Fixed Point Theory, 19(2018), No. 2, 823-836
DOI: 10.24193/fpt-ro.2018.2.60
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

POSITIVE SOLUTIONS FOR A SYSTEM OF p-LAPLACIAN BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we investigate the existence of positive solutions for a system of fourth order p-Laplacian boundary value problems

$$\begin{cases} -((-x''')^{p-1})' = f(t,x,x',y,y'), t \in [0,1], \\ -((-y''')^{p-1})' = g(t,x,x',y,y'), t \in [0,1], \\ x(0) = x'(1) = x''(0) = x'''(1) = 0, \\ y(0) = y'(1) = y''(0) = y'''(1) = 0, \end{cases}$$

where $p > 1$, $f, g \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)(\mathbb{R}^+ := [0,\infty))$. Under some new general conditions on f and g, we use the fixed point index to establish two existence theorems for the above system. The interesting point lies in the fact that the nonlinear term f, g can be allowed to depend on the first derivative of the unknown functions, and this derivative dependence in systems is seldom considered in the literature.
Key Words and Phrases: p-Laplacian equation; positive solution; fixed point index; derivative dependence.
2010 Mathematics Subject Classification: 34B18, 47H07, 47H11, 45M20, 26D15.

1. Introduction

The paper mainly concerns the existence of positive solutions for a system of fourth order p-Laplacian boundary value problems

$$\begin{cases} -((-x''')^{p-1})' = f(t,x,x',y,y'), t \in [0,1], \\ -((-y''')^{p-1})' = g(t,x,x',y,y'), t \in [0,1], \\ x(0) = x'(1) = x''(0) = x'''(1) = 0, \\ y(0) = y'(1) = y''(0) = y'''(1) = 0, \end{cases} \tag{1.1}$$

where $p > 1$, $f, g \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$.

Systems for nonlinear boundary value problems arise in many applications in engineering, science, economy, and other fields and some results have been established in the literature; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the references therein.

In [1, 2], the authors use the coincidence degree theory of Mawhin to study the existence of solutions for the two coupled systems of fractional differential equations

$$\begin{cases} D_{0+}^{\gamma}\phi_p(D_{0+}^{\alpha}u(t)) = f(t, v(t)), \\ D_{0+}^{\gamma}\phi_p(D_{0+}^{\beta}v(t)) = g(t, u(t)), \\ D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = D_{0+}^{\beta}v(0) = D_{0+}^{\beta}v(1) = 0, \end{cases}$$

and

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), v(t)),\ u(0) = 0,\ D^{\gamma}u(t)|_{t=1} = \sum_{i=1}^{n} a_i D^{\gamma}u(t)|_{t=\xi_i}, \\ D^{\beta}v(t) = g(t, u(t), v(t)),\ v(0) = 0,\ D^{\delta}v(t)|_{t=1} = \sum_{i=1}^{m} b_i D^{\delta}v(t)|_{t=\eta_i}. \end{cases}$$

In [3], the authors studied the existence of positive solutions of the singular fourth-order boundary value system with integral boundary conditions

$$\begin{cases} (\phi_{p_1}(u''(t)))'' = \lambda^{p_1-1}a_1(t)f_1(t, u(t), v(t)), 0 < t < 1, \\ (\phi_{p_2}(v''(t)))'' = \mu^{p_2-1}a_2(t)f_2(t, u(t), v(t)), 0 < t < 1, \\ u(0) = u(1) = \int_0^1 u(s)\mathrm{d}\xi_1(s), \\ v(0) = v(1) = \int_0^1 v(s)\mathrm{d}\xi_2(s), \\ \phi_{p_1}(u''(0)) = \phi_{p_1}(u''(1)) = \int_0^1 \phi_{p_1}(u''(s))\mathrm{d}\eta_1(s), \\ \phi_{p_2}(v''(0)) = \phi_{p_2}(v''(1)) = \int_0^1 \phi_{p_2}(v''(s))\mathrm{d}\eta_2(s). \end{cases}$$

In [4], the authors studied the existence of positive solutions for the coupled system of mixed higher-order nonlinear singular fractional differential equations with integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha_1}u(t) + a_1(t)f_1(t, u(t), v(t)) = 0, 0 < t < 1, \\ D_{0+}^{\alpha_2}v(t) + a_2(t)f_2(t, u(t)) = 0, 0 < t < 1, \\ u^{(j)}(0) = v^{(k)}(0) = 0, \\ u(1) = \int_0^1 h_1(t)u(t)\mathrm{d}t, v(1) = \int_0^1 h_2(t)v(t)\mathrm{d}t. \end{cases}$$

However we note that in most of these studies the nonlinear terms considered do not involve derivatives of the dependent variable. The papers [9, 10] tackle nonlinear terms that involve even order derivatives. In our paper, the nonlinear terms f, g in (1.1) depend on the first derivative of the unknown functions and our results extend and complement the rich literature on systems of boundary value problems.

2. Preliminaries

Let $E := C[0, 1]$, $\|u\| := \max_{t\in[0,1]} |u(t)|$, $P := \{u \in E : u(t) \geq 0, \forall t \in [0, 1]\}$. Then $(E, \|\cdot\|)$ is a real Banach space and P is a cone on E. Furthermore, the norm on $E \times E$ is defined by $\|(u, v)\| := \max\{\|u\|, \|v\|\}, (u, v) \in E \times E$, and $E \times E$ is a real Banach space and $P \times P$ is a cone on $E \times E$.

In what follows, we first convert the system (1.1) into equivalent integral equations. Let $u := -x'',\ v := -y''$. Then, together with the boundary conditions

$$x(0) = x'(1) = y(0) = y'(1) = 0,$$

we have

$$x(t) = \int_0^1 G_1(t,s)u(s)\mathrm{d}s := (L_1u)(t),\ y(t) = \int_0^1 G_1(t,s)v(s)\mathrm{d}s := (L_1v)(t),$$

where

$$G_1(t,s) = \begin{cases} t, 0 \le t \le s \le 1, \\ s, 0 \le s \le t \le 1. \end{cases}$$

Let

$$G_2(t,s) = \begin{cases} 1, 0 \le t \le s \le 1, \\ 0, 0 \le s \le t \le 1. \end{cases}$$

Then

$$x'(t) = \int_0^1 G_2(t,s)u(s)\mathrm{d}s := (L_2u)(t),\ y'(t) = \int_0^1 G_2(t,s)v(s)\mathrm{d}s := (L_2v)(t).$$

Consequently, we see that (1.1) is equivalent to

$$\begin{cases} -((u')^{p-1})' = f\left(t, (L_1u)(t), (L_2u)(t), (L_1v)(t), (L_2v)(t)\right), \\ -((v')^{p-1})' = g\left(t, (L_1u)(t), (L_2u)(t), (L_1v)(t), (L_2v)(t)\right), \\ u(0) = u'(1) = v(0) = v'(1) = 0. \end{cases}$$

Therefore, we obtain

$$\begin{cases} u(t) = \displaystyle\int_0^t \left(\int_s^1 f\left(\tau, (L_1u)(\tau), (L_2u)(\tau), (L_1v)(\tau), (L_2v)(\tau)\right)\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s, \\ v(t) = \displaystyle\int_0^t \left(\int_s^1 g\left(\tau, (L_1u)(\tau), (L_2u)(\tau), (L_1v)(\tau), (L_2v)(\tau)\right)\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s. \end{cases}$$

Let

$$A_1(u,v)(t) := \int_0^t \left(\int_s^1 f\left(\tau, (L_1u)(\tau), (L_2u)(\tau), (L_1v)(\tau), (L_2v)(\tau)\right)\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s,$$
$$A_2(u,v)(t) := \int_0^t \left(\int_s^1 g\left(\tau, (L_1u)(\tau), (L_2u)(\tau), (L_1v)(\tau), (L_2v)(\tau)\right)\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s,$$
$$A(u,v)(t) = (A_1(u,v), A_2(u,v))(t).$$

Note that if $f, g \in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+, \mathbb{R}^+)$, then $A_i : P\times P \to P (i = 1,2)$ and $A : P \times P \to P \times P$ are continuous and compact (completely continuous) operators, and the existence of positive solutions for (1.1) is equivalent to that of positive fixed points of A.

Remark 2.1. (i) $A_i (i = 1,2)$ is nonnegative and increasing about $t \in [0,1]$;

(ii) $\left(\int_s^1 f(\cdot)\,\mathrm{d}\tau\right)^{\frac{1}{p-1}}$ and $\left(\int_s^1 g(\cdot)\,\mathrm{d}\tau\right)^{\frac{1}{p-1}}$ are nonnegative and nonincreasing on $s \in [0,1]$.

Lemma 2.2. (see [13, Lemma 2.1]) *Let* $\kappa := 1 - 2/e$ *and* $\psi(t) := te^t, t \in [0,1]$. *Then* $\psi(t)$ *is nonnegative on* $[0,1]$ *and*

$$\kappa\psi(s) \le \int_0^1 G_1(t,s)\psi(t)\mathrm{d}t \le \psi(s). \tag{2.1}$$

Lemma 2.3. *Let* $P_0 = \{u \in P : u(t) \ge t\|u\|,\ \forall t \in [0,1]\}$. *Then* $A(P\times P) \subset P_0 \times P_0$.

Proof. Recall if h is nonnegative and nonincreasing on $[0,1]$ then for any $t \in [0,1]$, we have

$$\int_0^t h(s)\mathrm{d}s \ge t\int_0^1 h(s)\mathrm{d}s.$$

If $(u,v) \in P \times P$, then

$$\begin{aligned}\|A_1(u,v)\| &= A_1(u,v)(1)\\ &= \int_0^1 \left(\int_s^1 f\left(\tau,(L_1u)(\tau),(L_2u)(\tau),(L_1v)(\tau),(L_2v)(\tau)\right)\mathrm{d}\tau\right)^{\frac{1}{p-1}}\mathrm{d}s.\end{aligned}$$

Also note that $\left(\int_s^1 f(\cdot)\,\mathrm{d}\tau\right)^{\frac{1}{p-1}}$ is nonnegative and nonincreasing on $s \in [0,1]$, and we find

$$\begin{aligned}A_1(u,v)(t) &= \int_0^t \left(\int_s^1 f\left(\tau,(L_1u)(\tau),(L_2u)(\tau),(L_1v)(\tau),(L_2v)(\tau)\right)\mathrm{d}\tau\right)^{\frac{1}{p-1}}\mathrm{d}s\\ &\ge t\int_0^1 \left(\int_s^1 f\left(\tau,(L_1u)(\tau),(L_2u)(\tau),(L_1v)(\tau),(L_2v)(\tau)\right)\mathrm{d}\tau\right)^{\frac{1}{p-1}}\mathrm{d}s\\ &= t\|A_1(u,v)\|.\end{aligned}$$

Similarly we obtain

$$A_2(u,v)(t) \ge t\|A_2(u,v)\|.$$

This completes the proof. □

Lemma 2.4. (see [14]) *Let* $\Omega \subset E$ *be a bounded open set and* $A : \overline{\Omega}\cap P \to P$ *is a continuous and compact (completely continuous) operator. If there exists* $v_0 \in P\setminus\{0\}$ *such that* $v - Av \ne \lambda v_0$ *for all* $v \in \partial\Omega \cap P$ *and* $\lambda \ge 0$, *then* $i(A,\Omega\cap P, P) = 0$.

Lemma 2.5. (see [14]) *Let* $\Omega \subset E$ *be a bounded open set with* $0 \in \Omega$. *Suppose* $A : \overline{\Omega}\cap P \to P$ *is a continuous and compact (completely continuous) operator. If* $v \ne \lambda Av$ *for all* $v \in \partial\Omega\cap P$ *and* $0 \le \lambda \le 1$, *then* $i(A,\Omega\cap P,P) = 1$.

Lemma 2.6. (Jensen's inequalities, see [13, Lemma 2.6]) *Let* $\theta > 0$ *and* $\varphi \in C([0,1],\mathbb{R}^+)$. *Then*

$$\left(\int_0^1 \varphi(t)\mathrm{d}t\right)^\theta \le \int_0^1 (\varphi(t))^\theta \mathrm{d}t, \quad \text{if} \quad \theta \ge 1,$$

and

$$\left(\int_0^1 \varphi(t)\mathrm{d}t\right)^\theta \ge \int_0^1 (\varphi(t))^\theta \mathrm{d}t, \quad \text{if} \quad 0 < \theta \le 1.$$

3. Main results

For brevity, we denote by

$$w=(w_1,w_2,w_3,w_4)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,$$
$$p_*:=\min\{1,p-1\},\quad p^*:=\max\{1,p-1\},$$
$$\mathcal{G}(t,s):=\frac{1}{3}[G_1(t,s)+2G_2(t,s)]\in[0,1].$$

We now list our hypotheses.

(H1) There exist $a_1,b_1,c_1,d_1\geq 0$ and $l_1,l_2>0$ such that

$$f(t,w)\geq a_1(w_1+2w_2)^{p-1}+b_1(w_3+2w_4)^{p-1}-l_1,\forall(w,t)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times[0,1],$$
$$g(t,w)\geq c_1(w_1+2w_2)^{p-1}+d_1(w_3+2w_4)^{p-1}-l_2,\forall(w,t)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times[0,1],$$

and

$$\mathcal{K}_{12}>0,\mathcal{K}_{13}>0,\mathcal{K}_1:=\mathcal{K}_{11}\mathcal{K}_{14}-\mathcal{K}_{12}\mathcal{K}_{13}>0,$$

where

$$\mathcal{K}_{11}:=2^{\frac{p_*-p+1}{p-1}}3^{p_*-1}b_1^{\frac{p_*}{p-1}}\kappa,\ \mathcal{K}_{12}:=1-2^{\frac{p_*-p+1}{p-1}}3^{p_*-1}a_1^{\frac{p_*}{p-1}}\kappa,$$
$$\mathcal{K}_{13}:=1-2^{\frac{p_*-p+1}{p-1}}3^{p_*-1}d_1^{\frac{p_*}{p-1}}\kappa,\ \mathcal{K}_{14}:=2^{\frac{p_*-p+1}{p-1}}3^{p_*-1}c_1^{\frac{p_*}{p-1}}\kappa.$$

(H2) There exist $a_2,b_2,c_2,d_2\geq 0$ and $r>0$ such that

$$f(t,w)\leq a_2(w_1+2w_2)^{p-1}+b_2(w_3+2w_4)^{p-1},\forall(w,t)\in[0,r]\times[0,r]\times[0,r]\times[0,r]\times[0,1],$$
$$g(t,w)\leq c_2(w_1+2w_2)^{p-1}+d_2(w_3+2w_4)^{p-1},\forall(w,t)\in[0,r]\times[0,r]\times[0,r]\times[0,r]\times[0,1],$$

and

$$\mathcal{K}_{21}>0,\mathcal{K}_{24}>0,\mathcal{K}_2:=\mathcal{K}_{21}\mathcal{K}_{24}-\mathcal{K}_{22}\mathcal{K}_{23}>0,$$

where

$$\mathcal{K}_{21}:=1-2^{\frac{p^*-p+1}{p-1}}3^{p^*-1}a_2^{\frac{p^*}{p-1}},\ \mathcal{K}_{22}:=2^{\frac{p^*-p+1}{p-1}}3^{p^*-1}b_2^{\frac{p^*}{p-1}},$$
$$\mathcal{K}_{23}:=2^{\frac{p^*-p+1}{p-1}}3^{p^*-1}c_2^{\frac{p^*}{p-1}},\ \mathcal{K}_{24}:=1-2^{\frac{p^*-p+1}{p-1}}3^{p^*-1}d_2^{\frac{p^*}{p-1}}.$$

(H3) There exist $a_3,b_3,c_3,d_3\geq 0$ and $r>0$ such that

$$f(t,w)\geq a_3(w_1+2w_2)^{p-1}+b_3(w_3+2w_4)^{p-1},\forall(w,t)\in[0,r]\times[0,r]\times[0,r]\times[0,r]\times[0,1],$$
$$g(t,w)\geq c_3(w_1+2w_2)^{p-1}+d_3(w_3+2w_4)^{p-1},\forall(w,t)\in[0,r]\times[0,r]\times[0,r]\times[0,r]\times[0,1],$$

and

$$\mathcal{K}_{32}>0,\mathcal{K}_{33}>0,\mathcal{K}_3:=\mathcal{K}_{31}\mathcal{K}_{34}-\mathcal{K}_{32}\mathcal{K}_{33}>0,$$

where

$$\mathcal{K}_{31}:=2^{\frac{p_*-p+1}{p-1}}3^{p_*-1}b_3^{\frac{p_*}{p-1}}\kappa,\ \mathcal{K}_{32}:=1-2^{\frac{p_*-p+1}{p-1}}3^{p_*-1}a_3^{\frac{p_*}{p-1}}\kappa,$$
$$\mathcal{K}_{33}:=1-2^{\frac{p_*-p+1}{p-1}}3^{p_*-1}d_3^{\frac{p_*}{p-1}}\kappa,\ \mathcal{K}_{34}:=2^{\frac{p_*-p+1}{p-1}}3^{p_*-1}c_3^{\frac{p_*}{p-1}}\kappa.$$

(H4) There exist $a_4,b_4,c_4,d_4\geq 0$ and $l_3,l_4>0$ such that

$$f(t,w)\leq a_4(w_1+2w_2)^{p-1}+b_4(w_3+2w_4)^{p-1}+l_3,\forall(w,t)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times[0,1],$$
$$g(t,w)\leq c_4(w_1+2w_2)^{p-1}+d_4(w_3+2w_4)^{p-1}+l_4,\forall(w,t)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times[0,1],$$

and

$$\mathcal{K}_{41}>0,\mathcal{K}_{44}>0,\mathcal{K}_4:=\mathcal{K}_{41}\mathcal{K}_{44}-\mathcal{K}_{42}\mathcal{K}_{43}>0,$$

where

$$\mathcal{K}_{41} := 1 - 4^{\frac{p^*-p+1}{p-1}} 3^{p^*-1} a_4^{\frac{p^*}{p-1}},\ \mathcal{K}_{42} := 4^{\frac{p^*-p+1}{p-1}} 3^{p^*-1} b_4^{\frac{p^*}{p-1}},$$
$$\mathcal{K}_{43} := 4^{\frac{p^*-p+1}{p-1}} 3^{p^*-1} c_4^{\frac{p^*}{p-1}},\ \mathcal{K}_{44} := 1 - 4^{\frac{p^*-p+1}{p-1}} 3^{p^*-1} d_4^{\frac{p^*}{p-1}}.$$

We let $B_\rho := \{u \in E : \|u\| < \rho\}$ for $\rho > 0$ in the sequel.

Theorem 3.1. *Suppose that (H1) and (H2) are satisfied. Then* (1.1) *has at least one positive solution.*

Proof. Let

$$\mathcal{M}_1 := \{(u,v) \in P \times P : (u,v) = A(u,v) + \lambda(\varphi,\varphi) \text{ for some } \lambda \geq 0\},$$

where $\varphi(t) \in P_0$ is a fixed element. Clearly, Lemma 2.3 implies $u, v \in P_0$. Next we claim $\mathcal{M}_1$ is bounded. Indeed, $(u,v) \in \mathcal{M}_1$ implies $u = A_1(u,v) + \lambda\varphi,\ v = A_2(u,v) + \lambda\varphi$ and thus $u(t) \geq A_1(u,v)(t), v(t) \geq A_2(u,v)(t), \forall t \in [0,1]$. By definition we obtain

$$u(t) \geq \int_0^t \left(\int_s^1 f\left(\tau,(L_1u)(\tau),(L_2u)(\tau),(L_1v)(\tau),(L_2v)(\tau)\right)\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s, \forall (u,v) \in \mathcal{M}_1.$$

Note that $p_*, \frac{p_*}{p-1} \in [0,1]$. Now, by Jensen's inequality and (H1), we find

$$\begin{aligned}
u^{p_*}(t) &\geq \left[\int_0^t \left(\int_s^1 f\left(\tau,(L_1u)(\tau),(L_2u)(\tau),(L_1v)(\tau),(L_2v)(\tau)\right)\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s\right]^{p_*} \\
&\geq \int_0^t \int_s^1 f^{\frac{p_*}{p-1}}\left(\tau,(L_1u)(\tau),(L_2u)(\tau),(L_1v)(\tau),(L_2v)(\tau)\right)\mathrm{d}\tau\mathrm{d}s \\
&= \int_0^1 G_1(t,s) f^{\frac{p_*}{p-1}}\left(s,(L_1u)(s),(L_2u)(s),(L_1v)(s),(L_2v)(s)\right)\mathrm{d}s \\
&\geq \int_0^1 G_1(t,s)[a_1((L_1u)(s)+2(L_2u)(s))^{p-1} + b_1((L_1v)(s)+2(L_2v)(s))^{p-1} - l_1]^{\frac{p_*}{p-1}}\mathrm{d}s \\
&= \int_0^1 G_1(t,s)\Big[a_1\left(\int_0^1 [G_1(s,\tau)+2G_2(s,\tau)]u(\tau)\mathrm{d}\tau\right)^{p-1} \\
&\quad + b_1\left(\int_0^1 [G_1(s,\tau)+2G_2(s,\tau)]v(\tau)\mathrm{d}\tau\right)^{p-1} - l_1\Big]^{\frac{p_*}{p-1}}\mathrm{d}s \\
&\geq 2^{\frac{p_*-p+1}{p-1}} 3^{p_*} \int_0^1 G_1(t,s)\left[a_1^{\frac{p_*}{p-1}}\left(\int_0^1 \mathcal{G}(s,\tau)u(\tau)\mathrm{d}\tau\right)^{p_*} + b_1^{\frac{p_*}{p-1}}\left(\int_0^1 \mathcal{G}(s,\tau)v(\tau)\mathrm{d}\tau\right)^{p_*}\right]\mathrm{d}s \\
&\quad - \frac{l_1^{\frac{p_*}{p-1}}}{2} \\
&\geq 2^{\frac{p_*-p+1}{p-1}} 3^{p_*} \int_0^1 G_1(t,s)\left[a_1^{\frac{p_*}{p-1}} \int_0^1 \mathcal{G}(s,\tau)u^{p_*}(\tau)\mathrm{d}\tau + b_1^{\frac{p_*}{p-1}} \int_0^1 \mathcal{G}(s,\tau)v^{p_*}(\tau)\mathrm{d}\tau\right]\mathrm{d}s \\
&\quad - \frac{l_1^{\frac{p_*}{p-1}}}{2}
\end{aligned}$$

$$= 2^{\frac{p_*-p+1}{p-1}} 3^{p_*-1} \int_0^1 G_1(t,s) \int_0^1 [G_1(s,\tau) + 2G_2(s,\tau)] \left[a_1^{\frac{p_*}{p-1}} u^{p_*}(\tau) + b_1^{\frac{p_*}{p-1}} v^{p_*}(\tau)\right] \mathrm{d}\tau \mathrm{d}s - \frac{l_1^{\frac{p_*}{p-1}}}{2}. \tag{3.1}$$

Let

$$z_1(\tau) := a_1^{\frac{p_*}{p-1}} u^{p_*}(\tau) + b_1^{\frac{p_*}{p-1}} v^{p_*}(\tau).$$

Then multiplying both sides of (3.1) by $\psi(t)$, note (2.1), and we obtain

$$\begin{aligned}
&\int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t \\
&\geq 2^{\frac{p_*-p+1}{p-1}} 3^{p_*-1}\kappa \int_0^1 \psi(s) \left[\int_0^1 [G_1(s,\tau) + 2G_2(s,\tau)] z_1(\tau)\mathrm{d}\tau\right] \mathrm{d}s - \frac{l_1^{\frac{p_*}{p-1}}}{2} \\
&= 2^{\frac{p_*-p+1}{p-1}} 3^{p_*-1}\kappa \int_0^1 \int_0^s s e^s \tau z_1(\tau)\mathrm{d}\tau\mathrm{d}s + 2^{\frac{p_*-p+1}{p-1}} 3^{p_*-1}\kappa \int_0^1 \int_s^1 s e^s (s+2) z_1(\tau)\mathrm{d}\tau\mathrm{d}s \\
&\quad - \frac{l_1^{\frac{p_*}{p-1}}}{2} \\
&= 2^{\frac{p_*-p+1}{p-1}} 3^{p_*-1}\kappa \int_0^1 \int_\tau^1 s e^s \tau z_1(\tau)\mathrm{d}s\mathrm{d}\tau + 2^{\frac{p_*-p+1}{p-1}} 3^{p_*-1}\kappa \int_0^1 \int_0^\tau s e^s (s+2) z_1(\tau)\mathrm{d}s\mathrm{d}\tau \\
&\quad - \frac{l_1^{\frac{p_*}{p-1}}}{2} \\
&= 2^{\frac{p_*-p+1}{p-1}} 3^{p_*-1}\kappa \int_0^1 \left[a_1^{\frac{p_*}{p-1}} u^{p_*}(\tau) + b_1^{\frac{p_*}{p-1}} v^{p_*}(\tau)\right] \psi(\tau)\mathrm{d}\tau - \frac{l_1^{\frac{p_*}{p-1}}}{2}.
\end{aligned}$$

Similarly,

$$\int_0^1 v^{p_*}(t)\psi(t)\mathrm{d}t \geq 2^{\frac{p_*-p+1}{p-1}} 3^{p_*-1}\kappa \int_0^1 \left[c_1^{\frac{p_*}{p-1}} u^{p_*}(\tau) + d_1^{\frac{p_*}{p-1}} v^{p_*}(\tau)\right] \psi(\tau)\mathrm{d}\tau - \frac{l_2^{\frac{p_*}{p-1}}}{2}.$$

Hence,

$$\begin{bmatrix} \mathcal{K}_{11} & -\mathcal{K}_{12} \\ -\mathcal{K}_{13} & \mathcal{K}_{14} \end{bmatrix} \begin{bmatrix} \int_0^1 v^{p_*}(t)\psi(t)\mathrm{d}t \\ \int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t \end{bmatrix} \leq \begin{bmatrix} \frac{l_1^{\frac{p_*}{p-1}}}{2} \\ \frac{l_2^{\frac{p_*}{p-1}}}{2} \end{bmatrix}.$$

Also we have

$$\begin{bmatrix} \int_0^1 v^{p_*}(t)\psi(t)\mathrm{d}t \\ \int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t \end{bmatrix} \leq \frac{1}{\mathcal{K}_1} \begin{bmatrix} \mathcal{K}_{14} & \mathcal{K}_{12} \\ \mathcal{K}_{13} & \mathcal{K}_{11} \end{bmatrix} \begin{bmatrix} \frac{l_1^{\frac{p_*}{p-1}}}{2} \\ \frac{l_2^{\frac{p_*}{p-1}}}{2} \end{bmatrix}.$$

This implies there exist $N_1, N_2 > 0$ such that

$$\int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t \leq N_1, \ \int_0^1 v^{p_*}(t)\psi(t)\mathrm{d}t \leq N_2.$$

Recall that $u, v \in P_0$, and then

$$\int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t \geq \int_0^1 \|u\|^{p_*} t^{p_*}\psi(t)\mathrm{d}t := \delta_1\|u\|^{p_*},$$

where $\delta_1 = \int_0^1 t^{p_*}\psi(t)\mathrm{d}t > 0$. Consequently,

$$\|u\| \leq \sqrt[p_*]{\delta_1^{-1}N_1},\ \|v\| \leq \sqrt[p_*]{\delta_1^{-1}N_2} \tag{3.2}$$

for all $(u, v) \in \mathcal{M}_1$, which implies the boundedness of $\mathcal{M}_1$, as claimed.
Taking $R > \sup\{\|(u, v)\| : (u, v) \in \mathcal{M}_1\}$ and $R > r$(r is defined in (H2)), we have

$$(u, v) \neq A(u, v) + \lambda(\varphi, \varphi),\ \forall v \in \partial B_R \cap (P \times P),\ \lambda \geq 0.$$

Now by virtue of Lemma 2.4, we obtain

$$i(A, B_R \cap (P \times P), P \times P) = 0. \tag{3.3}$$

Let

$$\mathcal{M}_2 := \{(u, v) \in \overline{B}_r \cap (P \times P) : (u, v) = \lambda A(u, v) \text{ for some } \lambda \in [0, 1]\}.$$

We shall prove $\mathcal{M}_2 = \{0\}$. Indeed, if $(u, v) \in \mathcal{M}_2$, we have $u = \lambda A_1(u, v)$, $v = \lambda A_2(u, v)$ and thus $u(t) \leq A_1(u, v)(t), v(t) \leq A_2(u, v)(t),\ \forall t \in [0, 1]$. Hence

$$u(t) \leq \int_0^t \left(\int_s^1 f\left(\tau, (L_1u)(\tau), (L_2u)(\tau), (L_1v)(\tau), (L_2v)(\tau)\right)\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s, \forall (u, v) \in \mathcal{M}_2.$$

Note that $p^*, \frac{p^*}{p-1} \geq 1$. Now by (H2) and Jensen's inequality, we obtain

$$\begin{aligned}
&u^{p^*}(t)\\
&\leq \left[\int_0^t \left(\int_s^1 f\left(\tau, (L_1u)(\tau), (L_2u)(\tau), (L_1v)(\tau), (L_2v)(\tau)\right)\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s\right]^{p^*}\\
&\leq \int_0^t \int_s^1 f^{\frac{p^*}{p-1}}(\tau, (L_1u)(\tau), (L_2u)(\tau), (L_1v)(\tau), (L_2v)(\tau))\mathrm{d}\tau\mathrm{d}s\\
&= \int_0^1 G_1(t, s) f^{\frac{p^*}{p-1}}(s, (L_1u)(s), (L_2u)(s), (L_1v)(s), (L_2v)(s))\mathrm{d}s\\
&\leq \int_0^1 G_1(t, s)[a_2((L_1u)(s) + 2(L_2u)(s))^{p-1} + b_2((L_1v)(s) + 2(L_2v)(s))^{p-1}]^{\frac{p^*}{p-1}}\mathrm{d}s\\
&\leq 2^{\frac{p^*-p+1}{p-1}} \int_0^1 G_1(t, s)[a_2^{\frac{p^*}{p-1}}((L_1u)(s)+2(L_2u)(s))^{p^*} + b_2^{\frac{p^*}{p-1}}((L_1v)(s)+2(L_2v)(s))^{p^*}]\mathrm{d}s\\
&\leq 2^{\frac{p^*-p+1}{p-1}} 3^{p^*} \int_0^1 G_1(t, s)\left[a_2^{\frac{p^*}{p-1}}\left(\int_0^1 \mathcal{G}(s, \tau)u(\tau)\mathrm{d}\tau\right)^{p^*} + b_2^{\frac{p^*}{p-1}}\left(\int_0^1 \mathcal{G}(s, \tau)v(\tau)\mathrm{d}\tau\right)^{p^*}\right]\mathrm{d}s\\
&\leq 2^{\frac{p^*-p+1}{p-1}} 3^{p^*} \int_0^1 G_1(t, s)\left[a_2^{\frac{p^*}{p-1}} \int_0^1 \mathcal{G}(s, \tau)u^{p^*}(\tau)\mathrm{d}\tau + b_2^{\frac{p^*}{p-1}} \int_0^1 \mathcal{G}(s, \tau)v^{p^*}(\tau)\mathrm{d}\tau\right]\mathrm{d}s\\
&= 2^{\frac{p^*-p+1}{p-1}} 3^{p^*-1} \int_0^1 G_1(t, s)\left[\int_0^1 [G_1(s, \tau)+2G_2(s, \tau)]\left[a_2^{\frac{p^*}{p-1}} u^{p^*}(\tau)+b_2^{\frac{p^*}{p-1}} v^{p^*}(\tau)\right]\mathrm{d}\tau\right]\mathrm{d}s.
\end{aligned}$$

Multiplying both sides of the above by $\psi(t)$ and integrating over $[0,1]$, note (2.1) we get

$$\begin{aligned}
&\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t \\
&\le 2^{\frac{p^*-p+1}{p-1}} 3^{p^*-1} \int_0^1 \psi(s) \int_0^1 [G_1(s,\tau)+2G_2(s,\tau)] \left[a_2^{\frac{p^*}{p-1}} u^{p^*}(\tau) + b_2^{\frac{p^*}{p-1}} v^{p^*}(\tau)\right] \mathrm{d}\tau \mathrm{d}s \\
&= 2^{\frac{p^*-p+1}{p-1}} 3^{p^*-1} \int_0^1 \left[a_2^{\frac{p^*}{p-1}} u^{p^*}(t) + b_2^{\frac{p^*}{p-1}} v^{p^*}(t)\right] \psi(t)\mathrm{d}t.
\end{aligned}$$

Similarly,

$$\int_0^1 v^{p^*}(t)\psi(t)\mathrm{d}t \le 2^{\frac{p^*-p+1}{p-1}} 3^{p^*-1} \int_0^1 \left[c_2^{\frac{p^*}{p-1}} u^{p^*}(t) + d_2^{\frac{p^*}{p-1}} v^{p^*}(t)\right] \psi(t)\mathrm{d}t.$$

Hence,

$$\begin{bmatrix} \mathcal{K}_{21} & -\mathcal{K}_{22} \\ -\mathcal{K}_{23} & \mathcal{K}_{24} \end{bmatrix} \begin{bmatrix} \int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t \\ \int_0^1 v^{p^*}(t)\psi(t)\mathrm{d}t \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} \int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t \\ \int_0^1 v^{p^*}(t)\psi(t)\mathrm{d}t \end{bmatrix} \le \frac{1}{\mathcal{K}_2} \begin{bmatrix} \mathcal{K}_{24} & \mathcal{K}_{22} \\ \mathcal{K}_{23} & \mathcal{K}_{21} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t = 0, \quad \int_0^1 v^{p^*}(t)\psi(t)\mathrm{d}t = 0,$$

whence $u(t), v(t) \equiv 0, \forall (u,v) \in \mathcal{M}_2$. As a result, $\mathcal{M}_2 = \{0\}$, as claimed. Consequently,

$$(u,v) \neq \lambda A(u,v), \forall (u,v) \in \partial B_r \cap (P \times P), \lambda \in [0,1].$$

Now Lemma 2.5 yields

$$i(A, B_r \cap (P \times P), P \times P) = 1. \tag{3.4}$$

Combining this with (3.3) gives

$$i(A, (B_R \backslash \overline{B}_r) \cap (P \times P), P \times P) = 0 - 1 = -1.$$

Hence the operator A has at least one fixed point on $(B_R \backslash \overline{B}_r) \cap (P \times P)$ and therefore (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.2. *Suppose that (H3) and (H4) are satisfied. Then* (1.1) *has at least one positive solution.*

Proof. Let

$$\mathcal{M}_3 := \{(u,v) \in \overline{B}_r \cap (P \times P) : (u,v) = A(u,v) + \lambda(\varphi,\varphi) \text{ for some } \lambda \ge 0\},$$

where $\varphi \in P$ is a given element. Next we claim $\mathcal{M}_3 \subset \{0\}$. Indeed, if $(u, v) \in \mathcal{M}_3$, then we have $u \geq A_1(u, v), v \geq A_2(u, v)$ by definition. Consequently,

$$u(t) \geq \int_0^t \left(\int_s^1 f\left(\tau, (L_1 u)(\tau), (L_2 u)(\tau), (L_1 v)(\tau), (L_2 v)(\tau)\right) \mathrm{d}\tau \right)^{\frac{1}{p-1}} \mathrm{d}s, \forall (u, v) \in \mathcal{M}_3.$$

Note that $p_*, \frac{p_*}{p-1} \in [0, 1]$. Now by (H3) and Jensen's inequality, we obtain

$$\begin{aligned}
&u^{p_*}(t) \\
&\geq \left[\int_0^t \left(\int_s^1 f\left(\tau, (L_1 u)(\tau), (L_2 u)(\tau), (L_1 v)(\tau), (L_2 v)(\tau)\right) \mathrm{d}\tau \right)^{\frac{1}{p-1}} \mathrm{d}s \right]^{p_*} \\
&\geq \int_0^1 G_1(t, s) [a_3((L_1 u)(s) + 2(L_2 u)(s))^{p-1} + b_3((L_1 v)(s) + 2(L_2 v)(s))^{p-1}]^{\frac{p_*}{p-1}} \mathrm{d}s \\
&\geq 2^{\frac{p_* - p + 1}{p-1}} 3^{p_*} \int_0^1 G_1(t, s) \left[a_3^{\frac{p_*}{p-1}} \int_0^1 \mathcal{G}(s, \tau) u^{p_*}(\tau) \mathrm{d}\tau + b_3^{\frac{p_*}{p-1}} \int_0^1 \mathcal{G}(s, \tau) v^{p_*}(\tau) \mathrm{d}\tau \right] \mathrm{d}s \\
&= 2^{\frac{p_* - p + 1}{p-1}} 3^{p_* - 1} \int_0^1 G_1(t, s) \int_0^1 [G_1(s, \tau) + 2G_2(s, \tau)] \left[a_3^{\frac{p_*}{p-1}} u^{p_*}(\tau) + b_3^{\frac{p_*}{p-1}} v^{p_*}(\tau) \right] \mathrm{d}\tau \mathrm{d}s.
\end{aligned}$$

We multiply both sides of the above by $\psi(t)$ and integrate over $[0, 1]$, and use (2.1) to obtain

$$\begin{aligned}
&\int_0^1 u^{p_*}(t) \psi(t) \mathrm{d}t \\
&\geq 2^{\frac{p_* - p + 1}{p-1}} 3^{p_* - 1} \kappa \int_0^1 \psi(s) \int_0^1 [G_1(s, \tau) + 2G_2(s, \tau)] \left[a_3^{\frac{p_*}{p-1}} u^{p_*}(\tau) + b_3^{\frac{p_*}{p-1}} v^{p_*}(\tau) \right] \mathrm{d}\tau \mathrm{d}s \\
&= 2^{\frac{p_* - p + 1}{p-1}} 3^{p_* - 1} \kappa \int_0^1 \psi(t) \left[a_3^{\frac{p_*}{p-1}} u^{p_*}(t) + b_3^{\frac{p_*}{p-1}} v^{p_*}(t) \right] \mathrm{d}t.
\end{aligned}$$

Similarly,

$$\int_0^1 v^{p_*}(t) \psi(t) \mathrm{d}t \geq 2^{\frac{p_* - p + 1}{p-1}} 3^{p_* - 1} \kappa \int_0^1 \psi(t) \left[c_3^{\frac{p_*}{p-1}} u^{p_*}(t) + d_3^{\frac{p_*}{p-1}} v^{p_*}(t) \right] \mathrm{d}t.$$

Consequently,

$$\begin{bmatrix} \mathcal{K}_{31} & -\mathcal{K}_{32} \\ -\mathcal{K}_{33} & \mathcal{K}_{34} \end{bmatrix} \begin{bmatrix} \int_0^1 v^{p_*}(t) \psi(t) \mathrm{d}t \\ \int_0^1 u^{p_*}(t) \psi(t) \mathrm{d}t \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Also we have

$$\begin{bmatrix} \int_0^1 v^{p_*}(t) \psi(t) \mathrm{d}t \\ \int_0^1 u^{p_*}(t) \psi(t) \mathrm{d}t \end{bmatrix} \leq \frac{1}{\mathcal{K}_3} \begin{bmatrix} \mathcal{K}_{34} & \mathcal{K}_{32} \\ \mathcal{K}_{33} & \mathcal{K}_{31} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$\int_0^1 u^{p_*}(t)\psi(t)\mathrm{d}t = 0, \quad \int_0^1 v^{p_*}(t)\psi(t)\mathrm{d}t = 0,$$

whence $u(t), v(t) \equiv 0, \forall (u,v) \in \mathcal{M}_3$. Therefore, we claim $\mathcal{M}_3 \subset \{0\}$.
As a result, we have

$$(u,v) - A(u,v) \neq \lambda(\varphi,\varphi), \forall (u,v) \in \partial B_r \cap (P \times P), \lambda \geq 0.$$

Now Lemma 2.4 gives

$$i(A, B_r \cap (P \times P), P \times P) = 0. \tag{3.5}$$

Let

$$\mathcal{M}_4 := \{(u,v) \in P \times P : (u,v) = \lambda A(u,v) \text{ for some } \lambda \in [0,1]\}.$$

It follows from Lemma 2.3 that $u, v \in P_0$. Next we assert $\mathcal{M}_4$ is bounded. Indeed, if $(u,v) \in \mathcal{M}_4$, then

$$u \leq A_1(u,v), \quad v \leq A_2(u,v).$$

Hence

$$u(t) \leq \int_0^t \left(\int_s^1 f\left(\tau, (L_1u)(\tau), (L_2u)(\tau), (L_1v)(\tau), (L_2v)(\tau)\right) \mathrm{d}\tau \right)^{\frac{1}{p-1}} \mathrm{d}s, \forall (u,v) \in \mathcal{M}_4. \tag{3.6}$$

Note that $p^*, \frac{p^*}{p-1} \geq 1$. Now by (H4) and Jensen's inequality, we obtain

$$\begin{aligned}
&u^{p^*}(t)\\
&\leq \left[\int_0^t \left(\int_s^1 f\left(\tau, (L_1u)(\tau), (L_2u)(\tau), (L_1v)(\tau), (L_2v)(\tau)\right) \mathrm{d}\tau \right)^{\frac{1}{p-1}} \mathrm{d}s \right]^{p^*}\\
&\leq \int_0^1 G_1(t,s) \Big[a_4 \left((L_1u)(s) + 2(L_2u)(s)\right)^{p-1} + b_4 \left((L_1v)(s) + 2(L_2v)(s)\right)^{p-1} + l_3 \Big]^{\frac{p^*}{p-1}} \mathrm{d}s\\
&\leq 4^{\frac{p^*-p+1}{p-1}} 3^{p^*} \int_0^1 G_1(t,s) \left[a_4^{\frac{p^*}{p-1}} \int_0^1 \mathcal{G}(s,\tau) u^{p^*}(\tau)\mathrm{d}\tau + b_4^{\frac{p^*}{p-1}} \int_0^1 \mathcal{G}(s,\tau) v^{p^*}(\tau)\mathrm{d}\tau \right] \mathrm{d}s\\
&\quad + 2^{\frac{p^*-2p+2}{p-1}} l_3^{\frac{p^*}{p-1}}\\
&= 4^{\frac{p^*-p+1}{p-1}} 3^{p^*-1} \int_0^1 G_1(t,s) \int_0^1 [G_1(s,\tau) + 2G_2(s,\tau)] \left[a_4^{\frac{p^*}{p-1}} u^{p^*}(\tau) + b_4^{\frac{p^*}{p-1}} v^{p^*}(\tau) \right] \mathrm{d}\tau \mathrm{d}s\\
&\quad + 2^{\frac{p^*-2p+2}{p-1}} l_3^{\frac{p^*}{p-1}}.
\end{aligned} \tag{3.7}$$

Multiplying both sides of (3.7) by $\psi(t)$ and integrating over $[0,1]$, and (2.1) enables us to obtain

$$\begin{aligned}&\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t\\ &\le 4^{\frac{p^*-p+1}{p-1}}3^{p^*-1}\int_0^1\psi(s)\Big[\int_0^1[G_1(s,\tau)+2G_2(s,\tau)]\Big[a_4^{\frac{p^*}{p-1}}u^{p^*}(\tau)+b_4^{\frac{p^*}{p-1}}v^{p^*}(\tau)\Big]\mathrm{d}\tau\Big]\mathrm{d}s\\ &\quad+2^{\frac{p^*-2p+2}{p-1}}l_3^{\frac{p^*}{p-1}}\\ &=4^{\frac{p^*-p+1}{p-1}}3^{p^*-1}\int_0^1\left[a_4^{\frac{p^*}{p-1}}u^{p^*}(t)+b_4^{\frac{p^*}{p-1}}v^{p^*}(t)\right]\psi(t)\mathrm{d}t+2^{\frac{p^*-2p+2}{p-1}}l_3^{\frac{p^*}{p-1}}.\end{aligned}$$

Similarly,

$$\int_0^1 v^{p^*}(t)\psi(t)\mathrm{d}t\le 4^{\frac{p^*-p+1}{p-1}}3^{p^*-1}\int_0^1\left[c_4^{\frac{p^*}{p-1}}u^{p^*}(t)+d_4^{\frac{p^*}{p-1}}v^{p^*}(t)\right]\psi(t)\mathrm{d}t+2^{\frac{p^*-2p+2}{p-1}}l_4^{\frac{p^*}{p-1}}.$$

Consequently,

$$\begin{bmatrix}\mathcal{K}_{41} & -\mathcal{K}_{42}\\ -\mathcal{K}_{43} & \mathcal{K}_{44}\end{bmatrix}\begin{bmatrix}\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t\\ \int_0^1 v^{p^*}(t)\psi(t)\mathrm{d}t\end{bmatrix}\le\begin{bmatrix}2^{\frac{p^*-2p+2}{p-1}}l_3^{\frac{p^*}{p-1}}\\ 2^{\frac{p^*-2p+2}{p-1}}l_4^{\frac{p^*}{p-1}}\end{bmatrix}.$$

Also we have

$$\begin{bmatrix}\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t\\ \int_0^1 v^{p^*}(t)\psi(t)\mathrm{d}t\end{bmatrix}\le\frac{1}{\mathcal{K}_4}\begin{bmatrix}\mathcal{K}_{44} & \mathcal{K}_{42}\\ \mathcal{K}_{43} & \mathcal{K}_{41}\end{bmatrix}\begin{bmatrix}2^{\frac{p^*-2p+2}{p-1}}l_3^{\frac{p^*}{p-1}}\\ 2^{\frac{p^*-2p+2}{p-1}}l_4^{\frac{p^*}{p-1}}\end{bmatrix}.$$

This implies there exist $N_3, N_4>0$ such that

$$\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t\le N_3,\ \int_0^1 v^{p^*}(t)\psi(t)\mathrm{d}t\le N_4.$$

Recall that $u,v\in P_0$, and we see

$$N_3\ge\int_0^1 u^{p^*}(t)\psi(t)\mathrm{d}t\ge\int_0^1(t\|u\|)^{p^*}\psi(t)\mathrm{d}t:=\delta_2\|u\|^{p^*},$$

where $\delta_2=\int_0^1 t^{p^*}\psi(t)\mathrm{d}t$. Therefore,

$$\|u\|\le\sqrt[p^*]{\delta_2^{-1}N_3},\|v\|\le\sqrt[p^*]{\delta_2^{-1}N_4},\forall(u,v)\in\mathcal{M}_4.$$

Now the boundedness of $\mathcal{M}_4$, as asserted. Taking $R>\sup\{\|(u,v)\|:(u,v)\in\mathcal{M}_4\}$ and $R>r$ so we have

$$(u,v)\ne\lambda A(u,v),\forall(u,v)\in\partial B_R\cap(P\times P),\lambda\in[0,1].$$

Now Lemma 2.5 yields

$$i(A,B_R\cap(P\times P),P\times P)=1. \tag{3.8}$$

Combining this with (3.5) gives

$$i(A,(B_R\backslash\overline{B}_r)\cap(P\times P),P\times P)=1-0=1.$$

Hence the operator A has at least one fixed point on $(B_R\backslash\overline{B}_r)\cap(P\times P)$ and therefore (1.1) has at least one positive solution. This completes the proof. □

Remark 3.3. Using the inverse-positive matrix idea in this paper one can easily generalize to n-equations.

Acknowledgement. This research is supported by National Science Fund for Young Scholars of China (Grant No.11601048), Natural Science Foundation of Chongqing (Grant No.cstc2016jcyjA0181), the Science and Technology Research Program of Chongqing Municipal Education Commission (Grant No.KJ1703050), Natural Science Foundation of Chongqing Normal University (Grant No.15XLB011).

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Received: August 28, 2014; Accepted: March 12, 2017.