Fixed Point Theory, 19(2018), No. 2, 801-822
DOI: 10.24193/fpt-ro.2018.2.59
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ITERATIVE METHODS FOR SYSTEM OF VARIATIONAL INCLUSIONS INVOLVING ACCRETIVE OPERATORS AND APPLICATIONS

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Abstract. The purpose of this paper is to study existence and approximation of solutions of system of variational inclusions involving multi-valued $H-$accretive and single-valued accretive operators over two different closed convex subsets of a Banach space. The convergence analysis of two proposed iteration processes for approximating solutions will be conducted within the conceptual framework of the "altering point technique" without uniform convexity of underlying spaces. This technique should make existing or new results in solving system of variational inequalities and variational inclusions.

Key Words and Phrases: Accretive operator, altering points, Mann iteration method, Lipschitz mapping, resolvent operator, variational inclusion problem.
2010 Mathematics Subject Classification: 47H09, 47H10, 47J25.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $\mathcal{F} : C \to \mathcal{H}$ be a nonlinear operator. The class of all proper, lower semicontinuous, convex functions from $\mathcal{H}$ to $(-\infty, \infty]$ is denoted by $\Gamma_0(\mathcal{H})$. The normal cone for C at a point $u \in C$ is

$$N_C(u) = \{z \in \mathcal{H} : \langle u - v, z\rangle \geq 0 \text{ for all } v \in C\}.$$

Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ with $Dom(A) \subseteq C$ and $B : C \to \mathcal{H}$ be monotone operators. The inclusion problem is to find $z \in C$ such that

$$0 \in (A + B)z. \tag{1.1}$$

Many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning can be mathematically modeled in form of inclusion problem (1.1). For instance, a stationary solution to the initial value problem of the evolution equation

$$0 \in \frac{\partial u}{\partial t} + Fu, u_0 = u(0)$$

can be recast as (1.1) when the governing maximal monotone operator F is of the form $F = A + B$, see, for example, [5].

Consider $\psi \in \Gamma_0(\mathcal{H})$, and set $A = \partial\psi$. Then, the inclusion problem (1.1) is equivalent to the mixed variational inequality problem (in short, MVI) of finding $x^* \in C$ such that

$$\langle Bx^*, v - x^* \rangle + \psi(v) - \psi(x^*) \geq 0 \quad \text{for all } v \in C. \tag{1.2}$$

The central problem is to iteratively find the solution of the inclusion problem (1.1) when A and B are two monotone operators on $\mathcal{H}$. One method for finding solutions of problem (1.1) is *splitting method*, for which each iteration involves only the individual operators A and B, but not the sum $A + B$. Splitting methods for linear equations were introduced by Peaceman and Rachford [9] and Douglas and Rachford [2]. Extensions to nonlinear equations in Hilbert spaces were carried out by Lions and Mercier [5] (see also [13, 16]). Recenlty, in [8, 10, 11], the authors studied computation of zeros of accretive operators by using different approaches.

For $\psi = 0$, the mixed variational inequality problem (1.2) reduces to the variational inequality problem:

$$\text{Find } x^* \in C \text{ such that } \langle Bx^*, x - x^* \rangle \geq 0 \text{ for all } x \in C, \tag{1.3}$$

which is denoted by $VI(C, B)$.

In [17], Verma introduced a new system of monotone variational inequalities, and studied the approximation solvability of this system by using two-step projection method. In [12], the authors studied convergence of an iterative algorithm for systems of variational inequalities in $2-$uniformly smooth Banach space in view of extragradient technique. Recently, by using retraction technique, Yao, Liou and Kang [19] extended two-step projection method from the Hilbert space $\mathcal{H}$ to a uniformly convex and $2-$uniformly smooth Banach space X for computation of the unique solution of a system of variational inequality problems involving strongly accretive operators defined on a closed convex subset of X.

On the other hand, Fang and Huang [4] introduced a class of $H-$monotone operators and proposed a one-step iterative algorithm for finding a solution of variational inclusion problems. They showed that the sequence generated by this one-step iterative algorithm converges strongly to a solution of a variational inclusion problem for $H-$monotone and Lipschitz continuous operators. Later, Zeng, Guu and Yao [20] generalized this iterative method by introducing a two-step iterative algorithm.

The aim of this paper is to deal with a system of variational inclusion problems concerning two closed convex subsets C and D of a Banach space X. Let $G, H : X \to X$ be strongly accretive and Lipschitz continuous operators. Let $A : X \to 2^X$ be a H-accretive operator with $Dom(A) \subseteq C$ and $B : X \to 2^X$ be a G-accretive operator with

$Dom(B) \subseteq D$. Moreover, let $S : C \to X$ (resp. $T : D \to X$) be a strongly accretive with respect to H (resp. with respect to G) and Lipschitz continuous operator. We consider the following system of variational inclusion problem (abbreviated as SGVInclPB):

Find $(x^*, y^*) \in C \times D$ such that

$$\begin{cases} 0 \in Gy^* - Hx^* + \eta(Sx^* + By^*), \\ 0 \in Hx^* - Gy^* + \rho(Ty^* + Ax^*). \end{cases} \tag{1.4}$$

Concrete definitions for the notation will be given in Section 2. Many practical problems including problem SGVInclPB (1.4) can be formulated as an altering point problem ([14]):

Find $(x^*, y^*) \in C \times D$ such that

$$\begin{cases} T_1(x^*) = y^*, \\ T_2(y^*) = x^*, \end{cases} \tag{1.5}$$

where $T_1 : C \to D$ and $T_2 : D \to C$ are nonlinear operators.

Inspired and motivated by the results in [14, 15, 19], we will establish the existence of solutions and convergence results for problem (1.5) in a Banach space. Furthermore, we obtain some strong convergence theorems for system of variational inequalities and system of variational inclusion problems.

The paper is organized as follows. The next section includes some necessary preliminaries. In Section 3, we propose our iterative algorithms for altering point problem (1.5) and prove convergence results for the proposed algorithms. Section 4 contains applications of the convergence results of Section 3 in system of variational inequalities and system of variational inclusion problems. Our mathematical model (1.5) contains the mathematical models studied in [4, 17, 19, 20] as special cases. The results obtained in this paper significantly improve and extend the results of Verma [17] and Yao, Liou and Kang [19] in several aspects.

2. Preliminaries

Let X be a Banach space with norm $\|\cdot\|$. Define the norm $\|\cdot\|_1$ on $X \times X$ by

$$\|(x, y)\|_1 = \|x\| + \|y\| \text{ for all } (x, y) \in X \times X. \tag{2.1}$$

Note that $(X \times X, \|\cdot\|_1)$ is also a Banach space.

Lemma 2.1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality:*

$$a_{n+1} \leq k a_n + b_n \text{ for all } n \in \mathbb{N},$$

where $k \in (0, 1)$ and $\lim_{n\to\infty} b_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. [7] *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers and let $\{b_n\}$ be a sequence in $\mathbb{R}$ satisfying the inequality:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + b_n + c_n \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1]$. *Assume that* $\sum_{n=1}^{\infty} c_n < \infty$. *Then, the following statements hold:*

(a) If $b_n \leq K\alpha_n$ *for all* $n \in \mathbb{N}$ *and for some* $K \geq 0$*, then*

$$a_{n+1} \leq \delta_n a_1 + (1-\delta_n)K + \sum_{j=1}^{n} c_j \text{ for all } n \in \mathbb{N},$$

where $\delta_n = \prod_{j=1}^{n}(1-\alpha_j)$ *and hence* $\{a_n\}$ *is bounded.*

(b) If $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\limsup_{n\to\infty}(b_n/\alpha_n) \leq 0$*, then* $\{a_n\}_{n=1}^{\infty}$ *converges to zero.*

2.1. **Smoothness of Banach spaces.**

The Banach space X is said to be *smooth* provided the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in S_X, where $S_X = \{x \in X : ||x|| = 1\}$. In this case, the norm of X is said to be *Gâteaux differentiable*. It is said to be *uniformly Gâteaux differentiable* if for each $y \in S_X$, this limit is attained uniformly for $x \in S_X$. Let $\rho_X : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of X ([1]) defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(||x+y\| + \|x-y\|) - 1 : x \in S_X,\ \|y\| \leq t\right\}.$$

The Banach space X is said to be uniformly smooth if $\frac{\rho_X(t)}{t} \to 0$ as $t \to 0$, and it is said to be *q-uniformly smooth* if there exists a fixed constant $c > 0$ such that $\rho_X(t) \leq ct^q$. It is well-known that X is uniformly smooth if and only if the norm of X is uniformly Fréchet differentiable. If X is q-uniformly smooth, then $q \leq 2$ and X is uniformly smooth, and hence the norm of X is uniformly Fréchet differentiable, in particular, the norm of X is Fréchet differentiable. Typical example of uniformly smooth Banach spaces is L^p, where $p > 1$. More precisely, L^p is $\min\{p,2\}$-uniformly smooth for every $p > 1$. It is well known that every uniformly smooth space has uniformly Gâteaux differentiable norm (see, e.g., [1]). Concerned with the characteristic inequalities in $2-$uniformly smooth Banach spaces, Xu [18] proved the following result.

Lemma 2.3. *Let* X *be a real* $2-$*uniformly smooth Banach space* X*. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2c^2\|y\|^2 + 2\langle y, J(x)\rangle \text{ for all } x, y \in X,$$

where c *is a positive constant and* $J : X \to X^*$ *is a normalized duality mapping.*

2.2. **Retractions.**

A subset C of a Banach space X is said to be a *retract* of X if there exists a continuous mapping Q_C from X onto C such that $Q_C(x) = x$ for all x in C. We call such Q_C a *retraction* of X onto C. It follows that if a mapping Q_C is a retraction, then $Q_C(y) = y$ for all y in the range of Q_C. A retraction Q_C is said to be *sunny* if $Q_C(Q_C(x)+t(x-Q_C(x))) = Q_C(x)$ for each x in X and $t \geq 0$. If a sunny retraction Q_C is also nonexpansive, then C is said to be a *sunny nonexpansive retract* of X.

Let C be a nonempty subset of X and $x \in X$. An element $y_0 \in C$ is said to be a *best approximation* to x if $\|x - y_0\| = d(x, C)$, where $d(x, C) = \inf_{y\in C} \|x - y\|$. The set of all best approximations from x to C is denoted by

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}.$$

This defines a mapping P_C from X into 2^C, which is called the *nearest point projection mapping* (*metric projection mapping*) onto C. It is well known that if C is a nonempty closed convex subset of a Hilbert space $\mathcal{H}$, then the nearest point projection P_C from $\mathcal{H}$ onto C is the unique sunny nonexpansive retraction of $\mathcal{H}$ onto C. It is also known that $P_C(x) \in C$ and

$$\langle x - P_C(x), P_C(x) - y\rangle \geq 0 \quad \text{ for all } x \in \mathcal{H}, y \in C.$$

We need the following facts for proving our main results.

Lemma 2.4. [3, Lemma 13.1] *Let C be a convex subset of a smooth Banach space X, D be a nonempty subset of C and P be a retraction from C onto D. Then, the following statements are equivalent:*
(a) P is sunny and nonexpansive.
(b) $\langle x - Px, J(z - Px)\rangle \leq 0$ for all $x \in C, z \in D$.
(c) $\langle x - y, J(Px - Py)\rangle \geq \|Px - Py\|^2$ for all $x, y \in C$.

From (b), one can see that

$$P(x) = y \Longleftrightarrow \langle x - y, J(y - u)\rangle \geq 0 \quad \text{for all } u \in D.$$

2.3. **Altering points.**

The notion of altering points has been introduced by Sahu [14] as follows:

Definition 2.5. Let C and D be two nonempty subsets of a metric space X and let $S : C \to D$ and $T : D \to C$ be mappings. If there exist $x^* \in C$ and $y^* \in D$ such that

$$\begin{cases} S(x^*) = y^*, \\ T(y^*) = x^*, \end{cases}$$

then $x^* \in C$ and $y^* \in D$ are called altering points of mappings S and T.

Thus, $x^* \in C$ and $y^* \in D$ are altering points of ordered pair (S, T) if $S(x^*) = y^*$ and $T(y^*) = x^*$. We denote the set of altering points of mappings $S : C \to D$ and $T : D \to C$ by

$$\mathrm{Alt}(S, T) = \{(x^*, y^*) \in C \times D : S(x^*) = y^* \text{ and } T(y^*) = x^*\}.$$

Remark 2.6. If $S : C \to D$ and $T : D \to C$ are mappings such that $TS : C \to C$ has a fixed point $x^* \in C$, then there exists $y^* \in D$ such that $S(x^*) = y^*$ and hence $T(y^*) = x^*$.

Example 2.7. ([14, Example 3.2]) Let $X = \mathbb{R}$, $C = D = [0, 1]$ and define $S, T : X \to X$ by $Sx = Tx = 1 - x$. Note $TS : C \to C$ is defined by $TSx = T(1 - x) = x$. Thus, each point of C is a fixed point of TS . Then altering points $x^* \in C$ and $y^* \in D$ of S and T are given by $x^* + y^* = 1$. Indeed,

$$\mathrm{Alt}(S, T) = \{(x^*, y^*) \in C \times D : x^* + y^* = 1\}.$$

Remark 2.8. From Example 2.7, we conclude that the element $(x^*, y^*) \in \mathrm{Alt}(S, T)$ is not necessarily the point of the intersection of line segments $y = 1 - x, x \in [0, 1/2]$ and $y = 1 - x, x \in [1/2, 1]$ (e.g. $(x^*, y^*) = (0, 1)$).

Example 2.9. Let $X = \mathbb{R}$, $C = [0, 1]$, $D = [1, 2]$. Define $S : C \to D$ by $Sx = 1 + x$, $x \in C$, and $T : D \to C$ by $Tx = x^2/4$, $x \in D$. Note $TSx = T(1 + x) = (1 + x)^2/4$, $x \in C$ and $STx = S(x^2/4) = 1 + x^2/4$ for all $x \in D$. Then $(1, 2) \in \mathrm{Alt}(S, T)$. The graphical representation of altering points of mappings S and T are given in Figure 1.

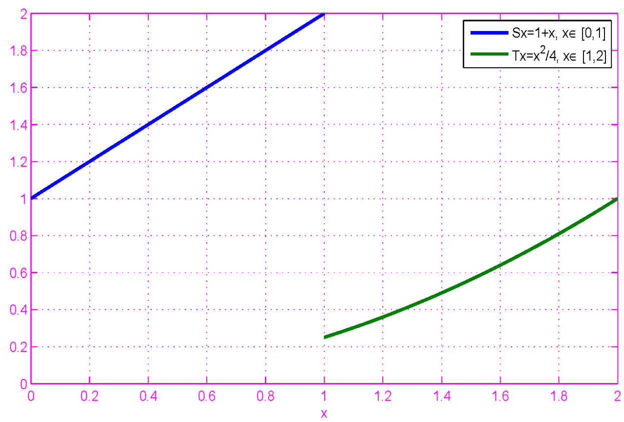

Figure 1. Graphical representation of altering points.

We remark that in Example 2.9, S and T are nonexpansive.

The following result plays a key role in the proof of our results.

Lemma 2.10. *Let C and D be nonempty closed convex subsets of a real smooth Banach space X. Let Q_C be the sunny nonexpansive retraction from X onto C and let Q_D be the sunny nonexpansive retraction from X onto D. Let $S : C \to X$ and $T : D \to X$ be nonlinear operators and let η and ρ be positive real numbers. Then the following statements are equivalent:*

(a) x^* and y^* are altering points of $Q_D(I - \eta S)$ and $Q_C(I - \rho T)$.

(b) $(x^*, y^*) \in C \times D$ is a solution of the following problem:

Find $(x^*, y^*) \in C \times D$ *such that*

$$\begin{cases} \langle \eta S(x^*) + y^* - x^*, J(x - y^*)\rangle \geq 0 & \textit{for all } x \in D, \\ \langle \rho T(y^*) + x^* - y^*, J(x - x^*)\rangle \geq 0 & \textit{for all } x \in C. \end{cases} \tag{2.2}$$

Proof. (a)$\Longrightarrow$(b). Suppose that x^* and y^* are altering points of $Q_D(I - \eta S)$ and $Q_C(I - \rho T)$. Note $Q_D(I - \eta S)(x^*) = y^*$. It follows from Lemma 2.4 that

$$\langle (I - \eta S)(x^*) - y^*, J(y^* - x)\rangle \geq 0 \quad \text{for all } x \in D,$$

i.e.,

$$\langle \eta S(x^*) + y^* - x^*, J(x - y^*)\rangle \geq 0 \quad \text{for all } x \in D.$$

On the other hand, $Q_C(I - \rho T)(y^*) = x^*$. Then, one can see that

$$\langle \rho T(y^*) + x^* - y^*, J(x - x^*)\rangle \geq 0 \quad \text{for all } x \in C.$$

(b)$\Longrightarrow$(a). Let $(x^*, y^*) \in C \times D$ be a solution of the problem (2.2). Note

$$\langle (I - \eta S)(x^*) - y^*, J(y^* - x)\rangle \geq 0 \quad \text{for all } x \in D.$$

It follows from Lemma 2.4 that $Q_D(I - \eta S)(x^*) = y^*$. Similarly, we can show that $Q_C(I - \rho T)(y^*) = x^*$. Therefore, x^* and y^* are altering points of $Q_D(I - \eta S)$ and $Q_C(I - \rho T)$. □

2.4. **Accretive operators.**

Definition 2.11. Let C be a nonempty subset of a real smooth Banach space X and let $T, H : C \to X$ be operators. Then T is said to be

(i) accretive if

$$\langle Tx - Ty, J(x - y)\rangle \geq 0 \text{ for all } x, y \in C;$$

(ii) strictly accretive if

$$\langle Tx - Ty, J(x - y)\rangle \geq 0$$

and the equality holds if and only if $y = x$;

(iii) strongly accretive if there exists a positive constant γ such that

$$\langle Tx - Ty, J(x - y)\rangle \geq \gamma \|x - y\|^2 \text{ for all } x, y \in C;$$

(iv) strongly accretive with respect to H if there exists a positive constant γ such that

$$\langle Tx - Ty, J(Hx - Hy)\rangle \geq \gamma \|x - y\|^2 \text{ for all } x, y \in C.$$

It is well known that when $X = \mathcal{H}$ is a real Hilbert space, the concept of accretive operator is identical with monotone operator.

Definition 2.12. Let $B : X \to 2^X$ be a multi-valued mapping and $H : X \to X$ be a mapping. We say that B is $H-$accretive if B is accretive and $(H + \lambda B)X = X$ holds, for all $\lambda > 0$.

Let $H : X \to X$ be a strongly accretive and Lipschitz continuous operator and let $B : X \to 2^X$ be an $H-$accretive operator. For the $H-$accretive operator B, we can associate its resolvent $J^B_{r,H}$ defined by

$$J^B_{r,H} \equiv (H + rB)^{-1} : X \to Dom(B),$$

where $r > 0$. We give some elementary properties of $J^B_{r,H}$.

Proposition 2.13. [6, Proposition 4.1] *Let X be a real Banach space. Let $H : X \to X$ be a strongly accretive and $L-$Lipschitz continuous operator and let $B : X \to 2^X$ be an $H-$accretive operator. Let $L, r > 0$. Then the resolvent operator $J^B_{r,H} : X \to Dom(B)$ has the following properties:*
(i) $\|J^B_{r,H}(x) - J^B_{r,H}(y)\| \leq \frac{1}{L}\|x - y\|$ *for all* $x, y \in R(H + rB)$;
(ii) $\|HJ^B_{r,H}(x) - HJ^B_{r,H}(y)\| \leq \|x - y\|$ *for all* $x, y \in X$;
(iii) $\|J^B_{r,H}H(x) - J^B_{r,H}H(y)\| \leq \|x - y\|$ *for all* $x, y \in X$.

Let C be a nonempty closed convex subset of X such that Dom(B) is contained in C. Then:
(i) $J^B_{r,H} \equiv (H + rB)^{-1} : X \to$Dom$(B) \subseteq C$.
(ii) $J^B_{r,H} \equiv (H + rB)^{-1} : C \to$Dom$(B) \subseteq C$.
If $H = I$, then the $H-$accretive operator $B : X \to 2^X$ is called $m-$accretive. For a $m-$accretive operator B, we can associate its resolvent J^B_r defined by $J^B_r : X \to Dom(B)$, where $r > 0$.

Proposition 2.14. *Let C and D be nonempty closed convex subsets of a real 2$-$uniformly smooth Banach space X. Let $F_C : X \to C$ be a $L-$Lipschitz continuous operator with $L > 0$ and let $T : D \to X$ be a $\kappa-$Lipschitzian and $\gamma-$strongly accretive operator. Suppose that there exists a positive constant ρ such that*

$$|\rho - \frac{\gamma}{2c^2\kappa^2}| < \frac{1}{2c^2\kappa^2}\sqrt{\gamma^2 - \left(1 - \frac{1}{L}\right)2c^2\kappa^2} \text{ and } \gamma^2 > \left(1 - \frac{1}{L}\right)2c^2\kappa^2. \tag{2.3}$$

Then $F_C(I - \rho T) : D \to C$ is a contraction with Lipschitz constant

$$L\sqrt{1 - 2\rho\gamma + 2c^2\kappa^2\rho^2}.$$

Proof. Let $x, y \in D$. Then, from Lemma 2.3, we have

$$\begin{aligned}\|(I - \rho T)x - (I - \rho T)y\|^2 &\leq \|x - y\|^2 + 2c^2\rho^2\|Tx - Ty\|^2 - 2\rho\langle Tx - Ty, J(x - y)\rangle \\ &\leq \|x - y\|^2 + 2c^2\kappa^2\rho^2\|x - y\|^2 - 2\rho\gamma\|x - y\|^2 \\ &= (1 - 2\rho\gamma + 2c^2\kappa^2\rho^2)\|x - y\|^2.\end{aligned}$$

Thus,

$$\|F_C(I - \rho T)x - F_C(I - \rho T)x\| \leq L\sqrt{1 - 2\rho\gamma + 2c^2\kappa^2\rho^2}\|x - y\|.$$

From (2.3), one sees that $0 \leq L\sqrt{1 - 2\rho\gamma + 2c^2\kappa^2\rho^2} < 1$.
Therefore, $F_C(I - \rho T) : D \to C$ is a contraction with Lipschitz constant

$$L\sqrt{1 - 2\rho\gamma + 2c^2\kappa^2\rho^2}.$$

□

Remark 2.15. If F_C is nonexpansive, then, for $\rho \in (0, \frac{\gamma}{c^2\kappa^2})$, the operator

$$F_C(I - \rho T) : D \to C$$

is a contraction with Lipschitz constant $\sqrt{1 - 2\rho\gamma + 2c^2\kappa^2\rho^2}$.

Proposition 2.16. *Let C and D be nonempty closed convex subsets of a real $2-$uniformly smooth Banach space X. Let $H : X \to X$ be a strongly accretive and L_1-Lipschitz continuous operator and let $G : X \to X$ be a strongly accretive and L_2-Lipschitz continuous operator. Let $T : D \to X$ be a γ-strongly accretive with respect to G and $\kappa-$Lipschitzian operator. Let $A : X \to 2^X$ be an $H-$accretive operator such that $Dom(A) \subseteq C$. Suppose that $L_1 > 0$ and that there exists a positive constant ρ such that*

$$|\rho - \frac{\gamma}{2c^2\kappa^2}| < \frac{1}{2c^2\kappa^2}\sqrt{\gamma^2 + 2c^2\kappa^2(L_1^2 - L_2^2)} \text{ and } \gamma^2 + 2c^2\kappa^2(L_1^2 - L_2^2) > 0. \quad (2.4)$$

Then $J^A_{\rho,H}(G - \rho T) : D \to C$ is a contraction with Lipschitz constant

$$\frac{1}{L_1}\sqrt{L_2^2 - 2\rho\gamma + 2c^2\kappa^2\rho^2}.$$

Proof. Set

$$\theta := \frac{1}{L_1}\sqrt{L_2^2 - 2\rho\gamma + 2c^2\kappa^2\rho^2}.$$

Proposition 2.13 implies that $J^A_{\rho,H} : X \to Dom(A) \subseteq C$ is $\frac{1}{L_1}-$Lipschitz continuous. Let $x, y \in D$. Then, from Lemma 2.3, we have

$$\begin{aligned} & \|(G - \rho T)x - (G - \rho T)y\|^2 \\ \leq \; & \|Gx - Gy\|^2 + 2c^2\rho^2\|Tx - Ty\|^2 - 2\rho\langle Tx - Ty, J(Gx - Gy)\rangle \\ \leq \; & L_2^2\|x - y\|^2 + 2c^2\kappa^2\rho^2\|x - y\|^2 - 2\rho\gamma\|x - y\|^2 \\ = \; & (L_2^2 - 2\rho\gamma + 2c^2\kappa^2\rho^2)\|x - y\|^2. \end{aligned}$$

Thus,

$$\|J^A_{\rho,H}(G - \rho T)x - J^A_{\rho,H}(G - \rho T)x\| \leq \theta\|x - y\|.$$

From (2.4), one sees that $0 \leq \theta < 1$. Therefore, $J^A_{\rho,H}(G - \rho T) : D \to C$ is a contraction with Lipschitz constant θ. ☐

2.5. **General system of variational inclusions in Banach spaces.**

Let C and D be nonempty closed convex subsets of a real smooth Banach space X, $F_C : X \to C$, $F_D : X \to D$ be operators, and $G, H : X \to X$ be strongly accretive and Lipschitz continuous operators. Let $S : C \to X$ be strongly accretive with respect to H and Lipschitz continuous and let $T : D \to X$ be strongly accretive with respect to G and Lipschitz continuous. Let $\eta, \rho > 0$. We consider the following altering point problem:

Find an element $(x^*, y^*) \in C \times D$ such that

$$\begin{cases} F_D(H - \eta S)(x^*) = y^*, \\ F_C(G - \rho T)(y^*) = x^*. \end{cases} \quad (2.5)$$

If $H = G = I$, then (2.5) reduces to the following altering point problem for operators $F_D(I - \eta S)$ and $F_C(I - \rho T)$:

Find an element $(x^*, y^*) \in C \times D$ such that

$$\begin{cases} F_D(I - \eta S)(x^*) = y^*, \\ F_C(I - \rho T)(y^*) = x^*. \end{cases} \tag{2.6}$$

The operators F_C and F_D play a key role in our mathematical modeling (2.5) and (2.6). Some special cases of the altering point problem (2.5) are as below:

(I) If $A : X \to 2^X$ is H-accretive such that $Dom(A) \subseteq C$ and $B : X \to 2^X$ is G-accretive such that $Dom(B) \subseteq D$, then for operators $F_C = J^A_{\rho,H}$ and $F_D = J^B_{\eta,G}$, the system (2.5) reduces to the following system of generalized variational inclusion problem involving accretive operators A, B, G, H, S and T (abbreviated as SGVInclPB):

Find $(x^*, y^*) \in C \times D$ such that

$$\begin{cases} 0 \in Gy^* - Hx^* + \eta(Sx^* + By^*), \\ 0 \in Hx^* - Gy^* + \rho(Ty^* + Ax^*). \end{cases} \tag{2.7}$$

(II) If $A, B : X \to 2^X$ are m-accretive operators such that $Dom(A) \subseteq C$ and $Dom(B) \subseteq D$, then for operators $F_C = J^A_\rho$ and $F_D = J^B_\eta$, the system (2.6) reduces to the following system of variational inclusion problem involving accretive operators A, B, S and T (abbreviated as SVInclPB):

Find $(x^*, y^*) \in C \times D$ such that

$$\begin{cases} 0 \in y^* - x^* + \eta(Sx^* + By^*), \\ 0 \in x^* - y^* + \rho(Ty^* + Ax^*). \end{cases} \tag{2.8}$$

If $X = \mathcal{H}$ is a real Hilbert space, $\phi, \psi \in \Gamma_0(\mathcal{H})$, and $A = \partial\phi$, $B = \partial\psi$, where $\partial\phi$ (resp. $\partial\psi$) is the subdifferential of ϕ (resp. ψ), then SVInclPB (2.8) reduces to a system of variational inclusion problem involving monotone operators S, T and functions ϕ, ψ (abbreviated as SVInclPH):

Find $(x^*, y^*) \in C \times D$ such that

$$\begin{cases} 0 \in y^* - x^* + \eta(Sx^* + \partial\psi(y^*)), \\ 0 \in x^* - y^* + \rho(Ty^* + \partial\phi(x^*)). \end{cases}$$

(III) If operators $F_C = Q_C$ and $F_D = Q_D$ are sunny nonexpansive retractions onto C and D, respectively, then, from Proposition 2.14, altering point problem (2.6) reduces to the following general system of nonlinear variational inequalities in Banach space X (abbreviated as GSNVIB$(C, D; S, T; \eta, \rho)$):

Find $(x^*, y^*) \in C \times D$ such that

$$\begin{cases} \langle \eta S(x^*) + y^* - x^*, J(x - y^*)\rangle \geq 0 \quad \text{for all } x \in D, \\ \langle \rho T(y^*) + x^* - y^*, J(x - x^*)\rangle \geq 0 \quad \text{for all } x \in C. \end{cases} \tag{2.9}$$

We denote by $\Omega[\text{GSNVIB}(C, D; S, T; \eta, \rho)]$ the set of solution of GSNVIB (2.9).

Yao, Liou and Kang [19] studied the following system of variational inequalities (abbreviated as SNVIB(S, T)) in Banach spaces:

Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \eta S(x^*) + y^* - x^*, J(x - y^*)\rangle \geq 0 & \text{for all } x \in C, \\ \langle \rho T(y^*) + x^* - y^*, J(x - x^*)\rangle \geq 0 & \text{for all } x \in C. \end{cases} \tag{2.10}$$

Remark 2.17. Due to the generality of GSNVIB (2.9), the two-step projection method studied by Yao, Liou and Kang [19] is not applicable for computation of solution of (2.9) if its solution exists.

If $X = \mathcal{H}$ is a real Hilbert space, then problem (2.9) reduces to the following system of variational inequality problem (abbreviated as SNVIH$(C, D; S, T; \eta, \rho)$):
Find $(x^*, y^*) \in C \times D$ such that

$$\begin{cases} \langle \eta S(x^*) + y^* - x^*, x - y^*\rangle \geq 0 & \text{for all } x \in D, \\ \langle \rho T(y^*) + x^* - y^*, x - x^*\rangle \geq 0 & \text{for all } x \in C. \end{cases} \tag{2.11}$$

If $C = D$, then problem (2.11) reduces to the following system of variational inequality problem (abbreviated as SNVIH(S, T)):
Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \eta S(x^*) + y^* - x^*,\ x - y^* \rangle \geq 0 & \text{for all } x \in C, \\ \langle \rho T(y^*) + x^* - y^*,\ x - x^* \rangle \geq 0 & \text{for all } x \in C. \end{cases} \tag{2.12}$$

In [17], Verma proved the strong convergence of the two-step projection methods for solving the problem SNVIH(S, T) (2.12).

3. Convergence theorems for altering point problems

As we have seen in Section 2.5 that various system of variational inequality problems and system of variational inclusion problems can be modeled as altering point problems. In this section, we first propose Mann type iteration process and a parallel iteration process for altering point problem (3.1), and then we establish the convergence theorems for the proposed iteration processes in Banach spaces without uniform convexity.

Let C and D be nonempty closed convex subsets of a Banach space X. Let $T_1 : C \to D$ and $T_2 : D \to C$ be contractions with Lipschitz constants θ_1 and θ_2, respectively. Since $T_2T_1 : C \to C$ is a contraction, there exists a unique element $(x^*, y^*) \in C \times D$ of the following altering point problem for operators T_1 and T_2:
Find $(x^*, y^*) \in C \times D$ such that

$$\begin{cases} T_1(x^*) = y^*, \\ T_2(y^*) = x^*. \end{cases} \tag{3.1}$$

We now introduce Mann iteration and parallel S-iteration process for computation of (x^*, y^*) :

(I) For arbitrary $x_0 \in C$, a sequence $\{(x_n, y_n)\}$ in $C \times D$ is generated by Mann iteration process:

$$\begin{cases} y_n = T_1(x_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_2(y_n) \text{ for all } n \in \mathbb{N}_0, \end{cases} \tag{3.2}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying appropriate conditions.

(II) For arbitrary $(x_0, y_0) \in C \times D$, a sequence $\{(x_n, y_n)\}$ in $C \times D$ is generated by the parallel S-iteration process:

$$\begin{cases} x_{n+1} = T_2[(1-\alpha_n)y_n + \alpha_n T_1(x_n)], \\ y_{n+1} = T_1[(1-\beta_n)x_n + \beta_n T_2(y_n)] \text{ for all } n \in \mathbb{N}_0, \end{cases} \tag{3.3}$$

where $\{(\alpha_n, \beta_n)\}$ is a sequence in $(0,1) \times (0,1)$ satisfying the suitable conditions.

For approximation of altering points of mappings $T_1 : C \to D$ and $T_2 : D \to C$, motivated by normal S-iteration process ([15]), the following parallel S-iteration process was introduced by Sahu [14]:

$$\begin{cases} x_{n+1} = T_2[(1-\alpha)y_n + \alpha T_1 x_n], \\ y_{n+1} = T_1[(1-\alpha)x_n + \alpha T_2 y_n] \text{ for all } n \in \mathbb{N}_0, \end{cases} \tag{3.4}$$

where $\alpha \in (0,1)$. Thus, the parallel S-iteration process (3.3) is a natural generalization of parallel S-iteration process (3.4).

First we establish strong convergence of the sequence $\{(x_n, y_n)\}$ generated by Mann iteration process (3.2) to the unique solution $(x^*, y^*) \in C \times D$ of altering point problem (3.1).

Theorem 3.1. *Let C and D be nonempty closed convex subsets of a Banach space X. Let $T_1 : C \to D$ and $T_2 : D \to C$ be contractions with Lipschitz constants θ_1 and θ_2, respectively. For arbitrary $x_0 \in C$, let $\{(x_n, y_n)\}$ be a sequence in $C \times D$ generated by Mann iteration process (3.2), where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then we have the following:*

(a) There exists a unique solution $(x^*, y^*) \in C \times D$ of altering point problem (3.1).

(b) $\{(x_n, y_n)\}$ converges strongly to (x^*, y^*) with the following error estimates:

$$\|x_{n+1} - x^*\| \le \prod_{j=0}^{n} (1 - (1-\theta_1\theta_2)\alpha_j)\|x_0 - x^*\| \textit{ for all } n \in \mathbb{N}_0$$

and

$$\|y_n - y^*\| \le \theta_1 \prod_{j=0}^{n-1} (1 - (1-\theta_1\theta_2)\alpha_j)\|x_0 - x^*\| \textit{ for all } n \in \mathbb{N}.$$

Proof. (a) Note $T_2T_1 : C \to C$ is a contraction with Lipschitz constant $\theta_1\theta_2$. Then, there exists a unique point $(x^*, y^*) \in C \times D$ such that x^* and y^* are altering points of mappings T_1 and T_2.

(b) Without loss of generality, we may assume that $\theta_1, \theta_2 \in (0,1)$. From (3.2), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n T_2(y_n) - x^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|T_2(y_n) - T_2(y^*)]\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \theta_2\alpha_n\|y_n - y^*\| \text{ for all } n \in \mathbb{N}_0. \end{aligned}$$

Note

$$\|y_n - y^*\| = \|T_1(x_n) - T_1(x^*)\| \le \theta_1 \|x_n - x^*\| \text{ for all } n \in \mathbb{N}_0. \tag{3.5}$$

Hence

$$\|x_{n+1} - x^*\| \le (1 - (1 - \theta_1\theta_2)\alpha_n)\|x_n - x^*\| \text{ for all } n \in \mathbb{N}_0.$$

It follows from Lemma 2.2 that $\{x_n\}$ converges strongly to x^*. From (3.5), one sees that $\{y_n\}$ converges strongly to y^*. Note

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1 - (1 - \theta_1\theta_2)\alpha_n)\|x_n - x^*\| \\ &\le \prod_{j=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_j)\|x_0 - x^*\| \text{ for all } n \in \mathbb{N}_0. \end{aligned} \tag{3.6}$$

Hence, from (3.5) and (3.6), we have

$$\begin{aligned} \|y_n - y^*\| &\le \theta_1 \|x_n - x^*\| \\ &\le \theta_1 \prod_{j=0}^{n-1}(1 - (1 - \theta_1\theta_2)\alpha_j)\|x_0 - x^*\| \text{ for all } n \in \mathbb{N}. \end{aligned}$$

□

In construction of the parallel S-iteration process (3.3), our approach is fundamentally different from the iteration processes in existing literature. We now establish strong convergence of sequence $\{(x_n, y_n)\}$ generated by the parallel S-iteration process (3.3) to the unique solution $(x^*, y^*) \in C \times D$ of altering point problem (3.1).

Theorem 3.2. *Let C and D be nonempty closed convex subsets of a Banach space X. Let $T_1 : C \to D$ and $T_2 : D \to C$ be contractions with Lipschitz constants θ_1 and θ_2, respectively. For arbitrary $(x_0, y_0) \in C \times D$, let $\{(x_n, y_n)\}$ be a sequence in $C \times D$ generated by the parallel S-iteration process (3.3), where $\{(\alpha_n, \beta_n)\}$ is a sequence in $(0,1) \times (0,1)$ satisfying the condition:*

$$\alpha_n - \theta_1\beta_n, \beta_n - \theta_2\alpha_n \in [0,1) \text{ for all } n \in \mathbb{N}_0. \tag{3.7}$$

Then we have the following:

(a) There exists a unique solution $(x^*, y^*) \in C \times D$ of the altering point problem (3.1).

(b) $\{(x_n, y_n)\}$ converges strongly to (x^*, y^*) with the following error estimate:

$$\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\|_1 \le \max\{\theta_1, \theta_2\}\|(x_n, y_n) - (x^*, y^*)\|_1 \text{ for all } n \in \mathbb{N}_0.$$

Proof. (a) It follows from Theorem 3.1(a).

(b) Set $\lambda := \max\{\theta_1, \theta_2\}$. From (3.3), we have

$$\begin{aligned} \|y_{n+1} - y^*\| &= \|T_1[(1 - \beta_n)x_n + \beta_n T_2(y_n)] - T_1(x^*)]\| \\ &\le \theta_1 \|(1 - \beta_n)x_n + \beta_n T_2(y_n) - x^*\| \\ &\le \theta_1[(1 - \beta_n)\|x_n - x^*\| + \beta_n\|T_2(y_n) - T_2(y^*)\|] \\ &\le \theta_1[(1 - \beta_n)\|x_n - x^*\| + \theta_2\beta_n\|y_n - y^*\|]. \end{aligned}$$

Similarly, we have

$$\|x_{n+1} - x^*\| \le \theta_2[(1 - \alpha_n)\|y_n - y^*\| + \theta_1\alpha_n\|x_n - x^*\|].$$

Adding the above two inequalities, we get

$$\begin{aligned}
&\|x_{n+1}-x^*\|+\|y_{n+1}-y^*\|\\
&\le \theta_1[(1-\beta_n)\|x_n-x^*\|+\theta_2\beta_n\|y_n-y^*\|]+\theta_2[(1-\alpha_n)\|y_n-y^*\|+\theta_1\alpha_n\|x_n-x^*\|]\\
&=\theta_1[1-(\beta_n-\theta_2\alpha_n)]\|x_n-x^*\|+\theta_2[1-(\alpha_n-\theta_1\beta_n)]\|y_n-y^*\|\\
&\le \lambda(\|x_n-x^*\|+\|y_n-y^*\|). \qquad (3.8)
\end{aligned}$$

From (2.1) and (3.8), we have

$$\|(x_{n+1},y_{n+1})-(x^*,y^*)\|_1\le\lambda\|(x_n,y_n)-(x^*,y^*)\|_1.$$

Noting that $\lambda\in(0,1)$, it follows from Lemma 2.1 that

$$\lim_{n\to\infty}\|(x_n,y_n)-(x^*,y^*)\|_1=0.$$

Thus, we obtain

$$\lim_{n\to\infty}\|x_n-x^*\|=\lim_{n\to\infty}\|y_n-y^*\|=0.$$

Therefore, $\{x_n\}$ and $\{y_n\}$ converge to x^* and y^*, respectively. □

4. Applications

To demonstrate the wide application of our convergence theory, a few examples are detailed below.

4.1. To approximate solutions of SGVInclPB(2.7) and SVInclPB (2.8).

Theorem 4.1. *Let C and D be nonempty closed convex subsets of a real $2-$uniformly smooth Banach space X. Let $H:X\to X$ be a strongly accretive and L_1-Lipschitz continuous operator and let $G:X\to X$ be a strongly accretive and L_2-Lipschitz continuous operator, where $L_1>0, L_2>0$. Let $S:C\to X$ be a κ_1-Lipschitzian and γ_1-strongly accretive operator and let $T:D\to X$ be a κ_2-Lipschitzian and γ_2-strongly accretive operator. Let $A:X\to 2^X$ be H-accretive and $B:X\to 2^X$ be G-accretive operators such that $\overline{Dom(A)}\subseteq C$ and $\overline{Dom(B)}\subseteq D$. Suppose that there exist positive constants η and ρ such that*

$$\left|\eta-\frac{\gamma_1}{2c^2\kappa_1^2}\right|<\frac{1}{2c^2\kappa_1^2}\sqrt{\gamma_1^2+2c^2\kappa_1^2(L_2^2-L_1^2)},\gamma_1^2+2c^2\kappa_1^2(L_2^2-L_1^2)>0$$

and

$$\left|\rho-\frac{\gamma_2}{2c^2\kappa_2^2}\right|<\frac{1}{2c^2\kappa_2^2}\sqrt{\gamma_2^2+2c^2\kappa_2^2(L_1^2-L_2^2)},\gamma^2+2c^2\kappa_2^2(L_1^2-L_2^2)>0.$$

Then we have the following:

(a) SGVIncIPB (2.7) has a unique solution $(x^*,y^*)\in C\times D$.

(b) For arbitrary $(x_0,y_0)\in C\times D$, if $\{(x_n,y_n)\}$ is a sequence in $C\times D$ generated by the parallel S-iteration process:

$$\begin{cases} x_{n+1}=J_\rho^{A,H}(G-\rho T)[(1-\alpha_n)y_n+\alpha_n J_\eta^{B,G}(H-\eta S)(x_n)],\\ y_{n+1}=J_\eta^{B,G}(H-\eta S)[(1-\beta_n)x_n+\beta_n J_\rho^{A,H}(G-\rho T)(y_n)] \text{ for all } n\in\mathbb{N}_0,\end{cases}$$

where $\{(\alpha_n,\beta_n)\}$ is a sequence in $(0,1)\times(0,1)$ satisfying the condition (3.7), then $\{(x_n,y_n)\}$ converges strongly to (x^*,y^*).

Proof. (a) Define $T_1 = J^B_{\eta,G}(H - \eta S)$ and $T_2 = J^A_{\rho,H}(G - \rho T)$. From Proposition 2.13, $J^A_{\rho,H} : X \to Dom(A) \subseteq C$ is $\frac{1}{L_1}$−Lipschitz continuous. It follows from Proposition 2.16 that $T_2 : D \to C$ is a contraction with Lipschitz constant

$$\frac{1}{L_1}\sqrt{L_2^2 - 2\rho\gamma_2 + 2c^2\kappa_2^2\rho^2}.$$

Similarly, $T_1 : C \to D$ is a contraction with Lipschitz constant

$$\frac{1}{L_2}\sqrt{L_1^2 - 2\rho\gamma_1 + 2c^2\kappa_1^2\rho^2}.$$

Thus, $T_2T_1 : C \to C$ is a contraction. By the Banach contraction principle, there exists a unique point $(x^*, y^*) \in C \times D$ such that x^* and y^* are altering points of mappings T_1 and T_2.

(b) It follows from Theorem 3.2. □

Theorem 4.2. *Let C and D be nonempty closed convex subsets of a real 2−uniformly smooth Banach space X. Let $A, B : X \to 2^X$ be m−accretive operators such that $\overline{Dom(A)} \subseteq C$ and $\overline{Dom(B)} \subseteq D$. Let $S : C \to X$ be a κ_1-Lipschitzian and γ_1-strongly accretive operator and let $T : D \to X$ be a κ_2-Lipschitzian and γ_2-strongly accretive operator. Assume that $0 < \eta < \frac{\gamma_1}{c\kappa_1^2}$ and $0 < \rho < \frac{\gamma_2}{c\kappa_2^2}$. Then we have the following:*

(a) SVInclPB (2.8) has a unique solution $(x^*, y^*) \in C \times D$.

(b) For arbitrary $x_0 \in C$, if $\{(x_n, y_n)\}$ is a sequence in $C \times D$ generated by Mann iteration process:

$$\begin{cases} y_n = J^B_\rho(I - \eta S)(x_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n J^A_\eta(I - \rho T)(y_n) \text{ for all } n \in \mathbb{N}_0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$, then it converges strongly to (x^*, y^*).

(c) For arbitrary $(x_0, y_0) \in C \times D$, if $\{(x_n, y_n)\}$ is a sequence in $C \times D$ generated by the parallel S-iteration process:

$$\begin{cases} x_{n+1} = J^A_\eta(I - \rho T)[(1 - \alpha_n)y_n + \alpha_n J^B_\rho(I - \eta S)(x_n)], \\ y_{n+1} = J^B_\rho(I - \eta S)[(1 - \beta_n)x_n + \beta_n J^A_\eta(I - \rho T)(y_n)] \text{ for all } n \in \mathbb{N}_0, \end{cases}$$

where $\{(\alpha_n, \beta_n)\}$ is a sequence in $(0, 1) \times (0, 1)$ satisfying the condition (3.7), then $\{(x_n, y_n)\}$ converges strongly to (x^*, y^*).

Proof. Define $T_1 = J^A_\eta(I - \eta S)$ and $T_2 = J^B_\rho(I - \rho T)$. Since $J^B_\rho : X \to$Dom$(B) \subseteq D$ is nonexpansive, it follows from Remark 2.15 that $T_1 : C \to D$ is a contraction with Lipschitz constant $\sqrt{1 - 2\eta\gamma_1 + 2c^2\eta^2\kappa_1^2}$. Similarly, $T_2 : D \to C$ is a contraction with Lipschitz constant $\sqrt{1 - 2\rho\gamma_2 + 2c^2\rho^2\kappa_2^2}$. Thus, Theorem 4.2 follows from Theorems 3.1 and 3.2. □

4.2. **To approximate solutions of GSNVIB(2.9).**

Theorem 4.3. *Let C and D be nonempty closed convex subsets of a real $2-$uniformly smooth Banach space X. Let $F_C : X \to C$ be a L_1-Lipschitz continuous operator and let $F_D : X \to D$ be a L_2-Lipschitz continuous operator such that $L_1 > 0, L_2 > 0$. Let $S : C \to X$ be a κ_1-Lipschitz continuous and γ_1-strongly accretive operator and let $T : D \to X$ be a κ_2-Lipschitz continuous and γ_2-strongly accretive operator. Suppose that there exist positive constants η and ρ such that*

$$|\eta - \frac{\gamma_1}{2c^2\kappa_1^2}| < \frac{1}{2c^2\kappa_1^2}\sqrt{\gamma_1^2 - 2c^2\kappa_1^2\left(1 - \frac{1}{L_2^2}\right)}, \quad \gamma_1^2 > 2c^2\kappa_1^2\left(1 - \frac{1}{L_2^2}\right)$$

and

$$|\rho - \frac{\gamma_2}{2c^2\kappa_2^2}| < \frac{1}{2c^2\kappa_2^2}\sqrt{\gamma_2^2 - 2c^2\kappa_2^2\left(1 - \frac{1}{L_1^2}\right)}, \quad \gamma_2^2 > 2c^2\kappa_2^2\left(1 - \frac{1}{L_1^2}\right).$$

Then we have the following:

(a) Altering point problem (2.5) has a unique solution $(x^*, y^*) \in C \times D$.

(b) For arbitrary $x_0 \in C$, if $\{(x_n, y_n)\}$ is a sequence in $C \times D$ generated by Mann iteration process:

$$\begin{cases} y_n = F_D(I - \eta S)(x_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n F_C(I - \rho T)(y_n) \text{ for all } n \in \mathbb{N}_0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$, then it converges strongly to (x^*, y^*).

Proof. Define $T_1 = F_D(I - \eta S)$ and $T_2 = F_C(I - \rho T)$. Set

$$\theta_1 := L_2\sqrt{1 - 2\eta\gamma_1 + 2c^2\eta^2\kappa_1^2} \text{ and } \theta_2 := \sqrt{1 - 2\rho\gamma_2 + 2c^2\rho^2\kappa_2^2}.$$

Since F_D is L_2-Lipschitz continuous, it follows from Proposition 2.14 that $T_1 : C \to D$ is θ_1-Lipschitz continuous. By assumption, we have

$$\theta_1 = L_2\sqrt{1 - 2\eta\gamma_1 + 2c^2\eta^2\kappa_1^2} < 1.$$

Similarly, $T_2 : D \to C$ is a contraction with Lipschitz constant θ_2. Therefore, Theorem 4.3 follows from Theorem 3.1. □

Theorem 4.4. *Let C and D be nonempty closed convex subsets of a real $2-$uniformly smooth Banach space X. Let Q_C and Q_D be sunny nonexpansive retractions onto C and D, respectively. Let $S : C \to X$ be a κ_1-Lipschitzian and γ_1-strongly accretive operator and let $T : D \to X$ be a κ_2-Lipschitzian and γ_2-strongly accretive operator. Assume that $0 < \eta < \frac{\gamma_1}{c^2\kappa_1^2}$ and $0 < \rho < \frac{\gamma_2}{c^2\kappa_2^2}$. Then we have the following:*

(a) GSNVIB (2.9) has a unique solution $(x^*, y^*) \in C \times D$.

(b) For arbitrary $x_0 \in C$, if $\{(x_n, y_n)\}$ is a sequence in $C \times D$ generated by Mann iteration process:

$$\begin{cases} y_n = Q_D(I - \eta S)(x_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Q_C(I - \rho T)(y_n) \text{ for all } n \in \mathbb{N}_0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$, then it converges strongly to (x^*, y^*).

Proof. Define $T_1 = Q_D(I - \eta S)$ and $T_2 = Q_C(I - \rho T)$. Since Q_D is nonexpansive, it follows from Proposition 2.14 that $T_1 : C \to D$ is a contraction with Lipschitz constant $\sqrt{1 - 2\eta\gamma_1 + 2c^2\eta^2\kappa_1^2}$. Similarly, $T_2 : D \to C$ is a contraction with Lipschitz constant $\sqrt{1 - 2\rho\gamma_2 + 2c^2\rho^2\kappa_2^2}$. Therefore, Theorem 4.4 follows from Theorem 4.3. □

We immediately obtain the following corollary.

Corollary 4.5. *Let C be a nonempty closed convex subset of a real $2-$uniformly smooth Banach space X. Let Q_C be sunny nonexpansive retraction onto C. Let $S : C \to X$ be a κ_1-Lipschitzian and γ_1-strongly accretive operator and let $T : C \to X$ be a κ_2-Lipschitzian and γ_2-strongly accretive operator. Assume that $0 < \eta < \frac{\gamma_1}{c^2\kappa_1^2}$ and $0 < \rho < \frac{\gamma_2}{c^2\kappa_2^2}$. Then we have the following:*

(a) SNVIB (2.10) has a unique solution $(x^*, y^*) \in C \times C$.

(b) For arbitrary $x_0 \in C$, if $\{(x_n, y_n)\}$ is a sequence in $C \times C$ generated by Mann iteration process:

$$\begin{cases} y_n = Q_C(I - \eta S)(x_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Q_C(I - \rho T)(y_n) \text{ for all } n \in \mathbb{N}_0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$, then it converges strongly to (x^*, y^*).

Corollary 4.5 guarantees the existence and approximation of unique solution of problem SNVIB (2.10) without uniform convexity. Therefore, Corollary 4.5 is a significant improvement of the result of Yao, Liou and Kang [19].

We now apply the parallel S-iteration process for finding solution of GSNVIB (2.9).

Theorem 4.6. *Let C and D be nonempty closed convex subsets of a real $2-$uniformly smooth Banach space X. Let Q_C and Q_D be sunny nonexpansive retractions onto C and D, respectively. Let $S : C \to X$ be a κ_1-Lipschitzian and γ_1-strongly accretive operator and let $T : D \to X$ be a κ_2-Lipschitzian and γ_2-strongly accretive operator. Assume that $0 < \eta < \frac{\gamma_1}{c^2\kappa_1^2}$ and $0 < \rho < \frac{\gamma_2}{c^2\kappa_2^2}$. Then we have the following:*

(a) GSNVIB (2.9) has a unique solution $(x^*, y^*) \in C \times D$.

(b) For arbitrary $(x_0, y_0) \in C \times D$, if $\{(x_n, y_n)\}$ is a sequence in $C \times D$ generated by the parallel S-iteration process:

$$\begin{cases} x_{n+1} = Q_C(I - \rho T)[(1 - \alpha_n)y_n + \alpha_n Q_D(I - \eta S)(x_n)], \\ y_{n+1} = Q_D(I - \eta S)[(1 - \beta_n)x_n + \beta_n Q_C(I - \rho T)(y_n)] \text{ for all } n \in \mathbb{N}_0, \end{cases}$$

where $\{(\alpha_n, \beta_n)\}$ *is a sequence in* $(0,1) \times (0,1)$ *satisfying the condition (3.7), then* $\{(x_n, y_n)\}$ *converges strongly to* (x^*, y^*).

Proof. Theorem 4.6(a) follows from Theorem 4.4(a). The part (b) follows from Theorem 3.2. □

Corollary 4.7. *Let* C *and* D *be nonempty closed convex subsets of a real Hilbert space* $\mathcal{H}$. *Let* P_C *and* P_D *be metric projections onto* C *and* D*, respectively. Let* $S : C \to \mathcal{H}$ *be a* κ_1*-Lipschitzian and* γ_1*-strongly monotone operator and let* $T : D \to \mathcal{H}$ *be a* κ_2*-Lipschitzian and* γ_2*-strongly monotone operator. Assume that*

$$0 < \eta < \frac{2\gamma_1}{\kappa_1^2} \text{ and } 0 < \rho < \frac{2\gamma_2}{\kappa_2^2}.$$

Define $T_1 = P_D(I - \eta S)$ *and* $T_2 = P_C(I - \rho T)$. *Then we have the following:*

(a) SNVIH (2.11) has a unique solution $(x^*, y^*) \in C \times D$.

(b) For arbitrary $(x_0, y_0) \in C \times D$*, if* $\{(x_n, y_n)\}$ *is a sequence in* $C \times D$ *generated by the parallel S-iteration process (3.3), where* $\{(\alpha_n, \beta_n)\}$ *is a sequence in* $(0,1) \times (0,1)$ *satisfying the condition (3.7), then it converges strongly to* (x^*, y^*).

5. Numerical examples

Now, we give a real numerical example in which the conditions satisfy the ones of Theorem 4.2 and some numerical experiment results to explain the main results in Theorem 4.2 as follows:

Example 5.1. Let $\mathcal{H} = \mathbb{R}$, $C = (-\infty, -1]$ and $D = [1, \infty)$. Define $S : C \to \mathcal{H}$ and $T : D \to \mathcal{H}$ by $Sx = -15 + 3x, x \in C$ and $Tx = 28 + 4x, x \in D$. Define $A, B : \mathcal{H} \to 2^{\mathcal{H}}$ by

$$A(x) = \begin{cases} 2x + \mathbb{R}^+ \text{ if } x = -1, \\ 2x \text{ if } x \in (-\infty, -1) \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 3x + \mathbb{R}^- \text{ if } x = 1, \\ 3x \text{ if } x \in (1, \infty). \end{cases}$$

Take $\eta = \frac{1}{6}$ and $\rho = \frac{1}{8}$. Then we have the following:

(a) $(-\frac{9}{4}, 1) \in C \times D$ is the unique solution of the following problem:

$$\text{Find } (x^*, y^*) \in C \times D \text{ such that } \begin{cases} 0 \in y^* - x^* + \eta(Sx^* + By^*), \\ 0 \in x^* - y^* + \rho(Ty^* + Ax^*). \end{cases} \tag{5.1}$$

(b) For arbitrary $x_0 \in C$, the sequence $\{(x_n, y_n)\}$ in $C \times D$ generated by Mann iteration process

$$\begin{cases} y_n = J_\rho^B(I - \eta S)(x_n), \\ x_{n+1} = (1 - \frac{1}{n+1})x_n + \frac{1}{n+1} J_\eta^A(I - \rho T)(y_n) \text{ for all } n \in \mathbb{N}_0 \end{cases} \tag{5.2}$$

converges strongly to (x^*, y^*).

(c) For arbitrary $(x_0, y_0) \in C \times D$, if $\{(x_n, y_n)\}$ is a sequence in $C \times D$ generated by the parallel S-iteration process:

$$\begin{cases} x_{n+1} = J_\eta^A(I - \rho T)[(1 - \alpha_n)y_n + \alpha_n J_\rho^B(I - \eta S)(x_n)], \\ y_{n+1} = J_\rho^B(I - \eta S)[(1 - \beta_n)x_n + \beta_n J_\eta^A(I - \rho T)(y_n)] \text{ for all } n \in \mathbb{N}_0, \end{cases} \tag{5.3}$$

where $\{(\alpha_n, \beta_n)\}$ is a sequence in $(0,1) \times (0,1)$ satisfying the condition (3.7), then $\{(x_n, y_n)\}$ converges strongly to (x^*, y^*).

Proof. (a) It is obvious that S and T are strongly monotone and that A and B are maximal monotone with

$$J_\eta^A(x) = \begin{cases} \frac{3x}{4} \text{ if } x \in (-\infty, -\frac{4}{3}), \\ -1 \text{ if } x \in [-\frac{4}{3}, \infty) \end{cases} \quad \text{and} \quad J_\rho^B(x) = \begin{cases} 1 \text{ if } x \in (-\infty, \frac{11}{8}], \\ \frac{8x}{11} \text{ if } x \in (\frac{11}{8}, \infty). \end{cases}$$

Hence

$$J_\eta^A(x - \rho Tx) = \begin{cases} \frac{3}{8}(x-7) \text{ if } x \in [1, \frac{13}{3}), \\ -1 \text{ if } x \in [\frac{13}{3}, \infty) \end{cases}$$

and

$$J_\rho^B(x - \eta Sx) = \begin{cases} 1 \text{ if } x \in (-\infty, -\frac{9}{4}], \\ \frac{4}{11}(x+5) \text{ if } x \in (-\frac{9}{4}, -1]. \end{cases}$$

Therefore, $(-\frac{9}{4}, 1) \in C \times D$ is the unique solution of the problem (5.1).

(b) Following the proof of Theorem 4.2 (b), we can easily obtain that $\{x_n\}$ converges strongly to $x^* = -\frac{9}{4}$ and $\{y_n\}$ converges to $y^* = 1$. The numerical experiment results using software Matlab 7.0 are given in Figure 2, which show the iteration process of the sequence $\{(x_n, y_n)\}$ generated by the iteration process (5.2) with $\alpha_n = 1/2$, and initial point $x_0 = -1$, respectively, $x_0 = -1.5$.

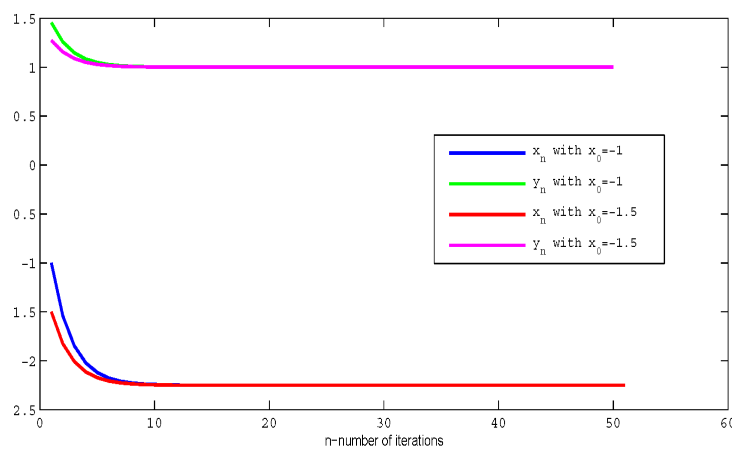

Figure 2. The iteration process of the sequence $\{(x_n, y_n)\}$ defined by (5.2).

(c) Following the proof of Theorem 4.2 (c), we easily obtain that $\{x_n\}$ converges to $x^* = -\frac{9}{4}$ and $\{y_n\}$ converges to $y^* = 1$. The numerical experiment results using software Matlab 7.0 are given in Figure 3, which show the iteration process of the sequence $\{(x_n, y_n)\}$ generated by the iteration process (5.3) with $\alpha_n = \beta_n = 1/2$, and initial point $(x_0, y_0) = (-2, 2)$ and $(x_0, y_0) = (-1.5, 1.5)$, respectively. $\square$

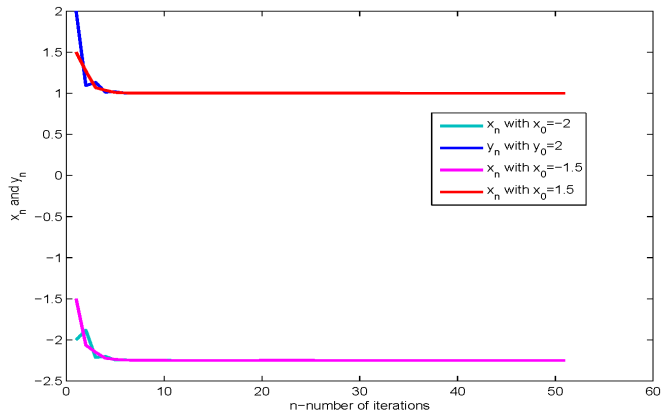

Figure 3. The iteration process of the sequence $\{(x_n, y_n)\}$ defined by (5.3)

Acknowledgements. The corresponding author was partially supported by the grant MOST 105-2115-M-037-001, and the grant from Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Taiwan.

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Received: March 14, 2016; Accepted: August 30, 2017.