformly in x, if it is continuous in t at every point $t_0 \in T$ uniformly in x.

Theorem 2.1. *Let the metric space T be connected,* $\operatorname{dom} U(\cdot, t) \ne \emptyset$ *for each $t \in T$, a number $k > 0$ be given. Denote the set of minimum points $x \in A$ of the function $U(\cdot, t)$ on the set A by $M(t)$ for every $t \in T$. Set*

$$\gamma(t) = \inf_{x \in A} U(x,t). \tag{2.1}$$

Assume that

(i) *the function $U(\cdot, t)$ is lower semi-continuous for every $t \in T$;*

(ii) *the function $U(\cdot, t)$ is bounded from below for every $t \in T$;*
(iii) *for every $t \in T$, for every $x \in A$, if*

$$\gamma(t) < U(x,t) \le \gamma(t) + k \operatorname{dist}_X(x, \partial A) \tag{2.2}$$

then there exists $x' \in A$ such that $x' \neq x$ and

$$U(x',t) + k\rho_X(x,x') \le U(x,t);$$

(iv) *the function U is continuous in t uniformly in x;*
(v) *$M(t) \cap \partial A = \emptyset$ for every $t \in T$;*
(vi) *for any point $\bar{t} \in T$ and a subsequence $\{t_i\}$ convergent to $\bar{t}$, if $M(t_i) \neq \emptyset$ and $h^+(M(t_i), \partial A) \to 0$ then there exists $x_i \in M(t_i)$ such that the sequence $\{x_i\}$ contains a convergent subsequence.*

Then if the set $M(t)$ is non-empty for some $t \in T$, then it is non-empty for all $t \in T$.

Proof. Set $\widetilde{U}(x,t) = U(x,t) - \gamma(t)$. Considering, if necessary, the function $\widetilde{U}$ instead of U, we can assume without loss of generality that $\gamma = 0$, $U(x,t) \ge 0$ for all $x \in A$, $t \in T$. Set

$$\overline{T} := \{t \in T :\ M(t) \neq \emptyset\}.$$

The set $\overline{T}$ is non-empty by assumption. It is enough to prove that $\overline{T}$ is both open and closed. At first, we will prove that $\overline{T}$ is open. Let's take an arbitrary point $\bar{t} \in \overline{T}$ and $\bar{x} \in M(\bar{t})$. It follows from (v) that $M(\bar{t}) \cap \partial A = \emptyset$. Thus, $\bar{x} \in \operatorname{int} A$. Hence, for some $\delta > 0$, we have $B_X(\bar{x}, \delta) \subset A$. In particular, this inclusion implies $\delta \le \operatorname{dist}_X(\bar{x}, \partial A)$. It follows from condition (iv) that there exists $\varepsilon > 0$ such that $|U(x,t) - U(x,\bar{t})| \le k\delta$ for every $x \in A$, for every $t \in B_T(\bar{t}, \varepsilon)$.

Let us prove that $B_T(\bar{t}, \varepsilon) \subset \overline{T}$. Take an arbitrary point $t \in B_T(\bar{t}, \varepsilon)$ and show that the function $U(\cdot, t)$ satisfies all the conditions of Theorem 1.3 on the ball $B_X(\bar{x}, \delta)$. Condition (i) holds obviously. Since $\rho_T(\bar{t}, t) \le \varepsilon$ and $U(\bar{x}, \bar{t}) = 0$, we have

$$U(\bar{x}, t) = U(\bar{x}, t) - U(\bar{x}, \bar{t}) \le |U(\bar{x}, t) - U(\bar{x}, \bar{t})| \le k\delta.$$

Therefore, $\frac{U(\bar{x},t)}{k} \le \delta$. So, condition (ii) of Theorem 1.3 holds.

Let us prove that condition (iii) of Theorem 1.3 is satisfied. Fix an arbitrary $t \in T$ and for convenience set $U(x) := U(x,t)$ for every $x \in A$. Take an arbitrary point $x \in A$ such that (1.4) holds. The triangle inequality implies

$$\operatorname{dist}_X(\bar{x}, \partial A) \le \rho_X(\bar{x}, x) + \operatorname{dist}_X(x, \partial A).$$

So,

$$U(x) \le k(\delta - \rho_X(x, \bar{x})) \le k(\operatorname{dist}_X(\bar{x}, \partial A) - \rho_X(x, \bar{x})) \le k \operatorname{dist}_X(x, \partial A).$$

Thus, it follows from (2.2) that there exists $x' \in A$ such that (1.5) holds. Let us prove that $x' \in B_X(\bar{x}, \delta)$. From (1.5), it follows that $\rho_X(x, x') \le \frac{U(x)}{k}$. Therefore,

$$\rho_X(\bar{x}, x') \le \rho_X(\bar{x}, x) + \rho_X(x, x') \le \rho_X(\bar{x}, x) + \frac{U(x)}{k} \le \delta.$$

The last inequality holds, since the choice of x implies that (1.4) holds. Thus, condition (iii) of Theorem 1.3 holds.

It follows from Theorem 1.3 that the set $M(t)$ is non-empty. Thus, it is proved that $B_T(\bar{t},\varepsilon) \subset \overline{T}$. Therefore, the set $\overline{T}$ is open. Let us prove that $\overline{T}$ is closed. Given a sequence $\{t_i\} \subset \overline{T}$ that converges to the point $\bar{t}$, prove that $\bar{t} \in \overline{T}$.

Let us show that for some $\delta > 0$ inequality $h^+(M(t_i), \partial A) \geq \delta$ holds for all sufficiently large i. By assuming the contrary and considering a subsequence, we have $h^+(M(t_i), \partial A) \to 0$. Again considering a subsequence, we deduce from condition (vi) that there exists a sequence of points $x_i \in M(t_i)$ and a point $\bar{x} \in \partial A$ such that $x_i \to \bar{x}$. It is obvious that $\bar{x} \in \partial A$. Let us show that $U(\bar{x},\bar{t}) = 0$. Assume that $a := U(\bar{x},\bar{t}) > 0$. Condition (i) implies that $U(x_i,\bar{t}) \geq 2a/3$ for sufficiently large i. Condition (iv) implies that $|U(x_i,t_i) - U(x_i,\bar{t})| \leq a/3$ for sufficiently large i. Therefore, $U(x_i,t_i) \geq a/3$ for sufficiently large i, whereas $U(x_i,t_i) = 0$ for each i. The contradiction proves that $U(\bar{x},\bar{t}) = 0$. This contradicts condition (v).

Thus, there exists $\delta > 0$ such that $h^+(M(t_i), \partial A) > \delta$ for all sufficiently large i, that we will exclusively consider. Therefore, there exist $x_i \in M(t_i)$ such that $B_X(x_i,\delta) \subset A$ for all i. Condition (iv) implies that $|U(x_i,t_i) - U(x_i,\bar{t})| \to 0$. Therefore, since $U(x_i,t_i) = 0$ for all i, we have $U(x_i,\bar{t}) \leq k\delta$ for some sufficiently large number i. The function $U(\cdot,\bar{t})$ on the ball $B_X(x_i,\delta) \subset A$ satisfies all conditions of Theorem 1.3. Indeed, conditions (i) and (ii) hold obviously, condition (iii) can be verified as above (by replacing $\bar{x}$ by x_i in the corresponding reasonings). Theorem 1.3 implies that the set $M(\bar{t})$ is non-empty. Therefore, $\overline{T}$ is closed.

Thus, the set $\overline{T}$ is simultaneously open, closed and non-empty. Therefore, $T = \overline{T}$, since T is connected. □

Remark 2.2. Note that (iv) implies that the function $\gamma(t) = \inf\limits_{x \in A} U(x,t)$ is continuous.

Remark 2.3. Theorem 2.1 is correct if T is not a metric but a topological space. In this case, the function U is assumed to be continuous in t, uniformly in x, if for every point $t_0 \in T$ and for every $\delta > 0$ there exists a neighbourhood V of the point t_0, such that $|U(x,t) - U(x,t_0)| < \delta$ for all $x \in A$ and $t \in V$.

Let us discuss condition (iv). It is quite burdensome but often it can be weakened. Denote

$$c_0 = \inf_{x \in A} \sup_{t \in T} U(x,t).$$

If T is compact then it is easy to observe that c_0 is finite, whereas in the general case it is not. For example, if $X = A = \mathbb{R}$, $T = \mathbb{R}$, and $U(x,t) = |x - t|$, then $c_0 = +\infty$.

For finite c_0 take an arbitrary $c > c_0$, set $c = +\infty$ if $c_0 = +\infty$. Consider the condition:

(iv)$_\mathbf{c}$ the function U is continuous in t uniformly in $x \in A_c := \{x \in A : U(x,t) \leq c \,\forall\, t \in T\}$.

Proposition 2.4. *Let $c > c_0$. Theorem 2.1 remains true if we replace condition (iv) by a weaker condition (iv)$_c$.*

Proof. Let for the function U all assumptions of Theorem 2.1 except (iv) hold and let $(iv)_c$ hold as well. Set

$$U_c(x,t) := \begin{cases} U(x,t), & \text{if} \quad x \in A_c; \\ +\infty, & \text{if} \quad x \in A \setminus A_c. \end{cases}$$

Its obvious that the set A_c is non-empty. Condition (i) implies that this set is closed. Moreover, it is a straightforward task to ensure that $(iv)_c$ implies that condition (iv) of Theorem 2.1 holds for the function U_c. Thus, the desired result follows from Theorem 2.1. □

Assume now that the set T is linearly connected. Then Theorem 2.1 still holds, if condition (iv) is replaced by the following weaker condition:

(iv)$_>$ for all $c \in \mathbb{R}$, the function U is continuous in t uniformly in $x \in A_c = \{x \in A : U(x,t) \le c \, \forall \, t \in T\}$.

Let us prove this. Fix $t \in T$ such that the set $M(t)$ is non-empty. Take an arbitrary $\tau \in T$. We have to prove that the set $M(\tau)$ is non-empty as well. Since T is linearly connected, there exists a continuous curve l that connects the points t and τ. The curve l is compact. Therefore, as it was noted above, there exists $c \in \mathbb{R}$, such that

$$c > \inf_{x \in A} \sup_{t \in l} U(x,t).$$

Condition $(iv)_>$ implies that condition $(iv)_c$ holds for the chosen c. Therefore, replacing the space T by its subspace l, from Proposition 2.4, we obtain that the set $M(\tau)$ is non-empty.

Let us consider more general assumptions that allow to replace condition (iv) by condition $(iv)_>$ in Theorem 2.1. Let the space T coincide with the union of sets T_i, $i \in I$, where I – is a set of indexes. Assume that each set T_i is either linearly connected or connected and compact. Moreover, let for arbitrary points t, $\tau \in T$ there exist such indexes $i_1, ..., i_m \in I$, that $t \in T_{i_1}$, $\tau \in T_{i_m}$ and $T_{i_j} \cap T_{i_{j+1}} \neq \emptyset$, $j = 1, ..., m-1$. In this case, in Theorem 2.1, condition (iv) can be replaced by condition $(iv)_>$. This fact holds true due to the aforementioned arguments.

Proposition 2.5. *Assume that all conditions of Theorem 2.1 hold and*

(vii) *there exist* $\bar{x} \in A$ *and* $\bar{t} \in T$ *such that* $U(\bar{x}, \bar{t}) \le \gamma(\bar{t}) + k \operatorname{dist}_X(\bar{x}, \partial A)$.

Then the set $M(t)$ *is non-empty for all* $t \in T$.

Proof. Set $\delta := \frac{U(\bar{x},\bar{t}) - \gamma(\bar{t})}{k}$. Condition (vii) implies that $B_X(\bar{x}, \delta) \subset A$. So, it follows from Theorem 1.3 that the set $M(\bar{t})$ is non-empty. Thus, Theorem 2.1 implies that the set $M(t)$ is non-empty for all $t \in T$. □

Let us study the properties of the set-valued mapping M.

Proposition 2.6. *Assume that all conditions of Theorem 2.1 hold. Then the set-valued mapping* M *is sequentially lower semi-continuous and closed (i.e. its graph is closed).*

Proof. As before, we can assume without loss of generality that $\gamma(t) = 0$, $U(x,t) \geq 0$ for all $x \in A$, $t \in T$. Fix an arbitrary $\bar{t} \in T$, $\bar{x} \in M(\bar{t})$, and a subsequence $\{t_i\}$ convergent to $\bar{t}$. It follows from condition (v) that $B_X(\bar{x},\delta) \subset A$ for some $\delta > 0$. Condition (iv) implies that $U(\bar{x},t_i) \to U(\bar{x},\bar{t}) = 0$. Thus, using arguments analogous to those in the proof of the openness of the set $\overline{T}$ in Theorem 2.1, we conclude that the function $U(\cdot,t_i)$ on the ball $B_X(\bar{x},\delta)$ satisfies all the conditions of Theorem 1.3 for sufficiently large i. This theorem implies that there exist points $\xi_i \in M(t_i)$ such that $\xi_i \to \bar{x}$. So, M is sequentially lower semi-continuous.

The closedness of the graph of M follows from the fact that if $x_i \to \bar{x}$, $t_i \to \bar{t}$ and $U(x_i,t_i) = 0$ for every i then $U(\bar{x},\bar{t}) = 0$. This property was shown in the proof of Theorem 2.1. □

Let ω be the modulus of continuity of the function U in the variable t, i.e.

$$\omega(\varepsilon) = \sup\{|U(x,t_1) - U(x,t_2)| : \, x \in A, \;\; t_1,t_2 \in T, \;\; \rho_T(t_1,t_2) \leq \varepsilon\}$$

for $\varepsilon > 0$ (here and everywhere below we assume that $(+\infty) - (+\infty) = 0$). Obviously

$$|U(x,t_1) - U(x,t_2)| \leq \omega(\rho_T(t_1,t_2)) \quad \forall\, t_1,t_2 \in T.$$

The condition $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0+$ means that the function U is continuous in t uniformly in x.

Let us define the internal δ-shell of the boundary of the set A as follows:

$$A(\delta) := \{x \in A : \, \mathrm{dist}_X(x,\partial A) \leq \delta\}.$$

Proposition 2.7. *Assume that all the conditions of Theorem 2.1, except (v), hold and*

(viii) *there exists $\delta > 0$, such that $M(t) \cap A(\delta) = \emptyset$ for all $t \in T$.*

Moreover, let $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0+$.

Then if the set $M(t)$ is non-empty for some $t \in T$, then the conclusion of Theorem 2.1 holds and

$$h_X(M(t_1),M(t_2)) \leq k^{-1} 2\omega(\rho_T(t_1,t_2)) \;\; \forall\, t_1,t_2 \in T : \, \omega(\rho_T(t_1,t_2)) \leq k\delta. \qquad (2.3)$$

Proof. Set $\gamma(t) := \inf\limits_{x \in A} U(x,t)$, $t \in T$. The assumptions of the proposition imply that the function γ is uniformly continuous and its modulus of continuity does not exceed ω. Set $\widetilde{U}(x,t) := U(x,t) - \gamma(t)$. Thus, passing to the function $\widetilde{U}$ from the function U, we will assume that $\gamma = 0$, $U(x,t) \geq 0$ for all $x \in A$, $t \in T$, and the modulus of continuity of the function U with respect to variable t does not exceed 2ω.

Condition (v) follows from condition (viii). Therefore, it is enough to prove that (2.3). For all $t \in T$, $x \in M(t)$, condition (viii) implies $B_X(x,\delta) \subset A$. Take arbitrary $t_1,t_2 \in T$ such that $2\omega(\rho_T(t_1,t_2)) \leq k\delta$. Let $x_1 \in M(t_1)$. Then $U(x_1,t_1) = 0$ and thus $U(x_1,t_2) \leq 2\omega(\rho_T(t_1,t_2)) \leq k\delta$. Moreover, as it was mentioned above, $B_X(x_1,\delta) \subset A$. Therefore, Theorem 1.2 implies that there exists $x_2 \in A$ such that $U(x_2,t_2) = 0$, $\rho_X(x_1,x_2) \leq k^{-1} 2\omega(\rho_T(t_1,t_2))$. Carrying out analogical arguments for the set $M(t_2)$, we have $h_X(M(t_1),M(t_2)) \leq 2k^{-1}\omega(\rho_T(t_1,t_2))$. □

Remark 2.8. If T is a convex subset of a normed space then inequality (2.3) holds for all $t_1, t_2 \in T$ (i.e. the conditon $\omega(\rho_T(t_1,t_2)) \le k\delta$ can be omitted in (2.3)).

Condition (vi) of Theorem 2.1 and condition (viii) of Proposition 2.7 are burdensome. They are caused by the possible non-compactness of the set $A \subset X$. Condition (vi) of Theorem 2.1 is required only for the proof of closedness of the set $\overline{T}$. In [9], the closedness of the set $\overline{T}$ was assumed instead of condition (vi). Moreover, condition (vi) automatically holds in certain important applications, for example, as it is shown below in specific problems related with fixed points.

Assume that A is compact. Then condition (vi) of Theorem 2.1 holds automatically. Moreover, Proposition 2.6 implies that the set-valued mapping M is sequentially lower semi-continuous and, therefore, continuous in Hausdorff metric. In addition, in this case, condition (viii) of Proposition 2.7 is a corollary of condition (v) of Theorem 2.1.

Theorem 2.1 gives the following "homotopic" corollary.

Proposition 2.9. *Let $T = [0,1]$, all the conditions of Proposition 2.7 hold, and there exist $x_0 \in \operatorname{dom} U(\cdot,0)$ such that $U(x_0,0) \le \gamma(0) + k\operatorname{dist}_X(x_0,\partial A)$.*

Then there exists a point $\xi \in A$ at which the function $U(\cdot,1)$ attains its minimum. Moreover, the set $M(t)$ is non-empty for all t and

$$h_X(M(t_1),M(t_2)) \le 2k^{-1}\omega(\rho_T(t_1,t_2)) \;\; \forall\, t_1,t_2 \in [0,1]. \tag{2.4}$$

Proof. Theorem 1.3 implies that the set $M(t)$ is non-empty. Inequality (2.4) follows from Proposition 2.7 and Remark 2.8. □

Unlike (2.3), estimate (2.4) is not local, since in Proposition 2.9, instead of an arbitrary connected metric space, the interval $[0,1]$ is considered. If the modulus of continuity of ω satisfies inequality $\omega(\varepsilon) \le c\varepsilon$ for each $\varepsilon > 0$ for a certain $c > 0$, then the function U is Lipschitz in t. Then (2.4) implies that the set-valued mapping $M(\cdot)$ is also Lipschitz.

To conclude this section, we provide examples illustrating the results above. The following example demonstrates the essentiality of assumption (vi) in Theorem 2.1.

Example 2.10. Let $T = [0,1/3]$, $X = [3,\infty) \times \mathbb{R}$, $A = [3,\infty) \times [0,1/2] \subset X$,

$$\rho_X(x,x') = |\lambda - \lambda'| + |\chi - \chi'| \quad \forall\, x = (\lambda,\chi), \quad x' = (\lambda',\chi') \in X. \tag{2.5}$$

Define a function $U : A \times T \to \mathbb{R}$ as follows. Set

$$a_0(\lambda) = 0,\;\; a_1(\lambda) = \frac{\lambda^3}{\lambda^4+1},\;\; a_2(\lambda) = \frac{1}{\lambda},\;\; a_3(\lambda) = \frac{1}{3},\;\; b(\lambda) = \lambda^4 \;\; \forall\, \lambda \in [3,\infty), \tag{2.6}$$

$$\mathcal{D}_i = \{(\lambda,t) :\; a_{i-1}(\lambda) \le t \le a_i(\lambda)\}, \quad i = \overline{1,3},$$

$$U(x,t) = \begin{cases} |\chi - t| + a_1(\lambda), & \text{if } (\lambda,t) \in \mathcal{D}_1\,; \\ |\chi - t| + b(\lambda)(a_2(\lambda) - t), & \text{if } (\lambda,t) \in \mathcal{D}_2\,; \\ |\chi - a_2(\lambda)|, & \text{if } (\lambda,t) \in \mathcal{D}_3\,. \end{cases}$$

It is a straightforward task to ensure that $\gamma(t) := \inf_{x\in A} U(x,t) = 0$ for every $t \in T$. When $t \ne 0$, the set of points of minimum of the function $U(\cdot,t)$ is the set

$$M(t) := \big\{x = (\lambda,\lambda^{-1}) :\; \lambda \ge t^{-1}\big\}.$$

The function $U(\cdot, 0)$ has no points of minimum, i.e. $M(0) = \emptyset$. So, the proposition of Theorem 2.1 does not hold for the function U.

Let us show that all the conditions of Theorem 2.1 except (vi) hold for U. Condition (i) holds, since the function U is continuous. Condition (ii) holds with $\gamma(t) \equiv 0$. Let us show that condition (iii) holds for $k = 1$. Take arbitrary $t \in T$, $x = (\lambda, \chi) \in A$, $x \notin M(t)$. Consider three cases: $(\lambda, t) \in \mathcal{D}_j$, $j = 1, 2, 3$.

Let $(\lambda, t) \in \mathcal{D}_1 \setminus \mathcal{D}_2$. If $\chi \neq t$ then for $x' = (\lambda, t)$ the following equation holds

$$U(x', t) + \rho(x, x') = U(x, t), \tag{2.7}$$

while if $\chi = t$ then inequality (2.2) is violated, since

$$\mathrm{dist}_X(x, \partial A) \leq \chi = t < a_1(\lambda) = U(x, t).$$

Thus, when $(\lambda, t) \in \mathcal{D}_1$ condition (iii) holds.

Let $(\lambda, t) \in \mathcal{D}_2$. If $\chi \neq t$ then (2.7) holds for $x' = (\lambda, t)$. If $\chi = t$ then set $x' := (1/t, t)$. Since $(1/t, t) \in \mathcal{D}_3$, we have

$$U(x', t) + \rho_X(x, x') = \frac{1}{t} - \lambda < \lambda^3 a_1(\lambda)\left(\frac{1}{t} - \lambda\right) < \lambda^3 t\left(\frac{1}{t} - \lambda\right) = U(x, t).$$

Thus, in the second case condition (iii) holds as well. Let now $(\lambda, t) \in \mathcal{D}_3 \setminus \mathcal{D}_2$. Then $x = (\lambda, \chi) \notin M(t)$. So, (2.7) holds for $x' := (\lambda, 1/\lambda)$. Moreover, $x \neq x'$, since $\chi \neq 1/\lambda$. Hence, condition (iii) holds.

Let us verify condition (iv). First, we prove that the function U is continuous at the point $t = 0$ in t uniformly in $x \in A$. For this purpose let us estimate the value $|U(x, 0) - U(x, t)|$ for each $t \in T$, $x \in A$. Consider three cases: $(\lambda, t) \in \mathcal{D}_j$, $j = 1, 2, 3$.

If $(\lambda, t) \in \mathcal{D}_1$ then $|U(x, 0) - U(x, t)| = \big||\chi - t| - |\chi|\big| \leq t < 2/\lambda$. If $(\lambda, t) \in \mathcal{D}_2$ then

$$\begin{aligned} |U(x, 0) - U(x, t)| &\leq |U(x, 0) - U(x, a_1(\lambda))| + |U(x, a_1(\lambda)) - U(x, t)| \\ &\leq a_1(\lambda) + (t - a_1(\lambda)) + b(\lambda)(t - a_1(\lambda)) = t + b(\lambda)(t - a_1(\lambda)) \\ &\leq a_2(\lambda) + b(\lambda)(a_2(\lambda) - a_1(\lambda)) < 2/\lambda. \end{aligned}$$

Finally, if $(\lambda, t) \in \mathcal{D}_3$ then $U(x, t) = U(x, a_2(\lambda))$, so

$$|U(x, 0) - U(x, t)| \leq |U(x, 0) - U(x, a_1(\lambda))| + |U(x, a_1(\lambda)) - U(x, a_2(\lambda))| < 2/\lambda.$$

Take an arbitrary $\varepsilon > 0$. For $x = (\lambda, \chi) \in [2/\varepsilon, +\infty) \times [0, 1/2]$, $t \in T$, we have $|U(x, 0) - U(x, t)| < 2/\lambda < \varepsilon$. Since the function U is continuous on the compact $[3, 2/\varepsilon] \times [0, 1/2] \times T$, this function is uniformly continuous on it. Therefore,

$$\exists\, \delta > 0 : \quad \forall\, x = (\lambda, \chi) \in [3, 2/\varepsilon] \times [0, 1/2] \quad \forall\, t \in [0, \delta) \quad |U(x, 0) - U(x, t)| < \varepsilon.$$

Hence, for all $(\lambda, \chi) \in A$, $t \in [0, \delta)$, we have $|U(x, 0) - U(x, t)| < \varepsilon$. Therefore, the function U is continuous in t at the point $t = 0$ uniformly in x.

Now let us prove that the function U is continuous in t at every point $\bar{t} \neq 0$ uniformly in $x \in A$. Take arbitrary $t_1, t_2 \in T$, $0 < t_1 < \bar{t} < t_2$. Note that if $x = (\lambda, \chi) \in A$ and $\lambda > 1/t_2$ then $(\lambda, \chi) \in \mathcal{D}_3$ and, therefore, the function $U(x, \cdot)$ is constant on $[t_1, t_2]$. Since the function U is continuous on the compact $[0, 1/t_2] \times [0, 1/2] \times [t_1, t_2]$, it is uniformly continuous on it. Therefore, U is continuous in t at the point $\bar{t}$ uniformly in $x \in A$. Condition (v) holds, since $(\lambda, 0) \notin M(t)$ and $(\lambda, 1/2) \notin M(t)$.

Theorem 2.1 implies that condition (vi) fails. Let us show this fact directly. For any decreasing sequence $t_i \to 0+$ we have $M(t_i) \neq \emptyset$, $h^+(M(t_i), \partial A) = t_i \to 0$. For any sequence of elements $x_i \in M(t_i)$, $x_i = (t_i, \lambda_i)$, we have $\lambda_i \geq t_i^{-1}$. Therefore, this sequence does not contain any convergent subsequence.

Consider an example showing that assumption (viii) on the separability of the set $M(t)$ from the boundary of the set A is essential in Propositions 2.7 and 2.9.

Example 2.11. Let $T = [0, 3^{-1}]$, $X = [2, \infty) \times \mathbb{R}$, $A = [2, \infty) \times [0, 2^{-1}] \subset X$. In the space $X = [2, \infty) \times \mathbb{R}$, define a metric using the function (2.5). Set $\bar{a} := a_1(3)$, $\bar{b} := b(3)$, where $a_1(\cdot)$ and $b(\cdot)$ are defined by formula (2.6). Define the function $U : A \times T \to \mathbb{R}$ as follows. If $(\lambda, \chi, t) \in [3, \infty) \times [0, 2^{-1}] \times T$ then $U(x, t)$ coincides with the value $U(x, t)$ from Example 2.10. If $(\lambda, \chi, t) \in [2, 3) \times [0, 2^{-1}] \times T$ then

$$U(x,t) = \begin{cases} \left|\chi - t - 1 + \frac{\lambda}{3}\right| + \bar{a}(\lambda - 2), & \text{if } (\lambda, t) \in \widetilde{\mathcal{D}}_1\,, \\ \left|\chi - t - 1 + \frac{\lambda}{3}\right| + \bar{b}\left|\frac{1}{3}(\lambda - 2) - t\right|, & \text{if } (\lambda, t) \in \widetilde{\mathcal{D}}_2\,; \end{cases}$$

where

$$\widetilde{\mathcal{D}}_1 = \{(\lambda, t) : 0 \leq t \leq \bar{a}(\lambda - 2)\}, \quad \widetilde{\mathcal{D}}_2 = \{(\lambda, t) : \bar{a}(\lambda - 2) < t \leq 1/3\}\,.$$

It is a straightforward task to ensure that $\gamma(t) := \inf_{x \in A} U(x, t) = 0$ for every $t \in T$. At the same time, the function $U(\cdot, t)$ attains its minimum at points of the set

$$M(t) = \left\{x = \left(\lambda, \frac{1}{\lambda}\right) : \lambda \geq \frac{1}{t}\right\} \cup \left\{\left(3t + 2, \frac{1}{3}\right)\right\} \quad \text{for} \quad t \neq 0, \quad M(0) = \left\{\left(2, \frac{1}{3}\right)\right\}.$$

Thus, the statements of propositions 2.7 and 2.9 are violated for the function U, since the set-valued mapping M is not continuous at zero.

Let us show that the function U satisfies all the conditions of Theorem 2.1. Condition (i) holds, since U is continuous and (ii) holds with $\gamma(t) \equiv 0$.

Let us show that condition (iii) holds for any positive $k \leq 3\bar{a}/(4 + 12\bar{a})$. For arbitrary $x = (\lambda, \chi) \in A$ and $t \in T$ such that $x \notin M(t)$ and $U(x, t) \leq k \operatorname{dist}_X(x, \partial A)$ construct $x' \in A$, $x' \neq x$ such that $U(x', t) + k\rho_X(x, x') \leq U(x, t)$. For the case when $\lambda \geq 3$, the corresponding construction of the point x' was made in Example 2.10 for $k = 1$. Since the values of $\operatorname{dist}_X(x, \partial A)$ in Examples 2.10 and 2.11 coincide and $k < 1$, the constructed point x' in 2.10 is the desired one.

Now let $\lambda \in [2, 3)$. Set $x' := (3t + 2, 1/3)$. If $(\lambda, t) \in \widetilde{D}_1$ then

$$\begin{aligned} U(x', t) + k\rho_X(x, x') &\leq k|\lambda - 3t - 2| + k\left|\chi - t - 1 + \frac{\lambda}{3}\right| + k\left|\frac{\lambda - 2}{3} - t\right| \\ &\leq k\left|\chi - t - 1 + \frac{\lambda}{3}\right| + \frac{4}{3}k(\lambda - 2) + 4kt \leq k\left|\chi - t - 1 + \frac{\lambda}{3}\right| + \frac{4}{3}k(\lambda - 2) \\ +4k\bar{a}(\lambda - 2) &\leq k\left|\chi - t - 1 + \frac{\lambda}{3}\right| + \bar{a}(\lambda - 2) \leq \left|\chi - t - 1 + \frac{\lambda}{3}\right| + \bar{a}(\lambda - 2) = U(x, t). \end{aligned}$$

If $(\lambda, t) \in \widetilde{D}_2$ then taking into account that $4k \leq \bar{b}$, we have

$$U(x', t) + k\rho_X(x, x') \leq k|\lambda - 3t - 2| + k\left|\chi - t - 1 + \frac{\lambda}{3}\right| + k\left|\frac{\lambda - 2}{3} - t\right|$$

$$= k\left|\chi - t - 1 + \frac{\lambda}{3}\right| + 4k\left|\frac{\lambda - 2}{3} - t\right| \leq \left|\chi - t - 1 + \frac{\lambda}{3}\right| + \bar{b}\left|\frac{\lambda - 2}{3} - t\right| = U(x, t).$$

Thus, condition (iii) holds.

Let us verify condition (iv). The restriction of the continuous function U to the compact $[2, 3] \times [0, 2^{-1}] \times T$ is uniformly continuous. The restriction of the function U to $T \times [3, \infty) \times [0, 2^{-1}]$ considered in Example 2.10 satisfies condition (iv). Therefore, condition (iv) holds for the function U. Conditions (v) and (vi) can be verified directly.

Thus, the function U satisfies all the conditions of Theorem 2.1. At the same time, the set-valued mapping $M(\cdot)$ is discontinuous at zero, and, therefore, the function U does not satisfy condition (viii).

3. Fixed points and coincidence points

Let us present a local analogue of the Caristi fixed point theorem. As before, we assume that X is a complete metric space with a metric ρ_X, T is a metric space with a metric ρ_T, $A \subset X$ is a closed subset with the boundary ∂A and the non-empty interior $\operatorname{int} A$. Let a number $r \in (0, +\infty]$, a point $\bar{x} \in X$, a mapping $g : B_X(\bar{x}, r) \to X$, and a function $U : B_X(\bar{x}, r) \to \mathbb{R} \cup \{+\infty\}$ be given.

Theorem 3.1. *Given $\gamma \in \mathbb{R}$ and $k > 0$, assume that*

- **(i)** *the function U is lower semi-continuous and $U(x) \geq \gamma$ for every $x \in B_X(\bar{x}, r)$;*
- **(ii)** *$\bar{x} \in \operatorname{dom} U$, $U(\bar{x}) \leq \gamma + rk$;*
- **(iii)** *for every $x \in B_X(\bar{x}, r)$, if*

$$U(x) \leq U(\bar{x}) - k\rho_X(x, \bar{x})$$

then

$$U(g(x)) + k\rho_X(x, g(x)) \leq U(x).$$

Then there exists a fixed point $\xi \in X$ of the mapping g, i.e. $g(\xi) = \xi$, such that

$$\rho_X(\bar{x}, \xi) \leq \frac{U(\bar{x}) - \gamma}{k}.$$

Proof. Set $\delta := (U(\bar{x}) - \gamma)/k$. Condition (ii) implies that $B_X(\bar{x}, \delta) \subset B_X(\bar{x}, r)$. Let us show that the mapping g has a fixed point $\xi \in B_X(\bar{x}, \delta)$. Consider the contrary, i.e. $g(x) \neq x$ for all $x \in B_X(\bar{x}, \delta)$. Then $U(x) > \gamma$ for every $x \in B_X(\bar{x}, \delta)$, since condition (iii) implies that either

$$U(x) \leq U(\bar{x}) - k\rho_X(x, \bar{x}) \quad \text{and} \quad U(x) > U(x) - k\rho_X(x, g(x)) \geq U(g(x)) \geq \gamma,$$

or

$$U(x) > U(\bar{x}) - k\rho_X(x, \bar{x}) \geq U(\bar{x}) - k\delta = U(\bar{x}) - k\frac{U(\bar{x}) - \gamma}{k} = \gamma.$$

It follows from (iii) that for every $x \in B_X(\bar{x}, \delta)$ such that $U(x) \leq U(\bar{x}) - k\rho_X(x, \bar{x})$, for $x' := g(x)$, we have $U(x') + k\rho_X(x, x') \leq U(x)$. Moreover, $x' \neq x$, since g has no fixed points in the ball $B_X(\bar{x}, \delta)$. Therefore, in the ball $B_X(\bar{x}, \delta)$, condition (iii) of Theorem 1.3 holds. Obviously, conditions (i) and (ii) of Theorem 1.3 hold as well. Therefore, Theorem 1.3 implies that there exists a point $\xi \in B_X(\bar{x}, \delta)$ such that $U(\xi) = \gamma$. However, it was shown above that $U(x) > \gamma$ for every $x \in B_X(\bar{x}, \delta)$. The obtained contradiction completes the proof. $\square$

The Caristi fixed point theorem (see [5]) directly follows from Theorem 3.1. Let us recall it.

Let a mapping $g : X \to X$ be given.

Caristi fixed point theorem (Theorem 2.1′ from [5]) *Assume that there exist a number $k > 0$ and a proper function $U : X \to \mathbb{R}_+ \cup \{+\infty\}$ (i.e. $\operatorname{dom} U \neq \emptyset$) such that U is lower semi-continuous and*

$$U(g(x)) + k\rho_X(x, g(x)) \leq U(x) \quad \forall\, x \in X. \tag{3.1}$$

Then for every point $\bar{x} \in \operatorname{dom} U$, there exists a fixed point $\xi \in X$ of the mapping g such that $\rho_X(\bar{x}, \xi) \leq (U(\bar{x}) - \gamma)/k$, $\gamma := \inf\limits_{x \in X} U(x)$.

The proof of this proposition consists in the direct application of Theorem 3.1 with $\delta = +\infty$. So, Theorem 3.1 is a local analogue of the Caristi fixed point theorem which has a global nature. Note also that the continuity of the mapping g is not assumed neither in the Caristi fixed point theorem nor in Theorem 3.1.

Let us proceed to generalized coincidence points of set-valued mappings. Let (Y, ρ_Y) be a metric space, let $G_1, G_2 : X \rightrightarrows Y$ be given set-valued mappings, i.e. mappings that assign to every point $x \in X$ non-empty closed subsets of the space Y. A point $\xi \in X$ is called a coincidence point of the mappings G_1 and G_2, if $G_1(\xi) \cap G_2(\xi) \neq \emptyset$ and a generalized coincidence point, if $\operatorname{dist}_Y(G_1(\xi), G_2(\xi)) = 0$. It is obvious that every coincidence point is a generalized coincidence point but not vice-versa. If at least one of the sets $G_1(\xi)$ or $G_2(\xi)$ is compact (for example, if G_1 and G_2 are "single-valued" mappings), then the generalized coincidence point ξ is a coincidence point.

Let a non-empty closed set $A \subset X$, the number $k > 0$, the point $\bar{x} \in A$ and set-valued mapping $G_1, G_2 : A \rightrightarrows Y$ be given. Let us denote the set of generalized points of the mappings G_1 and G_2 by Ξ, i.e.

$$\Xi := \{\xi \in A : \operatorname{dist}_Y(G_1(\xi), G_2(\xi)) = 0\}.$$

Theorem 3.2. *Given a number $k > 0$, assume that*

- **(i)** *the set-valued mappings G_1, G_2 are sequentially upper semi-continuous;*
- **(ii)** $\operatorname{dist}_Y(G_1(\bar{x}), G_2(\bar{x})) \leq k \operatorname{dist}_X(\bar{x}, \partial A)$*;*
- **(iii)** *for every $x \in A$, if*

$$0 < \operatorname{dist}_Y(G_1(x), G_2(x)) \leq \operatorname{dist}_Y(G_1(\bar{x}), G_2(\bar{x})) - k\rho_X(x, \bar{x})$$

then there exists a point $x' \in A$ such that $x' \neq x$ and

$$\operatorname{dist}_Y(G_1(x'), G_2(x')) + k\rho_X(x, x') \leq \operatorname{dist}_Y(G_1(x), G_2(x)).$$

Then the set of generalized coincidence points Ξ *is non-empty and*

$$\operatorname{dist}_X(\bar{x}, \Xi) \leq \frac{\operatorname{dist}_Y(G_1(\bar{x}), G_2(\bar{x}))}{k}.$$

The proof of this theorem uses the following auxiliary proposition (see [1]).

Lemma 3.3. *If the set-valued mappings* G_1 *and* G_2 *are sequentially upper semi-continuous, then the function* $U : x \mapsto \operatorname{dist}_Y(G_1(x), G_2(x))$, $x \in A$, *is lower semi-continuous.*

Proof. Let $x \in A$, $x_i \to x$. For each i take points $y_{1,i} \in G_1(x_i)$, $y_{2,i} \in G_2(x_i)$ such that $\rho_Y(y_{1,i}, y_{2,i}) \leq \operatorname{dist}_Y(G_1(x_i), G_2(x_i)) + i^{-1}$. Since G_1 and G_2 are sequentially upper semi-continuous, there exist $\tilde{y}_{1,i} \in G_1(x)$ and $\tilde{y}_{2,i} \in G_2(x)$ such that $\rho_Y(y_{1,i}, \tilde{y}_{1,i}) \to 0$ and $\rho_Y(y_{2,i}, \tilde{y}_{2,i}) \to 0$. Applying the triangle inequality we obtain

$$\operatorname{dist}_Y(G_1(x), G_2(x)) \leq \rho_Y(\tilde{y}_{1,i}, \tilde{y}_{2,i}) \leq \rho_Y(y_{1,i}, y_{2,i}) + \rho_Y(y_{1,i}, \tilde{y}_{1,i}) + \rho_Y(y_{2,i}, \tilde{y}_{2,i})$$
$$\leq \operatorname{dist}_Y(G_1(x_i), G_2(x_i)) + i^{-1} + \rho_Y(y_{1,i}, \tilde{y}_{1,i}) + \rho_Y(y_{2,i}, \tilde{y}_{2,i}) \quad \forall\, i.$$

Therefore, the function U is lower semi-continuous. □

Proof of Theorem 3.2. Set $U(x) := \operatorname{dist}_Y(G_1(x), G_2(x))$, $x \in A$, $\gamma := 0$,

$$\delta := \operatorname{dist}_X(\bar{x}, \partial A).$$

It is obvious that $B_X(\bar{x}, \delta) \subset A$. Let us show that the restriction of the function U to the ball $B_X(\bar{x}, \delta)$ satisfies all the conditions of Theorem 1.3 hold.

It is obvious that $U(x) \geq 0$ for every $x \in B_X(\bar{x}, \delta)$. Moreover, Lemma 3.3 implies that U is lower semi-continuous. Therefore, condition (i) of Theorem 1.3 holds. Assumptions (ii) and (iii) imply that conditions (ii) and (iii) of Theorem 1.3 hold. Thus, the restriction of U to $B_X(\bar{x}, \delta)$ satisfies all the conditions of Theorem 1.3. Theorem 1.3 implies that there exists a point $\xi \in B_X(\bar{x}, \delta)$ such that $U(\xi) = 0$ and $\rho_X(\bar{x}, \xi) \leq U(\bar{x})/k$. Obviously, ξ is a generalized coincidence point. The desired estimate follows the inequality $\rho_X(\bar{x}, \xi) \leq U(\bar{x})/k$. □

Theorem 3.2 is a analogue of Theorem 4 from [2] on generalized coincidence points which has a global nature. Theorem 4 from [2] is a direct corollary of Theorem 3.2. Let us recall the result from [2].

Given set-valued mappings $G_1, G_2 : X \rightrightarrows Y$, denote

$$W(\bar{x}) := \{x \in X : 0 < \operatorname{dist}_Y(G_1(x), G_2(x)) \leq \operatorname{dist}_Y(G_1(\bar{x}), G_2(\bar{x}))\} \quad \forall\, \bar{x} \in X.$$

Corollary 3.4. *(Theorem 4 from* [2]*) Assume that the mappings* G_1, G_2 *are sequentially upper semi-continuous, there exist a point* $\bar{x} \in X$ *and a number* $k > 0$ *such that for every* $x \in W(\bar{x})$ *there exists a point* $x' \in X$, $x' \neq x$, *satisfying the relation*

$$\operatorname{dist}_Y(G_1(x'), G_2(x')) + k\rho_X(x, x') \leq \operatorname{dist}_Y(G_1(x), G_2(x)). \tag{3.2}$$

Then the set of generalized coincidence points Ξ *of* G_1 *and* G_2 *is non-empty and*

$$\operatorname{dist}_X(\bar{x}, \Xi) \leq \frac{\operatorname{dist}_Y(G_1(\bar{x}), G_2(\bar{x}))}{k}.$$

In order to prove this proposition it is enough to take $\delta := k\mathrm{dist}_Y(G_1(\bar{x}), G_2(\bar{x}))$ and apply Theorem 3.2 to the restriction of the mappings G_1 and G_2 to the ball $B_X(\bar{x}, \delta)$.

Remark 3.5. If the assumptions of Corollary 3.4 hold then the mappings G_1 and G_2 may have no coincidence points. For example, let X be a complete metric space, $Y = \mathbb{R}^2_+$,

$$G_1(x) \equiv \{(t, 1/t) : t \geq 0\}, \quad G_2(x) \equiv \{(t, 2/t) : t \geq 0\}.$$

All the assumptions of Corollary 3.4 hold, since $\mathrm{dist}_Y(G_1(x), G_2(x)) \equiv 0$. However, G_1 and G_2 have no coincidence points, even though all the points $x \in X$ are generalized coincidence points.

Remark 3.6. In [2], it was shown that if for certain $\alpha > \beta \geq 0$ the mapping G_1 is α-covering and has a closed graph, while G_2 is β-Lipschitz with respect to the Hausdorff metric, then condition (3.2) holds for any positive $k < \alpha - \beta$.

Apply Corollary 3.4 to the fixed point problem we obtain the following result.

Theorem 3.2′. *Assume that the mapping $g : X \to X$ is continuous, $\bar{x} \in X$, there exists a number $k > 0$ such that*

$$0 < \rho_X(x, g(x)) \leq \rho_X(\bar{x}, g(\bar{x}))$$
$$\Rightarrow \exists\, x' \in X : \quad x' \neq x, \quad \rho_X(x', g(x')) + k\rho_X(x, x') \leq \rho_X(x, g(x)). \tag{3.3}$$

Then there exists a fixed point $\xi \in X$ of the mapping g such that

$$\rho_X(\bar{x}, \xi) \leq \rho_X(\bar{x}, g(\bar{x}))/k.$$

In [8] (see Chapter I, §1.3) the elementary implicit function theorem for contractive mappings was obtained. Let us consider its analogue for generalized coincidence points of set-valued mappings.

Let set-valued mappings $G_1, G_2 : A \times T \rightrightarrows Y$ be given (recall that T is a metric space with a metric ρ_T, $A \subset X$ is a non-empty closed subset, $\mathrm{int}A \neq \emptyset$). For every $t \in T$ denote the set of all generalized coincidence points of the mappings $G_1(\cdot, t)$ and $G_2(\cdot, t)$ by $\Xi(t)$, i.e.

$$\Xi(t) := \{\xi \in A : \mathrm{dist}_Y(G_1(\xi, t), G_2(\xi, t)) = 0\} \quad \forall\, t \in T.$$

We will say that the set-valued mapping $G : A \times T \rightrightarrows Y$ is continuous in t at a point $t_0 \in T$ uniformly in x, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $h_Y(G(x, t_0), G(x, t)) < \varepsilon$ for all $t \in B_T(t_0, \delta)$, for all $x \in A$. We will say that the set-valued mapping $G : A \times T \rightrightarrows Y$ is continuous in t uniformly in x, if it is continuous in t at every point $t_0 \in T$ uniformly in x.

Theorem 3.7. *Let the metric space T be connected, a number $k > 0$ be given. Assume that*

- **(i)** *the set-valued mappings $G_1(\cdot, t)$ and $G_2(\cdot, t)$ are sequentially upper semi-continuous for each $t \in T$;*

(ii) *for each* $t \in T$, *for each* $x \in A$, *if*

$$0 < \operatorname{dist}_Y(G_1(x,t), G_2(x,t)) \leq k \operatorname{dist}_X(x, \partial A) \tag{3.4}$$

then there exists $x' \in A$ *such that* $x' \neq x$ *and*

$$\operatorname{dist}_Y(G_1(x',t), G_2(x',t)) + k\rho_X(x,x') \leq \operatorname{dist}_Y(G_1(x,t), G_2(x,t);$$

(iii) *the set-valued mappings* G_1 *and* G_2 *are continuous in* t *uniformly in* x*;*

(iv) $\Xi(t) \cap \partial A = \emptyset$ *for each* $t \in T$*;*

(v) *for any point* $\bar{t} \in T$ *and a subsequence* $\{t_i\}$ *convergent to* t, *if* $\Xi(t_i)$ *is non-empty and* $h^+(\Xi(t_i), \partial A) \to 0$, *then there exists* $x_i \in \Xi(t_i)$ *such that the sequence* $\{x_i\}$ *contains a convergent subsequence.*

Then if the set $\Xi(t)$ *is non-empty for some* $t \in T$, *then it is non-empty for all* $t \in T$.

Proof. Set $U(x,t) := \operatorname{dist}_Y(G_1(x,t), G_2(x,t))$, $\gamma(t) := 0$, $t \in T$, $x \in A$. Let us show that the function U satisfies all the conditions of Theorem 2.1.

Lemma 3.3 and sequential upper semi-continuity of the functions G_1 and G_2 with respect to the variable x imply that condition (i) of Theorem 2.1 holds. The definition of the functions U and γ imply that condition (ii) of Theorem 2.1 holds. The definition of the functions U and γ and assumption (ii) imply that condition (iii) holds. Let us verify condition (iv). Assumption (iii) imply that for arbitrary $t_0 \in T$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $h_Y(G_1(x,t_0), G_1(x,t)) < \varepsilon/2$, $h_Y(G_2(x,t_0), G_2(x,t)) < \varepsilon/2$ hold for every $x \in A$, $t \in B_T(t_0, \delta)$. Therefore, by the definition of functions h_Y and dist_Y, we have

$$\operatorname{dist}_Y(G_1(x,t), G_2(x,t)) - \operatorname{dist}_Y(G_1(x,t_0), G_2(x,t_0))$$
$$\leq h_Y(G_1(x,t), G_1(x,t_0)) + h_Y(G_2(x,t), G_2(x,t_0)) < \varepsilon,$$
$$\operatorname{dist}_Y(G_1(x,t_0), G_2(x,t_0)) - \operatorname{dist}_Y(G_1(x,t), G_2(x,t))$$
$$\leq h_Y(G_1(x,t), G_1(x,t_0)) + h_Y(G_2(x,t), G_2(x,t_0)) < \varepsilon.$$

Therefore, $|U(x,t) - U(x,t_0)| < \varepsilon$ for every $t \in B_T(t_0, \delta)$ and $x \in A$. So, condition (iv) holds. Assumptions (iv) and (v) imply that conditions (v) and (vi) of Theorem 2.1 hold. Thus, all the conditions of Theorem 2.1 hold.

Theorem 2.1 implies that for each $t \in T$ there exists a point $\xi(t) \in T$ such that $U(t, \xi(t)) = \gamma(t) = 0$. Therefore, $\xi(t) \in \Xi(t)$ for every $t \in T$. □

Let us show that assumption (v) in Theorem 3.7 is essential.

Example 3.8. Let $T = [0, 1/3]$, $X = [3, \infty) \times \mathbb{R}$, $A = [3, \infty) \times [0, 1/2] \subset X$, the metric ρ_X is defined by (2.5), $U : A \times T \to \mathbb{R}$ is the function from Example 2.10. In Example 2.10, it was shown that all the conditions of Theorem 2.1 hold true, except condition (vi), and the function $U(\cdot, t)$ has points of minimum for every $t \in T$ except $t = 0$.

Let $G_1, G_2 : A \times T \rightrightarrows \mathbb{R}$, $G_1(x,t) := \{y \in \mathbb{R} : y \geq U(x,t)\}$, $G_2(x,t) := \{0\}$. It is obvious that the set-valued mappings G_1 and G_2 satisfy all the assumptions of Theorem 3.7 except (v), while the mappings $G_1(\cdot, 0)$ and $G_2(\cdot, 0)$ have no generalized coincidence points.

The corollary of the obtained result is the elementary implicit function theorem for single-valued contractive mappings (see [8], Chapter I, §1.3). Let us formulate and prove a proposition of greater generality.

Given a mapping $g : A \times T \to X$, denote the set of all fixed points of the mapping $g(\cdot, t)$ by $\Xi(t)$, i.e.

$$\Xi(t) := \{\xi \in A : \xi = g(t, \xi)\} \quad \forall\, t \in T.$$

Corollary 3.9. *Let T be a connected metric space, $\beta \in [0, 1)$ be given. Assume that the mapping g is uniformly continuous, for every fixed $t \in T$ the mapping $g(\cdot, t)$ is contractive with the constant β, and $\Xi(t) \cap \partial A = \emptyset$ for every $t \in T$. Then if the set $\Xi(t)$ is non-empty for some $t \in T$, then $\Xi(t)$ is non-empty and singleton for each $t \in T$.*

Proof. Let us show that the mappings $G_1(x, t) := \{g(x, t)\}$ and $G_2(x, t) := \{x\}$, $(x, t) \in A \times T$ satisfy all the conditions of Theorem 3.7 with the constant $k := 1 - \beta$.

Condition (i) holds, since g is continuous. Let us verify (ii). Let inequality (3.4) hold for some $(x, t) \in X \times T$. Set $x' := g(x, t)$.
We have $x' \neq x$, since $0 < \rho_X(x, g(x, t))$; $x' \in A$, since

$$\rho_X(x, x') = \rho_X(x, g(x, t)) \leq \beta \mathrm{dist}_X(x, \partial A) < (1 - \beta)\mathrm{dist}_X(x, \partial A)$$

and

$$\begin{aligned} \mathrm{dist}_Y(G_1(x', t), G_2(x', t)) + k\rho_X(x, x') &= \rho_X(g(x, t), g(g(x, t), t)) \\ +(1 - \beta)\rho_X(x, g(t, x)) &\leq \beta\rho_X(x, g(t, x)) + (1 - \beta)\rho_X(x, g(t, x)) \\ &= \rho_X(x, g(t, x)) = \mathrm{dist}_Y(G_1(x, t), G_2(x, t)). \end{aligned}$$

Thus, condition (ii) holds. Conditions (iii) and (iv) by the assumptions.

Let us verify condition (v). Assume that there exists a sequence $\{t_i\} \subset T$ such that $t_i \to \bar{t}$, $\Xi(t_i) \neq \emptyset$ for every i and $h_X^+(\Xi(t_i), \partial A) \to 0$ as $i \to \infty$. Take an arbitrary sequence $\{x_i\} \subset X$ such that $x_i \in \Xi(t_i)$ for every i. Since the set valued mapping g is contractive in x, we have

$$\rho_X(x, x_i) \leq \frac{\rho_X(g(x, t_i), x_i)}{1 - \beta} \quad \forall\, x \in A. \tag{3.5}$$

Let us prove that $\{x_i\}$ is a Cauchy sequence. Take an arbitrary $\varepsilon > 0$.
The uniform continuity of the mapping g implies that there exists a number N such that $\rho_X(g(x, t_i), g(x, t)) < \varepsilon(1 - \beta)/2$ for every $i > N$, $x \in X$. Thus, inequality (3.5) implies that

$$\begin{aligned} \rho_X(x_i, x_j) &\leq \frac{\rho_X(x_i, g(x_i, t_j))}{1 - \beta} = \frac{\rho_X(g(x_i, t_i), g(x_i, t_j))}{1 - \beta} \\ &\leq \frac{\rho_X(g(x_i, t_i), g(x_i, t)) + \rho_X(g(x_i, t), g(x_i, t_j))}{1 - \beta} < \varepsilon \end{aligned}$$

for every $i, j > N$. Thus, $\{x_i\}$ is a Cauchy sequence. Completeness of X and closedness of A imply that $\{x_i\}$ converges to some point $\bar{x} \in A$. Since $h_X^+(\Xi(t_i), \partial A) \to 0$, we have $\bar{x} \in \partial A$. Since the mapping g is continuous, passing to the limit as $i \to \infty$ in the equation $x_i = g(t_i, x_i)$, we obtain $\bar{x} = g(\bar{t}, \bar{x})$. Since $\bar{x} \in \partial A$, the last equality contradicts the assumption $\Xi(\bar{t}) \cap \partial A = \emptyset$. Therefore, the considered sequence $\{t_i\}$ does not exist, so, condition (v) of the theorem holds.

Theorem 3.7 implies that $\Xi(t) \neq \emptyset$ for all t. The fact that the set $\Xi(t)$ is singleton directly follows from the fact that g is a contraction in x. □

4. Comparison of fixed point theorems

Everywhere in this section X is a complete metric space with a metric ρ_X.

Let us start with an example in which the assumptions of Theorem 3.2′ hold, while it is onerous to verify the assumptions of the other fixed point theorems.

Example 4.1. Given continuously differentiable functions $\sigma_1, \sigma_2 : \mathbb{R}^2 \to \mathbb{R}$ of the variable $x = (x_1, x_2) \in \mathbb{R}^2$, assume that there exists $\mu < 1$ such that

$$\frac{\partial \sigma_1}{\partial x_1}(x) \leq \mu, \ \frac{\partial \sigma_1}{\partial x_2}(x) \geq 0, \ \frac{\partial \sigma_2}{\partial x_1}(x) \leq 0, \ \frac{\partial \sigma_2}{\partial x_2}(x) \leq \mu,$$

for every $x \in \mathbb{R}^2$, and both functions σ_1, σ_2 are bounded.
Define a mapping $g : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$g(x) = \left(x_2^3 + \sigma_1(x), -x_1^3 + \sigma_2(x)\right), \ x \in \mathbb{R}^2.$$

Let us show that the mapping g satisfies all the assumptions of Theorem 3.2′. Set $U(x) := \|g(x) - x\|, \ x \in \mathbb{R}^2$. Here $\|x\|$ is the Euclidean norm of a vector $x \in \mathbb{R}^2$. Let us prove that for every $d \geq 0$, the set $L_d := \{x \in \mathbb{R}^2 : U(x) \leq d\}$ is bounded. For every $x \in L_d$, we have $|x_2^3 + \sigma_1(x) - x_1| \leq d$ and $| - x_1^3 + \sigma_2(x) - x_2| \leq d$, so $|x_2^3| \leq d + |\sigma_1(x)| + |x_1|$ and $|x_1^3| \leq d + |\sigma_2(x)| + |x_2|$. Summing up these inequalities we obtain

$$|x_1|^3 + |x_2|^3 \leq 2d + |x_1| + |x_2| + |\sigma_1(x)| + |\sigma_2(x)|$$

$$\frac{1}{2}\|x\|^3 \leq 2d + 2\|x\| + |\sigma_1(x)| + |\sigma_2(x)| \quad \Rightarrow \quad \frac{1}{2}\|x\|^3 - 2\|x\| - (a + 2d) \leq 0,$$

where $a > 0$ is such that $|\sigma_1(x)| + |\sigma_2(x)| \leq a$ for every x. Thus, we have

$$\frac{1}{2}\|x\|^3 - 2\|x\| - (a + 2d) \leq 0 \ \forall \, x \in L_d. \tag{4.1}$$

Denote by $R = R(d)$ the maximal solution of the inequality $\frac{1}{2}r^3 - 2r - (a+2d) \leq 0$. In virtue of (4.1) we have $\|x\| \leq R$ for each $x \in L_d$ and, therefore, the set L_d is bounded.

Set $f(x) := x - g(x)$. Let us prove that for every $d \geq 0$ there exists $c(d) > 0$ such that

$$\left\| \left(\frac{\partial f}{\partial x}(x) \right)^{-1} \right\| \leq c(d) \text{ for every } x \in L_d.$$

Since

$$\det\left(\frac{\partial f}{\partial x}(x)\right) = \det \begin{pmatrix} 1 - \frac{\partial \sigma_1}{\partial x_1}(x) & -3x_2^2 - \frac{\partial \sigma_1}{\partial x_2}(x) \\ 3x_1^2 - \frac{\partial \sigma_2}{\partial x_1}(x) & 1 - \frac{\partial \sigma_2}{\partial x_2}(x) \end{pmatrix} \geq (1-\mu)^2 + 9x_1^2x_2^2 \geq (1-\mu)^2,$$

the matrix $\dfrac{\partial f}{\partial x}(x)$ is non-degenerate for $x \in \mathbb{R}^2$. So, the desired value $c(d)$ exists in virtue of compactness of the set L_d.

Let us verify the assumptions of Theorem 3.2′. Take arbitrary $\bar{x} \in \mathbb{R}^2$.
Set $d := U(\bar{x})+1$. For arbitrary x from the open set $\widetilde{X} := \{x \in \mathbb{R}^2 : 0 < U(x) < d+1\}$ denote $e := \frac{f(x)}{\|f(x)\|}$, $A := \frac{\partial f}{\partial x}(x)$. Since $\|e\| = 1$, we have

$$\left\|\frac{\partial U}{\partial x}(x)\right\| = \left\|eA\right\| = \frac{\|A^{-1}\|\|eA\|}{\|A^{-1}\|} \geq \frac{\|eAA^{-1}\|}{\|A^{-1}\|} = \frac{1}{\|A^{-1}\|} \geq \frac{1}{c(d+1)}.$$

Therefore, in virtue of Remark 1.10 to Lemma 1.9, the function U satisfy Carisiti-like condition (1.1) on the set $\widetilde{X}$ with arbitrary $k < (c(d+1))^{-1}$. Therefore, (3.3) holds for all $x \in \widetilde{X}$ and specifically, for all x such that $0 < \|x - g(x)\| \leq \|\bar{x} - g(\bar{x})\|$. So, all the assumptions of Theorem 3.2′ hold. Hence there exists a fixed point of the mapping g.

Let us discuss the Caristi's theorem and Theorem 3.2′ and compare them.

The Caristi's theorem is not only a sufficient condition but also a necessary condition for the existence of a fixed point for a mapping $g : X \to X$, in the case when the set of fixed points of this mapping is closed. Indeed, let the set of fixed points of the mapping g be non-empty and closed. Set $U(x) := 0$ for $x \in X$ such that $x = g(x)$ and $U(x) := +\infty$ for $x \in X$ such that $x \neq g(x)$. Obviously, this function is lower semi-continuous, bounded from below and satisfies condition (3.1). So, the assumptions of the Caristi's theorem hold if for the mapping $g : X \to X$ the set of fixed points is closed (for example, if g is continuous) and non-empty. Therefore, only those mappings are of interest for which there exists a function U such that not only condition (3.1) holds but also $x \neq g(x)$ for a certain $x \in \operatorname{dom} U$. Let us give an example of a corresponding mapping for which function U with the desired property exists, although there exists no function U that is finite on X satisfying condition (3.1).

Example 4.2. Let $g : [-1, 1] \to [-1, 1]$, $g(x) \equiv -x^3$. Set

$$\widetilde{U}(-1) = \widetilde{U}(1) = +\infty, \quad \widetilde{U}(x) = \frac{2}{1 - |x|} \quad \forall\, x \in (-1, 1).$$

Then condition (3.1) holds for $U = \widetilde{U}$, $\widetilde{U}(x)$ is finite for all $x \in (-1, 1)$ and the only fixed point of g is zero.

At the same time, if a function U satisfies condition (3.1), we have $U(-1) = U(1) = +\infty$. Indeed, if $U(-1)$ is finite, then (3.1) implies $U(1) < U(-1)$ and $U(-1) < U(1)$. If $U(1)$ is finite we obtain the contradiction also.

Generally speaking, the class of mappings g such that there fixed points set is non-empty and closed and any aforementioned function U satisfy the relation $U(x) = +\infty$ for each x such that $g(x) \neq x$, is quite wide. It includes all expanding mappings g. Recall that the mapping $g : X \to X$ is said to be an expanding map, if for some $\lambda > 1$ the following inequality holds

$$\rho_X(g(x), g(u)) \geq \lambda \rho_X(x, u) \quad \forall\, x, u \in X. \tag{4.2}$$

It is known that a surjective expanding mapping of a complete metric space has a unique fixed point.

Proposition 4.3. *Let $g : X \to X$ be an expanding mapping, a function U be bounded below, $k > 0$, and (3.1) hold. Then $U(x) = +\infty$ for each x such that $g(x) \neq x$. If, in addition, the mapping g is surjective then (3.3) holds for some $k > 0$.*

Proof. Assume that for a certain $\lambda > 1$, inequality (4.2) holds and U satisfies (3.1). Let $g(x) \neq x$. Inequality (3.1) implies $k\rho_X(x, g(x)) \leq U(x) - U(g(x))$ and $k\rho_X(g^{(n)}(x), g^{(n+1)}(x)) \leq U(g^{(n)}(x)) - U(g^{(n+1)}(x))$. Here n is an arbitrary non-negative integer, g^i is the i-th iteration of the mapping g, $g^0(x) \equiv x$. Summing up these inequalities, we have

$$k\sum_{i=0}^{n-1} \rho_X(g^{i-1}(x), g^i(x)) \leq U(x) - U(g^n(x)) \leq U(x) - \gamma \quad \forall\, n \geq 2, \tag{4.3}$$

where $\gamma = \inf\limits_{x\in X} U(x)$.

At the same time, (4.2) implies $\rho_X(g^{i-1}(x), g^i(x)) \geq \lambda^{i-1}\rho_X(x, g(x))$ for every $i \geq 2$. Since $\lambda > 1$ and $\rho_X(x, g(x)) > 0$ the obtained inequality implies $U(x) = +\infty$.

Let us additionally assume that the mapping g is surjective. It is a straightforward task to ensure that (3.3) holds for $x' := g^{-1}(x)$ and $k := \lambda - 1$. □

Thus, Theorem 3.2′ can be substantially applied to the class of surjective expanding mappings, whereas the Caristi's theorem is non-applicable. The following example shows that there exist continuous mappings for which the Caristi's theorem can be substantially applied unlike Theorem 3.2′.

Example 4.4. Let

$$g : \mathbb{R}_+ \to \mathbb{R}_+, \quad g(x) \equiv \frac{x}{1+x}, \quad U(x) \equiv x, \quad k = 1.$$

Then $U(x) < +\infty$ for every $x \in \mathbb{R}_+$ and condition (3.1) holds, since

$$U(g(x)) + \rho_X(x, g(x)) = \frac{x}{1+x} + x - \frac{x}{1+x} = x = U(x).$$

At the same time, condition (3.3) fails for any $k > 0$. In fact, if (3.3) holds, then for every $x \neq 0$ there exists $x' < x$, such that

$$x' - \frac{x'}{1+x'} + k(x - x') \leq x - \frac{x}{1+x}.$$

Then

$$k \leq 1 - \frac{1}{x - x'}\left(\frac{x}{1+x} - \frac{x'}{1+x'}\right) = 1 - \frac{1}{(1+x)(1+x')}.$$

Passing to the limit as $x \to 0$, since $x' < x$, we obtain $k \leq 0$.

Consider general corollaries of the Caristi's theorem and Theorem 3.2′.

Corollary 4.5. *Let there exist $\beta \in [0, 1)$, such that*

$$\rho_X(g^2(x), g(x)) \leq \beta\rho_X(g(x), x) \quad \forall\, x \in X.$$

Then for every $x \in X$, there exists $\xi \in X$ such that

$$\xi = g(\xi) \text{ and } \rho_X(x, \xi) \leq \rho_X(x, g(x))/(1 - \beta).$$

This proposition directly follows from the Caristi's theorem and Theorem 3.2$'$. For derivation of Corollary 4.5 from the Caristi's theorem, it is enough to apply it to the mapping g and the function $U(x) \equiv \rho_X(x, g(x))$. For derivation of Corollary 4.5 from Theorem 3.2$'$ it is enough to apply it to the mapping g, assuming that $x' := g(x)$ for all $x \in X$.

Note that unlike the contraction mapping principle, the assumptions of the Caristi's theorem and Theorem 3.2$'$ are not sufficient for the uniqueness of the fixed point. Let us provide a corresponding example.

Example 4.6. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x) = (x_1, 0)$, $x = (x_1, x_2) \in \mathbb{R}^2$. Then the assumptions of the Caristi's theorem and Theorem 3.2$'$ hold, since the assumptions of Corollary 4.5 hold for $\beta = 0$. The set of fixed points is $\{x : x_2 = 0\}$.

As it was noted above the Caristi's theorem holds true without prior assumption of continuity of the mapping g. However, if g is continuous then for the function U we can take the function $\widehat{U}$, defined by the equality

$$\widehat{U}(x) = \sum_{n=0}^{\infty} \rho_X(g^n(x), g^{n+1}(x)). \tag{4.4}$$

Here we assume that $\widehat{U}(x) = +\infty$ if the series in (4.4) diverges. Function $\widehat{U}$ was considered in ([4], Chapter 5, §3) as a minimal function, satisfying condition (3.1).

Elements of the functional series (4.4) are non-negative continuous functions, since g is continuous. Thus, the function $\widehat{U}$ is lower semi-continuous. It also follows from (4.4) that $\widehat{U}(g(x)) \equiv \widehat{U}(x) - \rho_X(x, g(x))$ and, therefore, (3.1) holds with $U = \widehat{U}$ and $k = 1$. Thus, the function $\widehat{U}$ defined by the relationship (4.4) is non-negative, lower semi-continuous, it satisfies condition (3.1), and if x is a fixed point of g then $\widehat{U}(x) = 0$.

Let us show that if the mapping g is continuous then the condition of the Caristi's theorem is equivalent to the fact that the series in (4.4) converges for a certain $x \in X$.

Proposition 4.7. *If the mapping g is continuous, then the series in (4.4) converges at a point $x \in X$ if and only if there exists a lower semi-continuous proper function U, that is bounded from below and satisfies (3.1).*

Proof. Let such a function U exist. Then (4.3) implies that the series in (4.4) converges for every $x \in \mathrm{dom} U \neq \emptyset$. Conversely, if the series in (4.4) converges at a point x then the function $\widehat{U}$ defined by the relationship (4.4) is the desired one. □

Thus, if the mapping g is continuous, then the condition in the Caristi's theorem is equivalent to the fact that the sequence in (4.4) converges for a certain $x \in X$. Conversely the convergence of this series guarantees that the sequence of iterations $\{g^n(x)\}$ is a Cauchy sequence and, therefore, converges to a certain point $\xi \in X$. It is obvious that $g(\xi) = \xi$.

Proposition 4.7 demonstrates that if the mapping g is continuous, then for the function U in the Caristi's theorem it is natural to take $\widehat{U}$, defined by formula (4.4). For discontinuous mappings g the statement of Proposition 4.7 might fail, since the function $\widehat{U}$ may be not lower semi-continuous. Let us demonstrate this fact by the following examples. In these examples, the function $\widehat{U}$ is finite at every point x but is not lower semi-continuous. At the same time in the first example the mapping g has a unique fixed point, however the estimate of the distance to the fixed point from Theorem 3.2$'$ does not hold. In the second example, the mapping g has no fixed points.

Example 4.8. Let $g : [0, 2] \to [0, 2]$,

$$g(x) = \frac{x}{2}, \quad \text{if} \quad x \in [0, 1], \quad g(x) = \frac{1}{2} + \frac{x}{2}, \quad \text{if} \quad x \in (1, 2].$$

Then the function $\widehat{U}$, defined by the equality (4.4), has the form

$$\widehat{U}(x) = \sum_{n=0}^{\infty} \rho_X(g^n(x), g^{n+1}(x)) = \sum_{n=0}^{\infty} \frac{x-1}{2^{n+1}} = x - 1 \quad \text{if} \quad x \in (1, 2],$$

$$\widehat{U}(x) = \sum_{n=0}^{\infty} \rho_X(g^n(x), g^{n+1}(x)) = \sum_{n=0}^{\infty} \frac{x}{2^{n+1}} = x \quad \text{if} \quad x \in [0, 1].$$

It is obvious that the function $\widehat{U}$ is not lower semi-continuous at the point $x = 1$. The Caristi condition (3.1) holds for it by the construction. However, the propositions of Theorem 3.2$'$ and specifically the estimate of the distance to a fixed point does not hold for it for any $k > 0$. Let us show this. Set $\bar{x}_n := 1 + n^{-1}$, $n \neq 1$, $\gamma := 0$. The only fixed point of g is $\xi = 0$. Therefore, if $\rho_X(\bar{x}_n, \xi) \leq \dfrac{\widehat{U}(\bar{x}_n) - \gamma}{k}$, then $1 + \dfrac{1}{n} \leq \dfrac{1}{nk}$ and thus $k \leq \dfrac{1}{n+1}$ for every n. So, the estimate in Theorem 3.2$'$ does not hold for any $k > 0$.

Example 4.9. Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \dfrac{x}{2} + \dfrac{j}{2}$ for $x \in (j, j+1]$, $j \in \mathbb{Z}$.
When $x \in (j, j+1]$, we have

$$\widehat{U}(x) = \sum_{n=0}^{\infty} \rho_X(g^n(x), g^{n+1}(x)) = \sum_{n=0}^{\infty} \left(\frac{x}{2^{n+1}} - \frac{j}{2^{n+1}} \right) = x - j.$$

It is obvious that the function $\widehat{U}$ is not lower semi-continuous at the points $x \in \mathbb{Z}$. Note that there exists no function U satisfying all the conditions of Caristi's Theorem, since g does not have fixed points.

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