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ON GREGUS-ĆIRIĆ MAPPINGS ON WEIGHTED GRAPHS

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Abstract. In this paper, we introduce the concept of monotone Gregus-Ćirić-contraction mappings in weighted digraphs. Then we establish a fixed point theorem for monotone Gregus-Ćirić-contraction mappings defined in convex weighted digraphs.

Key Words and Phrases: Fixed point, Gregus-Ćirić-contraction, monotone mappings, weighted graph.
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1. Introduction

Banach's Contraction Principle [3] is perhaps the most widely applied fixed point theorem in all of analysis. Over the years, many mathematicians tried successfully to extend this fundamental theorem. In 1980, Gregus [8] proved the following result:

Theorem 1.1. *Let X be a Banach space and C be a nonempty closed and convex subset of X. Let $T : C \to C$ be a mapping satisfying*

$$\|T(x) - T(y)\| \leq a\|x - y\| + p\|T(x) - x\| + p\|T(y) - y\|,$$

for all $x, y \in C$, where $0 < a < 1$, $p \geq 0$ and $a + 2p = 1$. Then T has a unique fixed point.

Ćirić [6] obtained the following generalization of Gregus' theorem.

Theorem 1.2. *Let (X, d) be a complete convex metric space and C be a nonempty closed and convex subset of X. Let $T : C \to C$ be a mapping satisfying*

$$\begin{aligned} d(T(x), T(y)) \quad &\leq \quad a \ \max\Big\{d(x, y), c\ \Big[d(x, T(y)) + d(y, T(x))\Big]\Big\} \\ &\qquad + b\ \max\{d(x, T(x)), d(y, T(y))\}, \end{aligned} \tag{CG}$$

for all $x, y \in C$, where $0 < a < 1$, $a + b = 1$ and $0 \leq c \leq \frac{4-a}{8-a}$. Then T has a unique fixed point.

Remark 1.1. *If we assume that $a+b < 1$ and $c \leq \frac{1}{2}$, then any map T which satisfies the condition (CG) also satisfies the following condition:*

$$d(T(x), T(y)) \leq (a+b)\ \max\Big\{d(x,y), d(x,T(y)), d(y,T(x)), d(x,T(x)), d(y,T(y))\Big\}.$$

In other words, T is a Ćirić quasi-contraction mapping. This concept was introduced by Ćirić [5] as an extension to the contraction condition. In [5], he proved an analogue to the Banach Contraction Principle for this type of mappings without the use of convexity.

Recently, Djafari-Rouhani and Moradi [7] obtained the following improvement of Ćirić's result:

Theorem 1.3. *Let (X, d) be a complete convex metric space and $T : X \to X$ be a mapping satisfying*

$$\begin{aligned} d(T(x), T(y)) \quad &\leq \quad a\ \max\{d(x,y), c[d(x,T(y)) + d(y,T(x))]\} \\ &\quad\quad +b\ \max\{d(x,T(x)), d(y,T(y))\}, \end{aligned}$$

for all $x, y \in X$, where $0 < a < 1$, $a + b = 1$ and $0 \leq c < \frac{1}{2}$. Then T has a unique fixed point.

In fact, the authors in [7] gave a simple example which shows that the conclusion of Theorem 1.3 does not hold if $c > \frac{1}{2}$ and asked whether its conclusion holds when $c = \frac{1}{2}$. This problem is still open.

In this work, we generalize Theorem 1.3 to the case of monotone self-mappings defined on a weighted graph.

2. Preliminaries

A graph G is a nonempty set $V(G)$ of elements called vertices together with a possibly empty subset $E(G)$ of $V(G) \times V(G)$ called edges. A directed graph (digraph) is a graph with a direction assigned to each of its edges. In this paper, we assume that all digraphs are reflexive, i.e., $(x, x) \in E(G)$ for each $x \in V(G)$. Moreover, we assume that there exists a distance function d defined on the set of vertices $V(G)$. Throughout this work, we treat G as a weighted digraph by giving each edge the metric distance between its vertices.

Let x and y be in $V(G)$. A (directed) path from x to y is a finite sequence $\{x_i\}_{i=0}^N$ of vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, ..., N$.

Definition 2.1. *The digraph G is said to be transitive if $(x, z) \in E(G)$ whenever $(x, y) \in E(G)$ and $(y, z) \in E(G)$, for any $x, y, z \in V(G)$. In another words, G is transitive if for any two vertices x and y that are connected by a directed finite path, we have $(x, y) \in E(G)$.*

Definition 2.2. *Let G be a weighted digraph and d be a metric distance on $V(G)$. Let C be a nonempty subset of $V(G)$. A mapping $T : C \to C$ is called*

(1) *G-monotone if T is edge preserving, i.e., $(T(x), T(y)) \in E(G)$ whenever $(x, y) \in E(G)$, for any $x, y \in C$.*
(2) *G-monotone Gregus-Ćirić-mapping if T is G-monotone and there exist $a, b, c \in [0, +\infty)$ such that*

$$\begin{aligned} d(T(x), T(y)) \quad \leq \quad & a \ \max\Big\{d(x,y), c\Big[d(x, T(y)) + d(y, T(x))\Big]\Big\} \\ & +b \ \max\Big\{d(x, T(x)), d(y, T(y))\Big\}, \end{aligned}$$

for any $x, y \in C$ with $(x, y) \in E(G)$.
(3) *G-monotone Gregus-Ćirić-contraction if T is G-monotone Gregus-Ćirić-mapping for which $0 < a < 1$, $a + b = 1$ and $c \leq \frac{1}{2}$.*

The point $x \in C$ is called a fixed point of T if $T(x) = x$.

Note that in the example given by the authors in [7], the mapping $T(x) = x + 1$ is monotone for the order and may be seen as an example of a monotone Gregus-Ćirić mapping. Moreover, any monotone contraction is a monotone Gregus-Ćirić-contraction. The example studied by Ran and Reurings [11] gives an example of a monotone-contraction which fails to be a contraction.

The following definition is needed since we will be using the concept of increasing or decreasing sequences in the sense of a digraph.

Definition 2.3. *Let G be a digraph. A sequence $\{x_n\} \in V(G)$ is said to be*
(a) *G-increasing if $(x_n, x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$;*
(b) *G-decreasing if $(x_{n+1}, x_n) \in E(G)$, for all $n \in \mathbb{N}$;*
(c) *G-monotone if $\{x_n\}$ is either G-increasing or G-decreasing.*

Definition 2.4. *Let G be a weighted digraph and d be a metric distance on $V(G)$. A subset C of $V(G)$ is said to be G-complete if any G-monotone sequence $\{x_n\}$ in C which is Cauchy is convergent to a point in C.*

Remark 2.1. *Let G be a weighted digraph and d be a metric distance on $V(G)$. If $(V(G), d)$ is complete, then $V(G)$ is G-complete. The converse is not true. Indeed, let $X = \{(x, y) \in \mathbb{R}^2;\ 0 \leq x < 1 \text{ and } 0 \leq y \leq 1\}$ endowed with the Euclidean distance. Then (X, d) is not complete. Moreover, if we consider the weighted digraph G such that $V(G) = X$ and*

$$\Big((x,y),(a,b)\Big) \in V(G) \ \text{ if and only if } \ x = a \text{ and } \ y \leq b;$$

then $V(G)$ is G-complete. In fact, any G-monotone sequence is convergent.

As Jachymski did in [9], we introduce the following property:

Definition 2.5. *Let G be a weighted digraph and C be a nonempty subset of $V(G)$. We say that C has Property (*) if for any G-increasing (resp. G-decreasing) sequence $\{x_n\}$ in C which converges to x, there is a subsequence $\{x_{k_n}\}$ with $(x_{k_n}, x) \in E(G)$ (resp. $(x, x_{k_n}) \in E(G)$), for $n \in \mathbb{N}$.*

Note that if G is transitive, then Property (*) implies that for any G-increasing sequence $\{x_n\}$ (resp. G-decreasing) which converges to x, we have $(x_n, x) \in E(G)$ (resp. $(x, x_n) \in E(G)$), for every $n \in \mathbb{N}$.

3. Some basic results

Throughout this section, we consider G a weighted digraph with d a metric distance on $V(G)$. Let C be a nonempty subset of $V(G)$ and $T : C \to C$ be G-monotone Gregus-Ćirić-contraction mapping. Then there exist positive numbers a, b, c such that $0 < a < 1$, $a + b = 1$ and $c \leq \frac{1}{2}$ such that

$$\begin{array}{rcl} d(T(x),T(y)) & \leq & a \ \max\Big\{d(x,y), c\Big[d(x,T(y)) + d(y,T(x))\Big]\Big\} \\ & & + \ b \ \max\Big\{d(x,T(x)), d(y,T(y))\Big\}, \end{array}$$

for any $x, y \in C$ with $(x, y) \in E(G)$.

The following technical results will be crucial to the establishment of the main theorem of this work.

Lemma 3.1. *Under the above assumptions, we have*

$$d(x,y) \leq \frac{2-a}{1-a} \ \Big(d(x,T(x)) + d(y,T(y))\Big),$$

for any $x, y \in C$ with $(x, y) \in E(G)$ or $(y, x) \in E(G)$.

Proof. Without loss of generality, we assume $(x, y) \in E(G)$. Then we have

$$\begin{array}{rcl} d(T(x),T(y)) & \leq & a \ \max\Big\{d(x,y), c\Big[d(x,T(y)) + d(y,T(x))\Big]\Big\} \\ & & +b \ \max\Big\{d(x,T(x)), d(y,T(y))\Big\}. \end{array}$$

Since $c \leq \frac{1}{2}$, we get

$$\begin{array}{rcl} c\Big[d(x,T(y)) + d(y,T(x))\Big] & \leq & c\Big[d(x,T(x)) + 2d(T(x),T(y)) + d(y,T(y))\Big] \\ & \leq & d(x,T(x)) + d(T(x),T(y)) + d(y,T(y), \end{array}$$

which implies

$$\begin{array}{rcl} d(T(x),T(y)) & \leq & a \ \max\Big\{d(x,T(x)) + d(T(x),T(y)) + d(y,T(y)), \\ & & c\Big[d(x,T(y)) + d(y,T(x))\Big]\Big\} + b \ \max\Big\{d(x,T(x)), d(y,T(y))\Big\} \\ & \leq & a \ \Big\{d(x,T(x)) + d(T(x),T(y)) + d(y,T(y))\Big\} \\ & & +b \ \Big\{d(x,T(x)) + d(y,T(y))\Big\} \\ & \leq & (a+b)\Big(d(x,T(x)) + d(T(x),T(y))\Big) + a \ d(T(x),T(y)). \end{array}$$

Since $a + b = 1$, we get

$$d(T(x),T(y)) \leq \frac{1}{1-a} \ \Big(d(x,T(x)) + d(T(x),T(y))\Big).$$

Hence

$$\begin{array}{rcl} d(x,y) & \leq & d(x,T(x)) + d(T(x),T(y)) + d(y,T(y), \\ & \leq & \Big(1 + \frac{1}{1-a}\Big) \ \Big(d(x,T(x)) + d(T(x),T(y))\Big), \end{array}$$

which implies

$$d(x,y) \leq \frac{2-a}{1-a} \ \Big(d(x,T(x)) + d(y,T(y))\Big). \qquad \square$$

Lemma 3.2. *Under the above assumptions, if $x \in C$ is such that $(x, T(x)) \in E(G)$ or $(T(x), x) \in E(G)$, then the sequence $\Big\{d(T^n(x), T^{n+1}(x))\Big\}_{n\in\mathbb{N}}$ is decreasing.*

Proof. Without loss of generality, we assume $(x, T(x)) \in E(G)$.
Since T is G-monotone, we get $(T^n(x), T^{n+1}(x)) \in E(G)$, for any $n \in \mathbb{N}$. Fix $n \geq 1$. Then

$$\begin{aligned} d(T^n(x), T^{n+1}(x)) \quad &\leq \quad a \ \max\Big\{d(T^{n-1}(x), T^n(x)), c\ d(T^{n-1}(x), T^{n+1}(x))\Big\} \\ &\quad + b\ \max\Big\{d(T^{n-1}(x), T^n(x)), d(T^n(x), T^{n+1}(x))\Big\}. \end{aligned}$$

Assume that $d(T^{n-1}(x), T^n(x)) < d(T^n(x), T^{n+1}(x))$ holds. Since

$$\begin{aligned} d(T^{n-1}(x), T^{n+1}(x)) \quad &\leq \quad d(T^{n-1}(x), T^n(x)) + d(T^n(x), T^{n+1}(x)) \\ &< \quad 2\ d(T^n(x), T^{n+1}(x)), \end{aligned}$$

and $c \leq \frac{1}{2}$, we get

$$\begin{aligned} d(T^n(x), T^{n+1}(x)) \quad &< \quad a\ d(T^n(x), T^{n+1}(x)) + b\ d(T^n(x), T^{n+1}(x)) \\ &= \quad d(T^n(x), T^{n+1}(x)). \end{aligned}$$

This contradiction forces $d(T^n(x), T^{n+1}(x)) \leq d(T^{n-1}(x), T^n(x))$. Since n was taken arbitrarily, we conclude that $\Big\{d(T^n(x), T^{n+1}(x))\Big\}_{n\in\mathbb{N}}$ is decreasing. □

Lemma 3.3. *Under the above assumptions, assuming G is transitive, if $x \in C$ such that $(x, T(x)) \in E(G)$ or $(T(x), x) \in E(G)$, then there exists $n \geq 1$ such that*

$$d(T^n(x), T^{n+2}(x)) \leq \frac{2}{2-a}\ d(x, T(x)).$$

Proof. Here we mimic an argument used by Djafari-Rouhani and Moradi in their proof of [[7], Theorem 2.2]. Without loss of generality, we assume that $(x, T(x)) \in E(G)$. Since G is transitive and T is G-monotone, then $(T^n(x), T^{n+h}(x)) \in E(G)$, for any $n, h \in \mathbb{N}$. Fix $n \geq 1$. then we have

$$\begin{aligned} d(T^n(x), T^{n+2}(x)) \quad &\leq \quad a\ \max\Big\{d(T^{n-1}(x), T^{n+1}(x)), \\ &\quad c\ \Big[d(T^{n-1}(x), T^{n+2}(x)) +\ d(T^n(x), T^{n+1}(x))\Big]\Big\} \\ &\quad + b\ \max\Big\{d(T^{n-1}(x), T^n(x)), d(T^{n+1}(x), T^{n+2}(x))\Big\}. \end{aligned}$$

Assume that for some $n \geq 1$, we have

$$d(T^{n-1}(x), T^{n+1}(x)) \leq c\ \Big[d(T^{n-1}(x), T^{n+2}(x)) +\ d(T^n(x), T^{n+1}(x))\Big].$$

Since $\Big\{d(T^n(x), T^{n+1}(x))\Big\}_{n\in\mathbb{N}}$ is decreasing, we get

$$\begin{aligned} d(T^n(x), T^{n+2}(x)) \quad &\leq \quad ac\ \Big[d(T^{n-1}(x), T^{n+2}(x)) +\ d(T^n(x), T^{n+1}(x))\Big] \\ &\quad + b\ d(x, T(x)). \end{aligned}$$

Since $d(T^{n-1}(x), T^{n+2}(x)) \leq d(T^{n-1}(x), T^n(x)) + d(T^n(x), T^{n+2}(x))$, we get

$$\begin{array}{rcl} d(T^n(x), T^{n+2}(x)) & \leq & ac\left[2\ d(x, T(x)) +\ d(T^n(x), T^{n+2}(x))\right] \\ & & +\ b\ d(x, T(x)), \end{array}$$

which implies

$$d(T^n(x), T^{n+2}(x)) \leq \frac{2ac + b}{1 - ac}\ d(x, T(x)).$$

The function $f(c) = \frac{2ac+b}{1-ac}$ is increasing in the interval $[0, \frac{1}{2}]$. Hence

$$\frac{2ac + b}{1 - ac} \leq \frac{a + b}{1 - a/2} = \frac{2}{2 - a}.$$

Therefore, we have

$$d(T^n(x), T^{n+2}(x)) \leq \frac{2}{2 - a}\ d(x, T(x)).$$

Next, assume that for any $n \geq 1$, we have

$$c\ \left[d(T^{n-1}(x), T^{n+2}(x)) +\ d(T^n(x), T^{n+1}(x))\right] \leq d(T^{n-1}(x), T^{n+1}(x)).$$

In this case, we have

$$d(T^n(x), T^{n+2}(x)) \leq a\ d(T^{n-1}(x), T^{n+1}(x)) + b\ d(x, T(x)),$$

which easily implies

$$\begin{array}{rcl} d(T^n(x), T^{n+2}(x)) & \leq & a^{n-1}\ d(x, T^2(x)) + \frac{b}{1-a}\ d(x, T(x) \\ & = & a^{n-1}\ d(x, T^2(x)) + d(x, T(x). \end{array}$$

Since $d(x, T^2(x)) \leq d(x, T(x)) + d(T(x), T^2(x)) \leq 2\ d(x, T(x))$, we conclude that

$$d(T^n(x), T^{n+2}(x)) \leq (2\ a^{n-1}\ + 1)\ d(x, T(x),$$

for any $n \geq 1$. Since $0 < a < 1$, there exists $n \geq 1$ such that

$$2\ a^{n-1}\ + 1 \leq \frac{a}{2 - a} + 1 = \frac{2}{2 - a},$$

which implies

$$d(T^n(x), T^{n+2}(x)) \leq \frac{2}{2 - a}\ d(x, T(x)). \qquad \square$$

Our final basic result of this section is the following:

Lemma 3.4. *Let a, b, c be positive numbers such that $0 < a < 1$, $a + b = 1$ and $c < \frac{1}{2}$. Then if we choose $\beta \geq 0$ such that $2c < \beta < 1$, we have*

$$K = \alpha\ a\ \max\left\{\alpha + \frac{2\beta}{2 - a}, c\ \left[2 + \frac{2\beta}{2 - a}\right]\right\} + \beta^2\ a + b < 1,$$

where $\alpha = 1 - \beta$.

Proof. Note that $K < 1$ if and only if

$$\alpha\ \max\left\{\alpha + \frac{2\beta}{2-a}, c\left[2 + \frac{2\beta}{2-a}\right]\right\} + \beta^2\ < 1,$$

where we used $1 - b = a$ and $a > 0$. Since $1 - \beta^2 = \alpha(1+\beta)$ and $\alpha > 0$, we get $K < 1$ if and only if

$$\max\left\{\alpha + \frac{2\beta}{2-a}, c\left[2 + \frac{2\beta}{2-a}\right]\right\} < 1 + \beta.$$

Since $a < 1$, we get $1 < 2 - a$ which implies $\dfrac{2\beta}{2-a} < 2\beta$. Hence

$$\alpha + \frac{2\beta}{2-a} = 1 - \beta + \frac{2\beta}{2-a} < 1 + \beta.$$

Moreover, we have $\beta^2 < 1 < 2 - a$ and since $2c < \beta$, we get

$$2c\left[1 + \frac{\beta}{2-a}\right] < \beta\left[1 + \frac{\beta}{2-a}\right] = \beta + \frac{\beta^2}{2-a} < 1 + \beta.$$

Therefore, we have

$$\max\left\{\alpha + \frac{2\beta}{2-a}, c\left[2 + \frac{2\beta}{2-a}\right]\right\} < 1 + \beta,$$

which completes the proof that $K < 1$. □

In the next section, we discuss the existence of fixed points of G-monotone Gregus-Ćirić mappings defined in weighted graphs.

4. Fixed Points of G-Monotone Gregus-Ćirić-nonexpansive Mappings

As we said earlier, the fixed point results obtained for these type of mappings were done in the context of convex metric spaces. Convexity in metric spaces was initiated by Menger [10] in 1928. The terms "metrically convex" and convex metric space" are due to Blumenthal [4].
Let (M, d) be a metric space and let $\mathbb{R}$ denotes the real line. We say that a mapping $c : \mathbb{R} \to M$ is a metric embedding of $\mathbb{R}$ into M if

$$d(c(s), c(t)) = |s - t|,$$

for all real $s, t \in \mathbb{R}$.

(i) The image $c([a, b]) \subset M$ of a real interval under a metric embedding will be called a metric segment, also known as a geodesic in the literature.
(ii) Let $x, y \in M$. A metric segment $c([a, b])$ is said to join x and y if $c(a) = x$ and $c(b) = y$ and will be denoted by $[x, y]$.
(iii) $C \subset M$ is said to be convex whenever $[x, y] \subset C$ for any $x, y \in C$.

Assume that for any x and y in M, there exists a unique metric segment $[x, y]$. For any $\beta \in [0, 1]$, the unique point $z \in [x, y]$ such that

$$d(x, z) = (1 - \beta)d(x, y), \ \text{ and } \ d(z, y) = \beta d(x, y),$$

will be denoted by $\beta x \oplus (1-\beta) y$. Metric spaces having this property are usually called convex metric spaces or geodesic metric spaces [10, 13]. Moreover, in this section, we assume that

$$d(\beta x \oplus (1-\beta) y, z) \leq \beta d(x,z) + (1-\beta) d(y,z),$$

for any $x, y, z \in M$ and $\beta \in [0,1]$. This property of the metric convex combination was introduced by Takahashi in [13]. Normed vector spaces and hyperbolic metric spaces [12] are a natural example of convex metric spaces which satisfy all the above properties.

Throughout this section, we consider G a transitive weighted digraph with d a metric distance on $V(G)$. We assume that $V(G)$ is a convex metric space such that G-intervals are convex. Recall that a G-interval is any of the subsets

$$[x, \rightarrow) = \{y \in V(G);\ (x,y) \in E(G)\} \text{ or } (\leftarrow, x] = \{y \in V(G);\ (y,x) \in E(G)\}.$$

Now, we are ready to state the main fixed point result of this work.

Theorem 4.1. *Let C be a nonempty G-complete and convex subset of $V(G)$ which satisfies the Property (*). Let $T : C \rightarrow C$ be G-monotone Gregus-Ćirić-contraction mapping, i.e. there exist positive numbers a, b, c such that $0 < a < 1$, $a + b = 1$ and $c \leq \frac{1}{2}$ such that*

$$\begin{aligned} d(T(x), T(y)) \quad \leq \quad & a \ \max\Big\{d(x,y), c\Big[d(x, T(y)) + d(y, T(x))\Big]\Big\} \\ & + b \ \max\Big\{d(x, T(x)), d(y, T(y))\Big\}, \end{aligned}$$

for any $x, y \in C$ with $(x,y) \in E(G)$. Assume that $c < \frac{1}{2}$. Let $x \in C$ be such that $(x, T(x)) \in E(G)$ (or $(T(x), x) \in E(G)$). Then T has a fixed point ω such that $(x, \omega) \in E(G)$ (or $(\omega, x) \in E(G)$). Moreover, if Ω is another fixed point of T such that $(x, \Omega) \in E(G)$ (or $(\Omega, x) \in E(G)$), then we must have $\omega = \Omega$.

Proof. Without loss of generality, we assume that $(x, T(x)) \in E(G)$ and x is not a fixed point of T. In this case, we have $(T^n(x), T^{n+1}(x)) \in E(G)$, for any $n \in \mathbb{N}$. Lemma 3.3 implies the existence of $n \geq 1$ such that

$$d(T^n(x), T^{n+2}(x)) \leq \frac{2}{2-a}\ d(x, T(x)).$$

Let $\beta < 1$ be the number obtained in Lemma 3.4. Set

$$z = \alpha\ T^{n+1}(x) \oplus \beta\ T^{n+2}(x) \in C$$

since C is convex. Using the convexity of the G-intervals, we have $(T^{n+1}(x), z) \in E(G)$ and $(z, T^{n+2}(x)) \in E(G)$. Since T is G-monotone and G is transitive, we conclude that $(z, T(z)) \in E(G)$ and $(T^n(x), z) \in E(G)$. Moreover, we have

$$d(z, T(z)) \leq \alpha\ d(T^{n+1}(x), T(z)) + \beta\ d(T^{n+2}(x), T(z)).$$

Hence

$$\begin{aligned} d(T^{n+1}(x), T(z)) \quad \leq \quad & a \ \max\Big\{d(T^n(x), z), c\ \Big[d(T^{n+1}(x), z) + d(T^n(x), T(z))\Big]\Big\} \\ & + b \ \max\Big\{d(T^n(x), T^{n+1}(x)), d(z, T(z))\Big\}, \end{aligned}$$

and

$$\begin{aligned} d(T^{n+2}(x),T(z)) \leq a\ \max\Big\{d(T^{n+1}(x),z), c\ \Big[d(T^{n+2}(x),z)+d(T^{n+1}(x),T(z))\Big]\Big\} \\ +\ b\ \max\Big\{d(T^{n+1}(x),T^{n+2}(x)), d(z,T(z))\Big\}. \end{aligned}$$

First note that we have

$$\begin{aligned} d(T^n(x),z) &\leq \alpha\ d(T^n(x),T^{n+1}(x))+\beta\ d(T^n(x),T^{n+2}(x)) \\ &\leq \alpha\ d(x,T(x))+\beta\ \tfrac{2}{2-a}\ d(x,T(x)), \end{aligned}$$

and

$$\begin{aligned} d(T^{n+1}(x),z)+d(T^n(x),T(z)) &\leq \beta\ d(T^{n+1}(x),T^{n+2}(x))+d(T^n(x),z)+d(z,T(z)) \\ &\leq \beta\ d(x,T(x))+\alpha\ d(T^n(x),T^{n+1}(x)) \\ &\quad +\ \beta\ d(T^n(x),T^{n+2}(x))+d(z,T(z)), \\ &\leq d(x,T(x))+\beta\ \frac{2}{2-a}\ d(x,T(x))+d(z,T(z)), \end{aligned}$$

which implies

$$\begin{aligned} d(T^{n+1}(x),T(z)) &\leq a\ \max\Big\{\Big[\alpha+\tfrac{2\beta}{2-a}\Big]\ d(x,T(x)), c\ \Big[\Big(1+\tfrac{2\beta}{2-a}\Big)\ d(x,T(x)) \\ &\quad +d(z,T(z))\Big]\Big\}+\ b\ \max\Big\{d(T^n(x),T^{n+1}(x)), d(z,T(z))\Big\} \\ &\leq a\ \max\Big\{\Big[\alpha+\tfrac{2\beta}{2-a}\Big]\ d(x,T(x)), c\ \Big[\Big(1+\tfrac{2\beta}{2-a}\Big)\ d(x,T(x)) \\ &\quad +d(z,T(z))\Big]\Big\}+\ b\ \max\Big\{d(x,T(x)), d(z,T(z))\Big\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} d(T^{n+2}(x),T(z)) &\leq a\ \max\Big\{d(T^{n+1}(x),z), c\ \Big[d(T^{n+2}(x),z)+d(T^{n+1}(x),T(z))\Big]\Big\} \\ &\quad +\ b\ \max\Big\{d(T^{n+1}(x),T^{n+2}(x)), d(z,T(z))\Big\} \\ &\leq a\ \max\Big\{\beta\ d(T^{n+1}(x),T^{n+2}(x)), c\ \Big[\alpha\ d(T^{n+2}(x),T^{n+1}(x)) \\ &\quad +\ d(T^{n+1}(x),z)+d(z,T(z))\Big]\Big\}+b\ \max\Big\{d(x,T(x)), d(z,T(z))\Big\} \\ &\leq a\ \max\Big\{\beta\ d(x,T(x)), c\ \Big[\alpha\ d(x,T(x))+\beta\ d(T^{n+1}(x),T^{n+2}(x)) \\ &\quad +d(z,T(z))\Big]\Big\}+b\ \max\Big\{d(x,T(x)), d(z,T(z))\Big\} \\ &\leq a\ \max\Big\{\beta\ d(x,T(x)), c\ \Big[d(x,T(x))+d(z,T(z))\Big]\Big\} \\ &\quad +b\ \max\Big\{d(x,T(x)), d(z,T(z))\Big\}. \end{aligned}$$

Since

$$d(z,T(z)) \leq \alpha\ d(T^{n+1}(x),T(z))+\beta\ d(T^{n+2}(x),T(z)),$$

we get

$$\begin{aligned} d(z,T(z)) \leq \alpha\ a\ \max\Big\{\Big[\alpha+\tfrac{2\beta}{2-a}\Big]\ d(x,T(x)), c\ \Big[\Big(1+\tfrac{2\beta}{2-a}\Big)\ d(x,T(x))+d(z,T(z))\Big]\Big\} \\ +\ \beta\ a\ \max\Big\{\beta\ d(x,T(x)), c\ \Big[d(x,T(x))+d(z,T(z))\Big]\Big\} \\ +\ b\ \max\Big\{d(x,T(x)), d(z,T(z))\Big\}. \end{aligned}$$

Assume that $d(x,T(x)) < d(z,T(z))$. Then, we must have

$$\begin{aligned} d(z,T(z)) &< \alpha\ a\ \max\Big\{\Big[\alpha+\tfrac{2\beta}{2-a}\Big], c\ \Big[\Big(1+\tfrac{2\beta}{2-a}\Big)+1\Big]\Big\}d(z,T(z) \\ &\quad +\beta\ a\ \max\{\beta, 2\ c\}\ d(z,T(z))+\ b\ d(z,T(z)). \end{aligned}$$

Since $2c < \beta$, we get

$$d(z,T(z)) < \Big[\alpha\ a\ \max\Big\{\alpha+\frac{2\beta}{2-a}, c\ \Big(2+\frac{2\beta}{2-a}\Big)\Big\}+\ \beta^2\ a+\ b\Big]\ d(z,T(z)).$$

Using Lemma 3.4, we know that

$$K = \alpha\ a\ \max\Big\{\alpha+\frac{2\beta}{2-a}, c\ \Big(2+\frac{2\beta}{2-a}\Big)\Big\}+\ \beta^2\ a+\ b < 1,$$

which implies $d(z,T(z)) < K\ d(z,T(z))$ an obvious contradiction. Therefore, we must have $d(z,T(z)) \leq d(x,T(x))$. Hence

$$d(z,T(z)) \leq \Big[\alpha\ a\ \max\Big\{\alpha+\frac{2\beta}{2-a}, c\ \Big(2+\frac{2\beta}{2-a}\Big)\Big\}+\ \beta^2\ a+\ b\Big]\ d(x,T(x)),$$

i.e. $d(T(z),z) \leq K\ d(x,T(x))$. By induction, we will construct a sequence $\{z_n\}$ in C such that

(i) $z_0 = x$ and z_1 is the point constructed before;
(ii) $(z_n, z_{n+1}) \in E(G)$, for any $n \in \mathbb{N}$;
(iii) $d(z_{n+1}, T(z_{n+1})) \leq K\ d(z_n, T(z_n))$, for any $n \in \mathbb{N}$.

In particular, we have $d(z_{n+1}, T(z_{n+1})) \leq K^n\ d(x,T(x))$, for any $n \in \mathbb{N}$. Since G is transitive, then $(z_n, z_m) \in E(G)$ for any $n \leq m$. Using Lemma 3.1, we get

$$d(z_n, z_m) \leq \frac{2-a}{1-a}\ \Big(d(z_n, T(z_n)) + d(z_m, T(z_m))\Big).$$

Since $K < 1$, we conclude that $\{z_n\}$ is Cauchy and G-increasing. Hence it is convergent some point $\omega \in C$ because C is G-complete. Since C satisfies the Property (*), we conclude that $(z_n, \omega) \in E(G)$, for any $n \in \mathbb{N}$. In particular, we have $(x, \omega) \in E(G)$. Next, we prove that ω is a fixed point of T. Since $(z_n, \omega) \in E(G)$, for any $n \in \mathbb{N}$, we get

$$\begin{aligned} d(T(z_n), T(\omega)) &\leq a \max\Big\{d(z_n,\omega), c\ \Big[d(z_n, T(\omega)) + d(T(z_n),\omega)\Big]\Big\} \\ &\quad +\ b\ \max\Big\{d(z_n, T(z_n)), d(\omega, T(\omega))\Big\}. \end{aligned}$$

Since $\lim\limits_{n\to+\infty} d(z_n, T(z_n)) = \lim\limits_{n\to+\infty} d(z_n,\omega) = 0$, we get $\lim\limits_{n\to+\infty} d(T(z_n),\omega) = 0$, which implies

$$d(\omega, T(\omega)) \leq a \max\Big\{0, c\ \Big[d(\omega,T(\omega))+0\Big]\Big\}+\ b\ \max\Big\{0, d(\omega, T(\omega))\Big\},$$

i.e. $d(\omega, T(\omega)) \leq a\ c\ d(\omega, T(\omega)) +\ b\ d(\omega, T(\omega))$. Since $ac+b < a+b = 1$, we conclude that $d(\omega, T(\omega)) = 0$, i.e. $T(\omega) = \omega$. Finally, let Ω be another fixed point of T such that $(x, \Omega) \in E(G)$. Since T is G-monotone, we get $(T^n(x), \Omega) \in E(G)$. Using the convexity of the G-intervals, we get $(z_n, \Omega) \in E(G)$ for any $n \in \mathbb{N}$. Using Lemma 3.1, we get

$$d(z_n, \Omega) \leq \frac{2-a}{1-a} \Big(d(z_n, T(z_n)) + d(\Omega, T(\Omega)) \Big) = \frac{2-a}{1-a}\ d(z_n, T(z_n)),$$

for any $n \in \mathbb{N}$. If we let $n \to +\infty$, we conclude that $\{z_n\}$ converges to Ω. the uniqueness of the limit implies that $\omega = \Omega$. □

Remark 4.1. *If we assume that $a + b < 1$ and $c \leq \frac{1}{2}$, then the map T is a quasi-contraction mapping* [5]. *In this case, Theorem 4.1 is similar to the main fixed point result found in* [1, 2] *without any convexity assumption on the weighted graph.*

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