RELATION-THEORETIC METRICAL FIXED POINT THEOREMS UNDER NONLINEAR CONTRACTIONS

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Abstract. We establish fixed point theorems for nonlinear contractions on a metric space (not essentially complete) endowed with an arbitrary binary relation. Our results extend, generalize, modify and unify several known results especially those contained in Samet and Turinici [Commun. Math. Anal. 13, 82-97 (2012)] and Alam and Imdad [J. Fixed Point Theory Appl. 17(4), 693-702 (2015)]. Interestingly a corollary to one of our main results proved under symmetric closure of any binary relation remains a sharpened version of a theorem due to Samet and Turinici. Finally, we use examples to highlight the realized improvements in the results proved in this paper.
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1. Introduction and preliminaries

Banach contraction principle [6] is one of the most fruitful and applicable theorems in classical functional analysis. In the last nine decades, this principle has been generalized and improved by numerous researchers in the different directions viz:

- enlarging the class of underlying spaces,
- replacing contraction condition with relatively weaker contractive condition,
- weakening the involved metrical notions,

and such practice is still in business.

In 1986, the idea of order-theoretic fixed point results can be traced back to Turinici [19, 20]. In 2004, unknowingly, Ran and Reurings [17] rediscovered a slightly more natural order-theoretic version of Banach contraction principle. Recently, several authors utilized different types of binary relation viz: partial order (see Nieto and Rodríguez-López [16]), tolerance (see Turinici [22, 23]), strict order (see Ghods et al. [10]), transitive (see Ben-El-Mechaiekh [7]), preorder (see Turinici [21]) *etc* to prove their respective fixed point results. Samet and Turinici [18] established a fixed point theorem for nonlinear contraction by using a symmetric closure of any binary relation. Most recently, Alam and Imdad [3, 4] established a relation-theoretic version of Banach contraction principle employing amorphous relation which in turn generalizes

several well known relevant order-theoretic fixed point theorems. For more details, one can consult ([1, 2, 3, 4, 7, 10, 11, 16, 17, 18, 19, 20, 21, 22, 23] and references cited therein).

In the sequel, the following definitions will be utilized.

Definition 1.1 [14]. A binary relation on a non-empty set X is defined as a subset of $X \times X$ which will be denoted by $\mathcal{R}$. We say that "x is $\mathcal{R}$-related to y" if and only if $(x, y) \in \mathcal{R}$.

In what follows, $\mathcal{R}$, $\mathbb{N}$ and $\mathbb{N}_0$ respectively, stand for a non-empty binary relation, the set of natural numbers and the set of whole numbers.

Definition 1.2 [15]. A binary relation $\mathcal{R}$ defined on X is called complete if for all $x, y \in X$, either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$ which is denoted as $[x, y] \in \mathcal{R}$.

Definition 1.3 [14]. Let $\mathcal{R}$ be a binary relation defined on a non-empty set X. Then the symmetric closure of $\mathcal{R}$ is defined as the smallest symmetric relation containing $\mathcal{R}$ (*i.e.*, $\mathcal{S} := \mathcal{R} \cup \mathcal{R}^{-1}$). Often, it is denoted by $\mathcal{S}$ or $\mathcal{R}^s$.

Definition 1.4 [3]. Let f be a self-mapping defined on a non-empty set X. Then a binary relation $\mathcal{R}$ defined on X is called f-closed if

$$(x, y) \in \mathcal{R} \Rightarrow (fx, fy) \in \mathcal{R}, \text{ for all } x, y \in X.$$

Definition 1.5 [3]. Let $\mathcal{R}$ be a binary relation defined on a non-empty set X. Then a sequence $\{x_n\}$ in X is called $\mathcal{R}$-preserving if

$$(x_n, x_{n+1}) \in \mathcal{R}, \text{ for all } n \in \mathbb{N}.$$

Definition 1.6 [4]. Let (X, d) be a metric space and $\mathcal{R}$ a binary relation defined on X. We say that (X, d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges to a point in X.

Remark 1.7 [4]. Every complete metric space is $\mathcal{R}$-complete, where $\mathcal{R}$ denotes a binary relation. Particularly, if $\mathcal{R}$ is universal relation, then notions of completeness and $\mathcal{R}$-completeness coincide.

Definition 1.8 [4]. Let (X, d) be a metric space and $\mathcal{R}$ a binary relation defined on X. Then a mapping $f : X \to X$ is called $\mathcal{R}$-continuous at $x \in X$ if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, we have $f(x_n) \xrightarrow{d} f(x)$. Moreover, f is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at every point of X.

Remark 1.9 [4]. Every continuous mapping is $\mathcal{R}$-continuous, where $\mathcal{R}$ denotes a binary relation. Particularly, if $\mathcal{R}$ is universal relation, then notions of continuity and $\mathcal{R}$-continuity coincide.

Definition 1.10 [18]. Let (X, d) be a metric space and $\mathcal{S}$ be the symmetric closure of a binary relation $\mathcal{R}$ defined on X. We say that $(X, d, \mathcal{S})$ is regular if for any sequence $\{x_n\}$ with $(x_n, x_{n+1}) \in \mathcal{S}$ (for all $n \in \mathbb{N}$) and $x_n \xrightarrow{d} x$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in \mathcal{S}$, for all $k \in \mathbb{N}$.

Definition 1.11 [3]. Let (X, d) be a metric space and $\mathcal{R}$ a binary relation defined on X. Then $\mathcal{R}$ is called d-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x] \in \mathcal{R}$, for all $k \in \mathbb{N}$. Notice that, $(X, d, \mathcal{S})$ is regular if and only if $\mathcal{S}$ is d-self-closed.

Definition 1.12 [18]. Let (X, d) be a metric space and $\mathcal{R}$ a binary relation defined on X. Then a subset D of X is called $\mathcal{R}$-directed if for every pair of points $x, y \in D$, there is $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

Definition 1.13 [13]. Let (X, d) be a metric space, $\mathcal{R}$ a binary relation defined on X and x, y a pair of points in X. Then a finite sequence $\{z_0, z_1, z_2, ..., z_l\}$ in X is said to be a path of length l (where $l \in \mathbb{N}$) joining x to y in $\mathcal{R}$ if $z_0 = x, z_l = y$ and $(z_i, z_{i+1}) \in \mathcal{R}$ for each $i \in \{1, 2, 3, \cdots, l-1\}$. Observe that, a path of length l involves $(l+1)$ elements of X that need not be distinct in general.

Given a metric space (X, d), a self-mapping f on X and a binary relation $\mathcal{R}$ on X, we employ the following notations:

- $F(f)$: the set of all fixed points of f;
- $X(f, \mathcal{R})$: the collection of all points $x \in X$ such that $(x, fx) \in \mathcal{R}$;
- $\Upsilon(x, y, \mathcal{R})$: the family of all paths joining x to y in $\mathcal{R}$;
- $\mathcal{M}_f(x, y) := \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)] \right\}$; and
- $\mathcal{N}_f(x, y) := \max \left\{ d(x, y), \frac{1}{2}[d(x, fx) + d(y, fy)], \frac{1}{2}[d(x, fy) + d(y, fx)] \right\}$.

Remark 1.14. Observe that, for all $x, y \in X$, $\mathcal{N}_f(x, y) \leq \mathcal{M}_f(x, y)$.

Let Φ be the family of all mappings $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following properties

(Φ_1): φ is increasing;

(Φ_2): $\displaystyle\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for each $t > 0$, where φ^n is the n-th iterate of φ.

Recall that, the necessary condition of any real convergent series $\sum_n a_n$ is that

$$\lim_{n \to \infty} a_n = 0.$$

The following two lemmas are required in our subsequent discussion.

Lemma 1.15 [18]. *Let $\varphi \in \Phi$. Then for all $t > 0$, we have $\varphi(t) < t$.*

Lemma 1.16. *Let (X, d) be a metric space and f a self-mapping on X. Then for each $x \in X$,*
$$\mathcal{M}_f(x, fx) \leq \max \left\{ d(x, fx), d(fx, f^2x) \right\}.$$

Proof. Let x be an arbitrary element of X. Then

$$
\begin{aligned}
\mathcal{M}_f(x, fx) &= \max \left\{ d(x, fx), d(x, fx), d(fx, f^2x), \frac{1}{2}[d(x, f^2x) + d(fx, fx)] \right\} \\
&\leq \max \left\{ d(x, fx), d(fx, f^2x), \frac{1}{2}[d(x, fx) + d(fx, f^2x)] \right\} \\
&\leq \max \left\{ d(x, fx), d(fx, f^2x), \max\{d(x, fx), d(fx, f^2x)\} \right\} \\
&= \max \left\{ d(x, fx), d(fx, f^2x) \right\}.
\end{aligned}
$$

$\square$

For the sake of completeness, we record the following known relevant results:

Theorem 1.17 (Theorem 2.1, Samet and Turinici [18]). *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation defined on X with a symmetric closure $\mathcal{S} := \mathcal{R} \cup \mathcal{R}^{-1}$ and f a self-mapping on X. Assume that the following conditions hold:*

(i) (X, d) is complete;

(ii) there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{S}$;
(iii) $\mathcal{S}$ is f-closed;
(iv) $(X, d, \mathcal{S})$ is regular;
(v) there exists $\varphi \in \Phi$ such that

$$d(fx, fy) \leq \varphi(\mathcal{N}_f(x, y)),$$

for all $x, y \in X$ with $(x, y) \in \mathcal{S}$.
Then f has a fixed point. Moreover, if in addition, $F(f)$ is $\mathcal{S}$-directed, then f has a unique fixed point.

Theorem 1.18 (Alam and Imdad [3, 4]). *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and f a self-mapping on X. Suppose that the following conditions hold:*
(i) there exists $Y \subseteq X, fX \subseteq Y \subseteq X$ such that (Y, d) is $\mathcal{R}$-complete;
(ii) $X(f, \mathcal{R})$ is non-empty;
(iii) $\mathcal{R}$ is f-closed;
(iv) either f is $\mathcal{R}$-continuous or $\mathcal{R}|_Y$ is d-self-closed;
(v) there exists $\alpha \in [0, 1)$ such that
$$d(fx, fy) \leq \alpha d(x, y) \text{ for all } x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$
Then f has a fixed point. Moreover, if
(vi) $\Upsilon(x, y, \mathcal{R}^s)$ is non-empty, for each x, y in X,
then f has a unique fixed point.

Proposition 1.19 [3]. *If $\mathcal{R}$ is a binary relation defined on a non-empty set X, then (for all x, y in X)*

$$(x, y) \in \mathcal{R}^s \Longleftrightarrow [x, y] \in \mathcal{R}.$$

Proposition 1.20. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ defined on X, f a self-mapping on X and $\varphi \in \Phi$. Then the following conditions are equivalent:*

(A): $d(fx, fy) \leq \varphi(\mathcal{M}_f(x, y))$ *with* $(x, y) \in \mathcal{R}$,
(B): $d(fx, fy) \leq \varphi(\mathcal{M}_f(x, y))$ *with* $[x, y] \in \mathcal{R}$.

Proof. The implication $(B) \Rightarrow (A)$ is straightforward.
To show that $(A) \Rightarrow (B)$, choose $x, y \in X$ such that $[x, y] \in \mathcal{R}$. If $(x, y) \in \mathcal{R}$, then (B) immediately follows from (A). Otherwise, if $(y, x) \in \mathcal{R}$, then by (A) and the symmetry of the metric d, we have

$$d(fx, fy) = d(fy, fx) \leq \varphi\Big(\max\Big\{d(y, x), d(y, fy), d(x, fx), \frac{1}{2}[d(y, fx) + d(x, fy)]\Big\}\Big)$$
$$= \varphi\Big(\max\Big\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\Big\}\Big)$$
$$= \varphi(\mathcal{M}_f(x, y)).$$

Hence, $(A) \Rightarrow (B)$. $\qquad\square$

The main results of this paper are based on the following motivations and observations:

(1) the main result of Samet and Turinici [18] is improved by replacing the symmetric closure $\mathcal{S}$ of any binary relation with amorphous binary relation $\mathcal{R}$,

(2) the completeness of the whole space X involved in Theorem 1.17 (due to Samet and Turinici [18]) is replaced by relatively weaker notion namely: $\mathcal{R}$-completeness of any subspace $Y \subseteq X$, such that $fX \subseteq Y \subseteq X$. Observe that the completeness of the whole space is not needed,

(3) the regularity of X involved in Theorem 1.17, is replaced by d-self-closedness of $\mathcal{R}$,

(4) the contraction conditions involved in Theorems 1.17 and 1.18 are replaced by relatively weaker nonlinear generalized Ćirić contraction,

(5) a corollary to our main result deduced for the symmetric closure $\mathcal{S}$ of any binary relation remains a sharper version of Theorem 1.17,

(6) some examples are furnished to highlight the realized improvement in the results of this paper.

2. Main results

Now, we are equipped to prove our main result as follows:

Theorem 2.1. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ defined on X and f a self-mapping on X. Assume that the following conditions hold:*

(i) there exists $Y \subseteq X, fX \subseteq Y \subseteq X$ such that (Y, d) is $\mathcal{R}$-complete;

(ii) $X(f, \mathcal{R})$ is non-empty;

(iii) $\mathcal{R}$ is f-closed;

(iv) either f is $\mathcal{R}$-continuous or $\mathcal{R}|_Y$ is d-self-closed;

(v) there exists $\varphi \in \Phi$ such that

$$d(fx, fy) \leq \varphi(\mathcal{M}_f(x, y)),$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

Then f has a fixed point.

Proof. Since $X(f, \mathcal{R}) \neq \emptyset$. Let $x_0 \in X(f, \mathcal{R})$. Construct a Picard sequence $\{x_n\}$, with the initial point x_0, *i.e.*,

$$x_{n+1} = f(x_n), \quad \text{for all } n \in \mathbb{N}_0. \tag{2.1}$$

Since $(x_0, fx_0) \in \mathcal{R}$ and $\mathcal{R}$ is f-closed, we have

$$(fx_0, f^2x_0), (f^2x_0, f^3x_0), \cdots, (f^nx_0, f^{n+1}x_0), \cdots \in \mathcal{R}.$$

Thus,

$$(x_n, x_{n+1}) \in \mathcal{R}, \quad \text{for all } n \in \mathbb{N}_0,$$

so that the sequence $\{x_n\}$ is $\mathcal{R}$-preserving. From condition (v), we have (for all $n \in \mathbb{N}$)

$$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \leq \varphi(\mathcal{M}_f(x_{n-1}, x_n)) \tag{2.2}$$

On using Lemma 1.16, we have (for all $n \in \mathbb{N}$)

$$\mathcal{M}_f(x_{n-1}, x_n) \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \tag{2.3}$$

On using (2.2), (2.3) and the property (Φ_1), we obtain (for all $n \in \mathbb{N}$)

$$d(x_n, x_{n+1}) \leq \varphi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \tag{2.4}$$

Now, we show that the sequence $\{x_n\}$ is Cauchy in (X, d). In case $x_r = x_{r+1}$ for some $r \in \mathbb{N}_0$, then the result is immediate. Otherwise, $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Suppose

that $d(x_{s-1}, x_s) \leq d(x_s, x_{s+1})$, for some $s \in \mathbb{N}$. On using (2.4) and Lemma 1.15, we get

$$d(x_s, x_{s+1}) \leq \varphi(d(x_s, x_{s+1})) < d(x_s, x_{s+1}),$$

which is a contradiction. Thus $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ (for all $n \in \mathbb{N}$), so that

$$d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n)), \text{ for all } n \in \mathbb{N}.$$

By induction on n and the property (Φ_1), we get

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)), \text{ for all } n \in \mathbb{N}_0.$$

Now, for all $m, n \in \mathbb{N}_0$ with $m \geq n$, we have

$$
\begin{aligned}
d(x_n, x_m) \quad &\leq \quad d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&\leq \quad \varphi^n(d(x_0, x_1)) + \varphi^{n+1}(d(x_0, x_1)) + \cdots + \varphi^{m-1}(d(x_0, x_1)) \\
&= \quad \sum_{k=n}^{m-1} \varphi^k(d(x_0, x_1)) \\
&\leq \quad \sum_{k \geq n} \varphi^k(d(x_0, x_1)) \\
&\rightarrow \quad 0 \text{ as } n \rightarrow \infty,
\end{aligned}
$$

which amounts to saying that the sequence $\{x_n\}$ is Cauchy in X. Hence, $\{x_n\}$ is $\mathcal{R}$-preserving Cauchy sequence in X. Since, $\{x_n\} \subseteq fX \subseteq Y$ (due to (2.1) and (i)), therefore $\{x_n\}$ is $\mathcal{R}$-preserving Cauchy sequence in Y. As (Y, d) is $\mathcal{R}$-complete, there exists $p \in Y$ such that $x_n \xrightarrow{d} p$.

If f is $\mathcal{R}$-continuous, then

$$p = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} fx_n = f \lim_{n \to \infty} x_n = fp.$$

Hence p is the fixed point of f.

Alternatively, if $\mathcal{R}|_Y$ is d-self-closed, then for any $\mathcal{R}$-preserving sequence $\{x_n\}$ in Y with $x_n \xrightarrow{d} p$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, p] \in \mathcal{R}|_Y \subseteq \mathcal{R}$, for all $k \in \mathbb{N}_0$.

Write $\delta := d(fp, p) \geq 0$. Suppose on contrary that $\delta > 0$. On using condition (v), Proposition 1.20 and $[x_{n_k}, p] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$, we get

$$d(x_{n_k+1}, fp) = d(fx_{n_k}, fp) \leq \varphi(\mathcal{M}_f(x_{n_k}, p)), \qquad (2.5)$$

where

$$\mathcal{M}_f(x_{n_k}, p) = \max \left\{ d(x_{n_k}, p), d(x_{n_k}, x_{n_k+1}), d(p, fp), \frac{1}{2}[d(x_{n_k}, fp) + d(p, x_{n_k+1})] \right\}.$$

If $\mathcal{M}_f(x_{n_k}, p) = d(p, fp) = \delta$, then (2.5) reduces to

$$d(x_{n_k+1}, fp) \leq \varphi(\delta),$$

which on making $k \to \infty$, gives arise

$$\delta \leq \varphi(\delta),$$

which is a contradiction. Otherwise, if

$$\mathcal{M}_f(x_{n_k}, p) = \max\left\{d(x_{n_k}, p), d(x_{n_k}, x_{n_k+1}), \frac{1}{2}[d(x_{n_k}, fp) + d(p, x_{n_k+1})]\right\},$$

then due to the fact that $x_n \xrightarrow{d} p$, there exists $h = h(\delta)$ such that

$$\mathcal{M}_f(x_{n_k}, p) \leq \frac{2}{3}\delta, \text{ for all } k \geq h.$$

As φ is increasing, we have

$$\varphi(\mathcal{M}_f(x_{n_k}, p)) \leq \varphi(\frac{2}{3}\delta), \text{ for all } k \geq h. \tag{2.6}$$

On using (2.5) and (2.6), we get

$$d(x_{n_k+1}, fp) = d(fx_{n_k}, fp) \leq \varphi(\frac{2}{3}\delta), \text{ for all } k \geq h.$$

Letting $k \to \infty$ and using Lemma 1.15, we get

$$\delta \leq \varphi(\frac{2}{3}\delta) < \frac{2}{3}\delta < \delta,$$

which is again a contradiction. Hence, $\delta = 0$, so that

$$d(fp, p) = \delta = 0 \Rightarrow fp = p,$$

which concludes the proof. □

In particular, on setting $Y = X$ in Theorem 2.1, we deduce a corollary which is an improved version of Theorem 1.17 (up to fixed point) due to the involvement of relatively weaker notions in the considerations of completeness, regularity and contraction condition:

Corollary 2.2. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ defined on X and f a self-mapping on X. Assume that the following conditions hold:*
  (i) (X, d) is $\mathcal{R}$-complete;
  (ii) $X(f, \mathcal{R})$ is non-empty;
  (iii) $\mathcal{R}$ is f-closed;
  (iv) either f is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed;
  (v) there exists $\varphi \in \Phi$ such that

$$d(fx, fy) \leq \varphi(\mathcal{M}_f(x, y)),$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.
Then f has a fixed point.

In view of Remarks 1.7 and 1.9, we deduce the following relatively more natural consequence of Theorem 2.1.

Corollary 2.3. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ defined on X and f a self-mapping on X. Assume that the following conditions hold:*
  (i) there exists $Y \subseteq X, fX \subseteq Y \subseteq X$ such that (Y, d) is complete;
  (ii) $X(f, \mathcal{R})$ is non-empty;
  (iii) $\mathcal{R}$ is f-closed;

(iv) *either f is continuous or $\mathcal{R}|_Y$ is d-self-closed;*
(v) *there exists $\varphi \in \Phi$ such that*

$$d(fx, fy) \leq \varphi(\mathcal{M}_f(x, y)),$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.
Then f has a fixed point.

In view of Remark 1.14, the following consequence of Theorem 2.1 is predictable.

Corollary 2.4. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ defined on X and f a self-mapping on X. Assume that the following conditions hold:*

(i) *there exists $Y \subseteq X, fX \subseteq Y \subseteq X$ such that (Y, d) is $\mathcal{R}$-complete;*
(ii) *$X(f, \mathcal{R})$ is non-empty;*
(iii) *$\mathcal{R}$ is f-closed;*
(iv) *either f is $\mathcal{R}$-continuous or $\mathcal{R}|_Y$ is d-self-closed;*
(v) *there exists $\varphi \in \Phi$ such that*

$$d(fx, fy) \leq \varphi\Big(\max\Big\{d(x, y), \frac{1}{2}[d(x, fx) + d(y, fy)], \frac{1}{2}[d(x, fy) + d(y, fx)]\Big\}\Big),$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.
Then f has a fixed point.

Now, we prove the following results ensuring the uniqueness of the fixed point (corresponding to Corollary 2.4):

Theorem 2.5. *In addition to the hypotheses of Corollary 2.4, suppose that the following condition holds:*

$$(vi): \quad fX \text{ is } \mathcal{R}^s\text{-directed.}$$

Then f has a unique fixed point.

Proof. By Corollary 2.4, $F(f)$ is non-empty. If $F(f)$ is singleton, then there is nothing to prove. Otherwise, to accomplish the proof, take two arbitrary elements p, q in $F(f)$, so that

$$fp = p \text{ and } fq = q.$$

Now, we are required to show that $p = q$. Since $F(f) \subseteq fX$ and fX is $\mathcal{R}^s$-directed, therefore there exists $z \in X$ such that $[p, z] \in \mathcal{R}$ and $[q, z] \in \mathcal{R}$. Now, we construct a Picard sequence $\{z_n\}$ corresponding to $z_0 = z$, so that $z_n = f^n z_0$ for all $n \in \mathbb{N}_0$.

Our claim is that $\lim_{n \to \infty} d(p, z_n) = 0$. If $d(p, z_s) = 0$, for some $s \in \mathbb{N}_0$, then the result is immediate. Otherwise, suppose $d(p, z_n) > 0$, for all $n \in \mathbb{N}_0$. Since $[p, z_n] \in \mathcal{R}$, for all $n \in \mathbb{N}_0$ (due to the fact that $\mathcal{R}$ is f-closed and $[p, z] \in \mathcal{R}$), therefore on using Proposition 1.20 and hypothesis (v), we have

$$d(p, z_{n+1}) = d(fp, fz_n) \leq \varphi(\mathcal{N}_f(p, z_n)), \tag{2.7}$$

where,

$$
\begin{aligned}
\mathcal{N}_f(p, z_n) &= \max\Big\{d(p, z_n), \frac{1}{2}[d(p, p) + d(z_n, z_{n+1})], \frac{1}{2}[d(p, z_n) + d(p, z_{n+1})]\Big\} \\
&\leq \max\Big\{d(p, z_n), \frac{1}{2}[d(p, z_n) + d(p, z_{n+1})]\Big\} \\
&\leq \max\Big\{d(p, z_n), d(p, z_{n+1})\Big\}.
\end{aligned}
$$

Using the property (Φ_1) and (2.7) (for all $n \in \mathbb{N}_0$), we get

$$
\begin{aligned}
d(p, z_{n+1}) &\leq \varphi\big(\max \{d(p, z_n), d(p, z_{n+1})\}\big) \\
&= \varphi\big(d(p, z_n)\big),
\end{aligned}
$$

otherwise, the fact that $\varphi\big(d(p, z_{n+1})\big) < d(p, z_{n+1})$ is contradicted. So, by induction on n and increasingness of φ, we get

$$
d(p, z_n) \leq \varphi^n(d(p, z_0)), \text{ for all } n \in \mathbb{N}_0,
$$

which on making $n \to \infty$ and using the property (Φ_2), we get

$$
\lim_{n \to \infty} d(p, z_n) = 0. \tag{2.8}
$$

Similarly, we can prove that

$$
\lim_{n \to \infty} d(q, z_n) = 0. \tag{2.9}
$$

Using (2.8) and (2.9), we have

$$
\begin{aligned}
d(p, q) &\leq d(p, z_n) + d(z_n, q) \\
&\to 0, \text{ as } n \to \infty \\
\Rightarrow p &= q.
\end{aligned}
$$

Thus, f has a unique fixed point. $\qquad\square$

Remark 2.6. In Theorem 2.5, we have used a relatively more natural condition "fX is $\mathcal{R}^s$-directed" instead of "$F(f)$ is $\mathcal{R}^s$-directed" which is too restrictive. Our proof continue to hold even if we take "$F(f)$ is $\mathcal{R}^s$-directed". Sometimes it is difficult to find $F(f)$.

Theorem 2.7. *Theorem 2.5 remains true, if we replace the condition (vi) by the following condition:*

$$
(vi)' : \mathcal{R}|_{fX} \text{ is complete.}
$$

Proof. From Corollary 2.4, $F(f) \neq \emptyset$. If $F(f)$ is singleton, then there is nothing to prove. Otherwise, take two arbitrary but distinct elements p, q in $F(f)$, so that

$$
fp = p \text{ and } fq = q.
$$

Since $\mathcal{R}|_{fX}$ is complete, therefore $[p, q] \in \mathcal{R}$. Using Proposition 1.20 and condition (v), we get

$$
\begin{aligned}
d(p, q) = d(fp, fq) &\leq \varphi\big(\max \{d(p, q), \tfrac{1}{2}[d(p, fp) + d(q, fq)], \tfrac{1}{2}[d(p, fq) + d(q, fp)]\}\big) \\
&= \varphi\big(\max \{d(p, q), \tfrac{1}{2}[d(p, p) + d(q, q)], \tfrac{1}{2}[d(p, q) + d(q, p)]\}\big) \\
&= \varphi\big(d(p, q)\big).
\end{aligned}
$$

which contradicts to the fact that $\varphi\big(d(p, q)\big) < d(p, q)$ (in view of Lemma 1.15). Hence $p = q$. Thus f has a unique fixed point. $\qquad\square$

In view of Remark 2.6, on choosing $\mathcal{R}$ to be the symmetric closure $\mathcal{S}$ of any arbitrary binary relation in Theorem 2.5, we obtain the following sharpened version of Theorem 1.17.

Corollary 2.8. *Let (X, d) be a metric space equipped with the symmetric closure $\mathcal{S}$ of any arbitrary binary relation defined on X and f a self-mapping on X. Assume that the following conditions hold:*

   (i) there exists $Y \subseteq X, fX \subseteq Y \subseteq X$ such that (Y, d) is $\mathcal{S}$-complete;
   (ii) $X(f, \mathcal{S})$ is non-empty;
   (iii) $\mathcal{S}$ is f-closed;
   (iv) either f is $\mathcal{S}$-continuous or $\mathcal{S}|_Y$ is d-self-closed;
   (v) there exists $\varphi \in \Phi$ such that

$$d(fx, fy) \leq \varphi\big(\max\big\{ d(x,y), \tfrac{1}{2}[d(x,fx) + d(y,fy)], \tfrac{1}{2}[d(x,fy) + d(y,fx)] \big\} \big),$$

for all $x, y \in X$ with $(x, y) \in \mathcal{S}$.
Then f has a fixed point. Moreover, if in addition, $F(f)$ is $\mathcal{S}$-directed, then f has a unique fixed point.

Notice that the hypothesis '$\mathcal{S}$ is f-closed' is equivalent to 'the comparative property of f' and '$\mathcal{S}|_Y$ is d-self-closed' is equivalent to 'the regular property of $(Y, d, \mathcal{S})$'.

Let Λ be the collection of all increasing continuous mappings $\psi : [0, \infty) \to [0, \infty)$ such that

   (Λ_1): For all $t > 0$, $0 < \psi(t) < t$;
   (Λ_2): $g(t) = \frac{t}{t - \psi(t)}$ is strictly decreasing function on $(0, \infty)$;
   (Λ_3): $\int_0^T g(t)dt < \infty$, for all $T > 0$.

Lemma 2.9 [5]. *We have $\Lambda \subset \Phi$.*

In view of Lemma 2.9, we obtain the following consequence of Theorem 2.5:

Corollary 2.10. *The conclusion of Theorem 2.5 remains true if we replace the condition (v) by the following (besides retaining rest of the hypotheses):*

$$(v)' \text{ there exists } \psi \in \Lambda \text{ such that } d(fx, fy) \leq \psi\big(\mathcal{N}_f(x, y)\big),$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

Notice that, Corollary 2.10 is a sharpened version of Corollary 2.7 of Samet and Turinici [18].

3. Consequences

As consequences of our earlier established results, we derive several well known results of the existing literature.

3.1. Relation-theoretic fixed point results.
On setting $\varphi(t) = kt$, with $k \in [0, 1)$, we derive the following corollaries which are immediate consequences of Theorem 2.5.

Corollary 3.1. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ defined on X and f a self-mapping on X. Assume that the following conditions hold:*

   (i) there exists $Y \subseteq X, fX \subseteq Y \subseteq X$ such that (Y, d) is $\mathcal{R}$-complete;
   (ii) $X(f, \mathcal{R})$ is non-empty;
   (iii) $\mathcal{R}$ is f-closed;
   (iv) either f is $\mathcal{R}$-continuous or $\mathcal{R}|_Y$ is d-self-closed;

(v) there exists $k \in [0, 1)$ such that

$$d(fx, fy) \le k\big(\max \big\{ d(x, y), \frac{1}{2}[d(x, fx) + d(y, fy)], \frac{1}{2}[d(x, fy) + d(y, fx)] \big\} \big),$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

Then f has a fixed point. Moreover, if in addition, fX is $\mathcal{R}^s$-directed, then f has a unique fixed point.

Remark 3.2. Corollary 3.1 is a sharpened version of Corollary 2 (corresponding to condition (11)) due to Ahmadullah et al. [1].

Corollary 3.3. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ defined on X and f a self-mapping on X. Assume that the following conditions hold:*

(i) there exists $Y \subseteq X, fX \subseteq Y \subseteq X$ such that (Y, d) is $\mathcal{R}$-complete;

(ii) $X(f, \mathcal{R})$ is non-empty;

(iii) $\mathcal{R}$ is f-closed;

(iv) either f is $\mathcal{R}$-continuous or $\mathcal{R}|_Y$ is d-self-closed;

(v) there exist $a, b, c \ge 0$ with $a + 2b + 2c < 1$ such that

$$d(fx, fy) \le a d(x, y) + b[d(x, fx) + d(y, fy)] + c[d(x, fy) + d(y, fx)],$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

Then f has a fixed point. Moreover, if in addition, fX is $\mathcal{R}^s$-directed, then f has a unique fixed point.

Remark 3.4. Corollary 3.3 remains a sharpened version of a relation-theoretic Ćirić fixed point theorem proved in Ahmadullah et al. [1] *viz.* Corollary 2, corresponding to (13).

The following result was obtained by Alam and Imdad [4]:

Corollary 3.5. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ defined on X and f a self-mapping on X. Assume that the following conditions hold:*

(i) there exists $Y \subseteq X, fX \subseteq Y \subseteq X$ such that (Y, d) is $\mathcal{R}$-complete;

(ii) $X(f, \mathcal{R})$ is non-empty;

(iii) $\mathcal{R}$ is f-closed;

(iv) either f is $\mathcal{R}$-continuous or $\mathcal{R}|_Y$ is d-self-closed;

(v) there exists $k \in [0, 1)$ such that

$$d(fx, fy) \le k d(x, y),$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

Then f has a fixed point. Moreover, if in addition, fX is $\mathcal{R}^s$-directed, then f has a unique fixed point.

Corollary 3.6. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ defined on X and f a self-mapping on X. Assume that the following conditions hold:*

(i) there exists $Y \subseteq X, fX \subseteq Y \subseteq X$ such that (Y, d) is $\mathcal{R}$-complete;

(ii) $X(f, \mathcal{R})$ is non-empty;

(iii) $\mathcal{R}$ is f-closed;

(iv) either f is $\mathcal{R}$-continuous or $\mathcal{R}|_Y$ is d-self-closed;

(v) there exists $k \in [0, 1/2)$ such that

$$d(fx, fy) \le k[d(x, fx) + d(y, fy)],$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

Then f has a fixed point. Moreover, if in addition, fX is $\mathcal{R}^s$-directed, then f has a unique fixed point.

Remark 3.7. Corollary 3.6 remains an improved version of a relation-theoretic Kannan fixed point theorem (*i.e.*, Corollary 2 corresponding to (2.9)) established in Ahmadullah et al. [1].

Corollary 3.8. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ defined on X and f a self-mapping on X. Assume that the following conditions hold:*

   (i) there exists $Y \subseteq X, fX \subseteq Y \subseteq X$ such that (Y, d) is $\mathcal{R}$-complete;

   (ii) $X(f, \mathcal{R})$ is non-empty;

   (iii) $\mathcal{R}$ is f-closed;

   (iv) either f is $\mathcal{R}$-continuous or $\mathcal{R}|_Y$ is d-self-closed;

   (v) there exists $k \in [0, 1/2)$ such that

$$d(fx, fy) \leq k[d(x, fy) + d(y, fx)],$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

Then f has a fixed point. Moreover, if in addition, fX is $\mathcal{R}^s$-directed, then f has a unique fixed point.

Remark 3.9 Corollary 3.8 remains a sharpened version of a relation-theoretic Chatterjea fixed point theorem proved in Ahmadullah et al. [1], Corollary 2 corresponding to (2.10).

3.2. **Fixed point results in abstract space.** Under the universal relation (*i.e.*, $\mathcal{R} = X \times X$), Theorems 2.5 and 2.7 unify to the following lone corollary:

Corollary 3.10. *Let (X, d) be a metric space and f a self-mapping on X. Assume that the following conditions hold:*

   (i) there exists $Y \subseteq X$, $fX \subseteq Y \subseteq X$ such that (Y, d) is complete;

   (iii) there exists $\varphi \in \Phi$ such that

$$d(fx, fy) \leq \varphi\big(\max\big\{d(x, y), \frac{1}{2}[d(x, fx) + d(y, fy)], \frac{1}{2}[d(x, fy) + d(y, fx)]\big\}\big),$$

for all $x, y \in X$.

Then f has a unique fixed point.

   Now, we consider the following special cases:

   - On setting $\varphi(t) = kt$ (with $k \in [0, 1)$) in Corollary 3.10, we deduce an improved version of Ćirić fixed point theorem [9].
   - Corollary 3.5 deduces to a sharpened form of Banach contraction principle [6], under the universal relation.
   - If $\mathcal{R} = X \times X$, Corollary 3.6 deduces to an improved version of Kannan fixed point theorem [12].
   - Under the universal relation, Corollary 3.8 deduces to an improved version of Chatterjea fixed point theorem [8].

4. ILLUSTRATIVE EXAMPLES

In this section, we furnish some examples to highlight the realized improvement in our results proved in this paper.

Example 4.1. Consider a usual metric space (X, d), where $X = (-1, 4)$. Now, define a binary relation

$$\mathcal{R} = \{(x, y) \in X^2 \mid x \geq y\},$$

an increasing mapping $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{t}{2}$ and a self-mapping $f : X \to X$ by

$$f(x) = \begin{cases} \frac{x}{2}, & x \in (-1, 2]; \\ 1, & x \in (2, 4). \end{cases}$$

Let $Y = [-\frac{1}{2}, 2)$, so that $fX = (-\frac{1}{2}, 1] \subset Y$ and Y is $\mathcal{R}$-complete but X is not $\mathcal{R}$-complete. Evidently, $\mathcal{R}$ is f-closed and f is $\mathcal{R}$-continuous.

By routine calculations, one can verify hypothesis (v) of Theorems 2.5 and 2.7. Moreover, as fX is $\mathcal{R}^s$-*directed* and $\mathcal{R}|_{fX}$ is complete, therefore all the hypotheses of Theorems 2.5 and 2.7 are satisfied, ensuring the uniqueness of the fixed point. Observe that, $x = 0$ is the only point fixed of f.

With a view to establish genuineness of our results, notice that

$$\mathcal{R} = \{(x, y) \in X^2 \mid x \geq y\}$$

is not symmetric and $\mathcal{R}$ can not be a symmetric closure of any binary relation. Also (X, d) is not complete and even not $\mathcal{R}$-complete, which shows that the condition (i) of Theorem 1.17 (due to Samet and Turinici [18]) and even Corollary 2.2 is not satisfied. Thus, our Theorems 2.1, 2.5 and 2.7 are applicable to the present example while Theorem 1.17 and even Corollary 2.2 are not, which substantiates the utility of Theorems 2.1, 2.5 and 2.7.

Example 4.2. Consider $X = [0, 3)$ equipped with usual metric, *i.e.*, $d(x, y) = |x - y|$ for all $x, y \in X$ and a binary relation $\mathcal{R} = \{(0, 0), (0, 1), (0, 2), (1, 1), (2, 2)\}$ on X, whose symmetric closure is $\mathcal{S} = \{(0, 0), (0, 1), (1, 0), (0, 2), (2, 0), (1, 1), (2, 2)\}$. Define an increasing function $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{3}{4}t$ and a self-mapping $f : X \to X$ by

$$f(x) = \begin{cases} 0, & x \in [0, 1]; \\ 1, & x \in (1, 3). \end{cases}$$

Let $Y = [0, 1]$, so that $fX = \{0, 1\} \subset Y$ and Y is $\mathcal{S}$-complete. Evidently, f is not continuous but $\mathcal{S}$ is f-closed and $\varphi \in \Phi$. Take any $\mathcal{S}$-preserving sequence $\{x_n\}$ in Y, *i.e.*,

$$(x_n, x_{n+1}) \in \mathcal{S}, \text{ for all } n \in \mathbb{N} \text{ with } x_n \xrightarrow{d} x.$$

Here, one may notice that $(x_n, x_{n+1}) \in \mathcal{S}|_Y$, for all $n \in \mathbb{N}$ and there exists $N \in \mathbb{N}$ such that $x_n = x \in \{0, 1\}$, for all $n \geq N$. So, we need to choose a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k} = x$, for all $k \in \mathbb{N}$, which amounts to saying that $(x_{n_k}, x) \in \mathcal{S}|_Y$, for all $k \in \mathbb{N}$. Therefore, $\mathcal{S}|_Y$ is d-self-closed or $(Y, d, \mathcal{S})$ is regular.

Now, we check the condition (v) of Theorem 1.17 and Corollary 2.8. For this, we need to verify for $(x, y) \in \{(0, 2), (2, 0)\}$. Otherwise, we have $d(fx, fy) = 0$, hence condition (v) is obvious. If $(x, y) \in \{(0, 2), (2, 0)\}$, then $d(fx, fy) = 1 \leq \varphi(2) = \varphi(d(x, y))$ and hence f has a fixed point. As fX is $\mathcal{S}$-directed, therefore all the

hypotheses of Corollary 2.8 are satisfied. Observe that, $x = 0$ is the only fixed point of f.

Interestingly, Theorem 1.17 is not applicable to the present example as the underlying metric space (X, d) is not complete, which shows that our result (*i.e.*, Corollary 2.8) is an improvement over Theorem 1.17 (due to Samet and Turinici [18]).

Example 4.3. Let $X = [0, 4]$ equipped with usual metric, *i.e.*, for all $x, y \in X$, $d(x, y) = |x - y|$ and a binary relation $\mathcal{R} = \{(0, 0), (1, 1), (3, 3), (4, 4), (1, 2), (3, 4)\}$ on X. Define an increasing mapping $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{3}{4}t$ and a self-mapping $f : X \to X$ by

$$f(x) = \begin{cases} 0, & x \in [0, 1); \\ 3, & x \in [1, 2); \\ 4, & x \in [2, 4]. \end{cases}$$

Clearly, $fX = \{0, 3, 4\} \subset Y = X$ where Y is $\mathcal{R}$-complete, f is not continuous but $\mathcal{R}$ is f-closed and $\varphi \in \Phi$. On the lines of Example 4.2, one can verify that $\mathcal{R}|_Y$ is d-self-closed.

Now, with a view to check the condition (v) of Theorem 2.1, let $(x, y) = (1, 2)$ (as in rest of cases $d(fx, fy) = 0$), we have

$$d(f1, f2) = d(3, 4) = 1 < \frac{3}{2} = \varphi(2) = \varphi\left(\frac{d(1, f2) + d(2, f1)}{2}\right),$$

which shows that condition (v) (of Theorem 2.1) is satisfied, and henceforth f has a fixed point. Since $\{0, 3\} \subset fX$ but $(0, 3) \notin \mathcal{R}^s$, therefore $\mathcal{R}$ is not complete and hence fX is not $\mathcal{R}^s$-directed. Observe that, $x = 0, 4$ are the fixed points of f.

In the present example, observe that

$$(1, 2) \in \mathcal{R} \text{ but } d(f1, f2) = d(3, 4) \leq kd(1, 2), \ i.e., \ 1 \leq k$$

which shows that the contraction condition (v) of Theorem 1.18, due to Alam and Imdad [3, 4] is not satisfied. Also, $\mathcal{R}$ can not be a symmetric closure of any binary relation.

Thus, in all our results generalize, modify and unify the results of Samet and Turinici [18] and Alam and Imdad [3, 4].

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