Fixed Point Theory, 19(2018), No. 2, 785-800
DOI: 10.24193/fpt-ro.2018.2.58
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

EXISTENCE THEOREMS OF A NEW SET-VALUED MT-CONTRACTION IN b-METRIC SPACES ENDOWED WITH GRAPHS AND APPLICATIONS

JUKRAPONG TIAMMEE*,[1], SUTHEP SUANTAI** AND YEOL JE CHO***

*Department of Mathematics and Statistics, Faculty of Science and Technology
Chiang Mai Rajabhat University, Chiang Mai 50300, Thailand
E-mail: jukrapong.benz@gmail.com

**Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200, Thailand
E-mail: suthep.s@cmu.ac.th

***Department of Education and the RINS, Gyeongsang National University
Jinju 660-701, Korea
Center for General Education, China Medical University
Taichung, 40402, Taiwan
E-mail: yjcho@gnu.ac.kr

Abstract. A new concept of set-valued Mizoguchi-Takahashi G-contractions is introduced in this paper and some fixed point theorems for such mappings in b-metric spaces endowed with directed graphs are established under some sufficient conditions. Our results improve and extend those of [20] and [24]. We also give some examples supporting our main results. As an applications, we prove the existence of fixed points for multivalued mappings satisfying generalized MT-contractive condition in ϵ-chainable b-metric spaces and the existence of a solution for some integral equations.
Key Words and Phrases: Fixed point, Mizoguchi-Takahashi function, b-metric spaces, directed graph, set-valued map, integral equation, ϵ-chainable metric space.
2010 Mathematics Subject Classification: 47H04, 47H10, 54H25.

1. Introduction

In 1969, Nadler [21] extended the Banach contraction principle to a multi-valued contraction mapping and proved the following theorem.

Theorem 1.1 ([21]). *Let (X, d) be a complete metric space and T be a mapping from X into $CB(X)$. Assume that there exists $k \in [0, 1)$ such that*

$$H(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

[1] Corresponding author.

In 1988, Mizoguchi and Takahashi [20] extended the Nadler's theorem by using some auxiliary functions. They introduced the following contractive condition,

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y)$$

for all $x, y \in X$, where $\varphi : [0, \infty) \to [0, 1)$ is a function satisfying

$$\lim_{r \to t^+} \sup \varphi(r) < 1, \text{ for each } t \in [0, \infty) \tag{1.1}$$

and proved fixed point theorem for this type of mappings. We call a function satisfying (1) a MT-function.

After that, many authors extended and generalized Nadler's fixed point theorem for set-valued mappings in several directions (see [4], [5], [6], [13], [16], [23]).

In 2012, Du [14] gave characterizations of MT-functions and proved the followings fixed point theorem for some contractive set-valued mappings in a complete metric space.

Theorem 1.2 ([14]). *Let (X, d) be a complete metric space, $T : X \to CB(X)$ be a set-valued mapping, $g : X \to X$ be a continuous self-map and $\varphi : [0, \infty) \to [0, 1)$ be an MT-function. Assume that*

(a) *Tx is g-invariant (i.e. $g(Tx) \subseteq Tx$) for each $x \in X$;*
(b) *there exists a function $h : X \to [0, \infty)$ such that*

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + h(gy)d(gy, Tx) \quad \text{for all } x, y \in X.$$

Then there exists $v \in X$ such that $gv \in Tv$ and $v \in Tv$.

Afterward, Pathak, Agarwal and Cho [22] introduced a concept of P-functions and gave characterizations of the mappings in this class. By using this concept, they proved several fixed point and coincidence point theorems for a set-valued mapping satisfying some contractive conditions.

Recently, in 2014, Javahernia et al. [18] introduced the notion of a generalized Mizoguchi-Takahashi function as follows.

Definition 1.3 ([18]). A function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a *generalized Mizoguchi-Takahashi function* (or *generalized MT-function*) if the following conditions hold:

(1) $0 < \beta(u, v) < 1$ for all $u, v > 0$;

(2) $\limsup_{n \to \infty} \beta(u_n, v_n) < 1$, for any bounded sequence $\{u_n\} \subset (0, \infty)$ and any non-increasing sequence $\{v_n\} \subset (0, \infty)$.

We denote by Λ the set of all generalized MT-functions. By using the concept of generalized MT-function, they also proved some fixed point theorems for some contractive set-valued mappings.

On the other hand, the concept of a metric space has been generalized in many ways. In 1989, Bakhtin [2] introduced the concept of b-metric spaces which is a metric space satisfying a relaxed form of the triangle inequality and proved Banach's contraction principle in this space. Since Bakhtin's results, many authors have followed this concept and proved fixed point results for several types of single-valued and set-valued mappings in b-metric spaces (see [2],[7]-[12]).

On the other hand, in 2008, Jachymski [17] combined two concepts in fixed point theory and graph theory to study fixed point theorems for G-contraction mappings in a metric space endowed a directed graph. These results have been generalized by some authors in several ways (see [4], [13], [16], [24], [22], [23])

Let $G = (V(G), E(G))$ be a directed graph, where $V(G)$ is a set of vertices of graph and $E(G)$ is a set of its edges. Assume that G has no parallel edges. We denote by G^{-1} the directed graph obtained from G by reversing the direction of edges, that is,

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

Let x and y be two vertices in G, a *path* in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for each $i = 1, 2, \cdots, n$. A graph G is said to be *connected* if there exists a (directed) path between any two vertices of G. We denote

$$[x]_G^N = \{y \in X : \text{ there is a path in } G \text{ of length } N \text{ from } x \text{ to } y\}.$$

In 2014, Sultana and Vetrivel [24] introduced the notion of a Mizoguchi-Takahashi G-contraction as follows:

Definition 1.4 ([24])**.** Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. A mapping $T : X \to CB(X)$ is called a *Mizoguchi-Takahashi G-contraction* if, for any $x, y \in X$ with $x \neq y$ and $(x, y) \in E(G)$,

(1) $H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$, where $\alpha : [0, \infty) \to [0, 1)$ with

$$\limsup_{s \to t^+} \alpha(s) < 1$$

for all $t \in [0, \infty)$;

(2) if $u \in Tx$ and $v \in Ty$ are such that $d(u, v) \leq d(x, y)$, then $(u, v) \in E(G)$.

They obtained some fixed point theorems for a Mizoguchi-Takahashi G-contraction in a metric space as follows.

Theorem 1.5 ([24])**.** *Let (X, d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. Let $T : X \to CB(X)$ be a Mizoguchi-Takahashi G-contraction. Assume that there exist $N \in \mathbb{N}$ and $x_0 \in X$ such that:*

(1) $[x_0]_G^N \cap Tx_0 \neq \emptyset$;

(2) for any sequence $\{x_n\} \subset X$, if $x_n \to x$ and $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{k_n}\}_{k \in \mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Then there exists a sequence $\{x_n\}$ in X with $x_n \in [x_0]_G^N \cap T(x_{n-1})$ for all $n \in \mathbb{N}$ converging to a fixed point of T.

Motivated by the result of Jachymski, Javahernia et al. and Sultana et al., we introduced the concept of a new generalized Mizoguchi-Takahashi G-contraction in b-metric spaces and establish some fixed point theorems for this contraction in b-metric spaces endowed with a directed graph. Also, we give some examples to illustrate our main results and apply our main result to obtain some fixed point theorems for some contractions in ϵ-chainable metric spaces.

2. Preliminaries

Now, we give some basic definitions, lemmas and notations concerning b-metric spaces.

Definition 2.1. Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a *b-metric* if, for all $x, y, z \in X$, the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

Then (X, d) is called a *b-metric space* with coefficient s.

It is noted that every metric space is a b-metric space with $s = 1$, but the converse in not generally true.

Example 2.2. Let $X = [0, 1]$ and a mapping $d : X \times X \to \mathbb{R}^+$ be defined by

$$d(x, y) = |x - y|^2$$

for all $x, y \in X$. Clearly, (X, d) is a b-metric space with coefficient $s = 2$.

Example 2.3. Let $0 < p < 1$ and $L_p[0, 1]$ be the set of all real functions x on $[0, 1]$ such that $\int_0^1 |x(t)|^p dt < 1$. Define a mapping $d : L_p[0, 1] \times L_p[0, 1] \to \mathbb{R}^+$ by

$$d(x, y) = \int_0^1 |x(t) - y(t)|^p dt$$

for all $x, y \in L_p[0, 1]$. Then (X, d) is a b-metric space with coefficient $s = 2^p$.

Definition 2.4. Let (X, d) be a b-metric space.

(1) A sequence $\{x_n\}$ is said to be *convergent* to a point $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0;$$

(2) A sequence $\{x_n\}$ is called a *Cauchy sequence* if, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_m, x_n) < \epsilon$$

for all $m, n > N$;

(3) A b-metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point $x \in X$.

Next, we give some notions and lemmas concerning set-valued mappings on b-metric spaces. We denote by $CB(X)$ the class of all nonempty closed and bounded subsets of X and $CL(X)$ the class of all nonempty closed subset of X. For any $A, B \in CB(X)$, define the function $H : CB(X) \times CB(X) \to \mathbb{R}^+$ by

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

where

$$\delta(A, B) = \sup\{d(a, B) : a \in A\},$$
$$\delta(B, A) = \sup\{d(b, A) : b \in B\},$$

$$d(a, C) = \inf\{d(a, x) : x \in C\}.$$

Note that H is called the *Hausdorff b-metric* induced by the b-metric d.

Now, we recall the following properties from ([10], [11], [12]).

Lemma 2.5 ([10]). *Let (X, d) be a b-metric space with coefficient $s \geq 1$. For any $A, B, C \in CB(X)$ and $x, y \in X$, one has the following:*

(1) *$d(x, B) \leq d(x, b)$ for any $b \in B$;*
(2) *$\delta(A, B) \leq H(A, B)$;*
(3) *$d(x, B) \leq H(A, B)$ for any $x \in A$;*
(4) *$H(A, A) = 0$;*
(5) *$H(A, B) = H(B, A)$;*
(6) *$H(A, C) \leq s[H(A, B) + H(B, C)]$;*
(7) *$d(x, A) \leq s[d(x, y) + d(y, A)]$.*

Remark 2.6 ([10]). The function $H : CL(X) \times CL(X) \to \mathbb{R}^+$ is a generalized Pompeiu-Hausdorff b-metric, that is, $H(A, B) = +\infty$ if $\max\{\delta(A, B), \delta(B, A)\}$ does not exist.

Lemma 2.7 ([11]). *Let (X, d) be a b-metric space with coefficient $s \geq 1$. For any $A \in CL(X)$ and $x \in X$, one has*

$$d(x, A) = 0 \iff x \in \bar{A} = A,$$

where $\bar{A}$ denote the closure of the set A.

Lemma 2.8 ([12]). *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $A, B \in CL(X)$ with $H(A, B) > 0$. Then, for each $h > 1$ and $a \in A$, there exists $b \in B$ such that $d(a, b) \leq hH(A, B)$.*

3. Main results

We first need the following class of functions for our main results.

Definition 3.1. Let $s \geq 1$. A function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a *s-generalized Mizoguchi-Takahashi function* (or *s-generalized MT-function*) if the following conditions hold:

(1) $0 < \beta(u, v) < 1$ for all $u, v > 0$;

(2) for any bounded sequence $\{u_n\} \subset (0, \infty)$ and any non-increasing sequence $\{v_n\} \subset (0, \infty)$, we have

$$\limsup_{n\to\infty} \beta(u_n, v_n) < \frac{1}{s}.$$

We denote by Λ_s the set of all s-generalized MT-functions.

Example 3.2. Let $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a generalized MT-function, then β is 1-generalized MT-function.

Example 3.3. Let $s \geq 1$ and $g(x) = \dfrac{ln(x + 5)}{s(x + 4)}$ for all $x > -4$. Define a function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\beta(u, v) = \begin{cases} \dfrac{u}{s(v^2 + 1)}, & 1 < u < v, \\ g(v), & \text{otherwise.} \end{cases}$$

It is easily to see that β is a s-generalized MT-function.

Definition 3.4. Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. A mapping $T : X \to CB(X)$ is called a *s-generalized Mizoguchi-Takahashi G-contraction* if there exists $\beta \in \Lambda_s$ such that, for any $x, y \in X$ with $x \neq y$ and $(x, y) \in E(G)$,

(1) $H(Tx, Ty) \leq \beta(H(Tx, Ty), d(x, y)) \cdot d(x, y)$;

(2) if $u \in Ty$ and $v \in Ty$ are such that $d(u, v) \leq d(x, y)$, then $(u, v) \in E(G)$.

Now, we prove our main results.

Theorem 3.5. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Suppose that $T : X \to CB(X)$ is a s-generalized Mizoguchi-Takahashi G-contraction. Assume that there exist $N \in \mathbb{N}$ and $x_0 \in X$ such that*

(1) $[x_0]_G^N \cap Tx_0 \neq \emptyset$;

(2) for any sequence $\{x_n\}$ in X, if $x_n \to x$ and $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Then there exists $x \in X$ such that $x \in Tx$.

Proof. By (1), there exists $x_1 \in [x_0]_G^N \cap Tx_0$. Then there is a path $(y^i)_{i=0}^N$ in G from x_0 to x_1, i.e., $y^0 = x_0$, $y^N = x_1$ and $(y^{i-1}, y^i) \in E(G)$ for each $i = 1, 2, \cdots, N$. We can choose $k_1^1 > 0$ such that

$$\begin{aligned}&\beta^{k_1^1}(H(Ty^0, Ty^1), d(y^0, y^1)) + \beta(H(Ty^0, Ty^1), d(y^0, y^1)) \cdot d(y^0, y^1)\\ &< d(y^0, y^1). \qquad (3.1)\end{aligned}$$

Since $(y^0, y^1) \in E(G)$, we obtain

$$H(Ty^0, Ty^1) \leq \beta(H(Ty^0, Ty^1), d(y^0, y^1)) \cdot d(y^0, y^1). \qquad (3.2)$$

Since $x_1 \in Ty^0$, by Lemma 2.8, there exists $y_1^1 \in Ty^1$ such that

$$d(x_1, y_1^1) \leq H(Ty^0, Ty^1) + \beta^{k_1^1}(H(Ty^0, Ty^1), d(y^0, y^1)),$$

which implies, by (3.2), that

$$\begin{aligned}&d(x_1, y_1^1)\\ &\leq \beta(H(Ty^0, Ty^1), d(y^0, y^1)) \cdot d(y^0, y^1) + \beta^{k_1^1}(H(Ty^0, Ty^1), d(y^0, y^1)). \qquad (3.3)\end{aligned}$$

From (3.1), it follows that $d(x_1, y_1^1) < d(y^0, y^1)$ and $(x_1, y_1^1) \in E(G)$.

In the same argument, set $y_1^0 = x_1$, for each $i = 1, 2, \cdots, N$, we can choose k_1^i with $k_1^{i-1} < k_1^i$ and $y_1^i \in Ty^i$ such that

$$\begin{aligned}&d(y_1^{i-1}, y_1^1)\\ &\leq H(Ty^{i-1}, Ty^i) + \beta^{k_1^i}(H(Ty^{i-1}, Ty^i), d(y^{i-1}, y^i))\\ &\leq \beta(H(Ty^{i-1}, Ty^i), d(y^{i-1}, y^i)) \cdot d(y^{i-1}, y^i) + \beta^{k_1^i}(H(Ty^{i-1}, Ty^i), d(y^{i-1}, y^i))\\ &< d(y^{i-1}, y^i)\end{aligned}$$

and so $(y_1^{i-1}, y^i) \in E(G)$ for each $i = 2, 3, \cdots, N$. If we denote $x_2 = y_1^N \in Ty^N$, then $(y_1^i)_{i=0}^N$ is a path from x_1 to x_2. Hence $x_2 \in [x_1]_G^N \cap Tx_1$.

Next, we can choose $k_2^1 > k_1^1$ such that

$$\begin{aligned}&\beta^{k_2^1}(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)) + \beta(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)) \cdot d(y_1^0, y_1^1)\\ &< d(y_1^0, y_1^1).\end{aligned} \tag{3.4}$$

Since $(y_1^0, y_1^1) \in E(G)$, we obtain

$$H(Ty_1^0, Ty_1^1) \le \beta(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)) \cdot d(y_1^0, y_1^1). \tag{3.5}$$

By Lemma 2.8, there exists $y_2^1 \in Ty_1^1$ such that

$$d(x_2, y_2^1) \le H(Ty_1^0, Ty_1^1) + \beta^{m_1^1}(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)),$$

which implies, by (3.5), that

$$\begin{aligned}&d(x_2, y_2^1)\\ &\le \beta(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)) \cdot d(y_1^0, y_1^1) + \beta^{k_2^1}(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)).\end{aligned} \tag{3.6}$$

From (3.4), it follows that $d(x_2, y_2^1) < d(y_1^0, y_1^1)$ and $(x_2, y_2^1) \in E(G)$.

In the same argument, set $y_2^0 = x_2$, for each $i = 1, 2, \cdots, N$, we can choose $k_2^i > 0$ and $y_2^i \in Ty_1^i$ such that

$$\begin{aligned}&d(y_2^{i-1}, y_2^1)\\ &\le H(Ty_1^{i-1}, Ty_1^i) + \beta^{k_2^i}(H(Ty_1^{i-1}, Ty_1^i), d(y_1^{i-1}, y_1^i))\\ &\le \beta(H(Ty_1^{i-1}, Ty_1^i), d(y_1^{i-1}, y_1^i)) \cdot d(y_1^{i-1}, y_1^i) + \beta^{k_2^i}(H(Ty_1^{i-1}, Ty_1^i), d(y_1^{i-1}, y_1^i))\\ &< d(y_1^{i-1}, y_1^i).\end{aligned}$$

Then $(y_2^{i-1}, y_2^i) \in E(G)$ for each $i = 2, 3, \cdots, N$. If we denote $x_3 = y_2^N \in Ty_1^N = Tx_2$, then $(y_2^i)_{i=0}^N$ is a path from x_2 to x_3. Hence $x_3 \in [x_2]_G^N \cap Tx_2$.

Continuing this process for each $n \in \mathbb{N}$, we get $x_{n+1} \in [x_n]_G^N \cap Tx_n$ by producing a path $(y_n^i)_{i=0}^N$ from x_n to x_{n+1}, i.e., $y_n^0 = x_n$, $y_n^N = x_{n+1}$ and $y_n^i \in Ty_{n-1}^i$ and

$$\begin{aligned}&d(y_n^{i-1}, y_n^i)\\ &\le H(Ty_{n-1}^{i-1}, Ty_{n-1}^i) + \beta^{k_n^i}(H(Ty_{n-1}^{i-1}, Ty_{n-1}^i), d(y_{n-1}^{i-1}, y_{n-1}^i))\\ &\le \beta(H(Ty_{n-1}^{i-1}, Ty_{n-1}^i), d(y_{n-1}^{i-1}, y_{n-1}^i)) + \beta^{k_n^i}(H(Ty_{n-1}^{i-1}, Ty_{n-1}^i), d(y_{n-1}^{i-1}, y_{n-1}^i))\\ &< d(y_{n-1}^{i-1}, y_{n-1}^i),\end{aligned}$$

where $k_n^i > k_{n-1}^i$ for each $i = 1, 2, \cdots, N$.

For each $i = 1, 2, \cdots, N$, we denote $d_0^i = d(y^{i-1}, y^i)$ and $d_n^i = d(y_n^{i-1}, y_n^i)$ for each $n \ge 1$. From the above inequality, it follows that, for each $i = 1, 2, \cdots, N$, $\{d_n^i\}_{n \in \mathbb{N}}$ is a monotone non-increasing sequence of nonnegative real numbers. Then $d_n^i \to r^i \ge 0$ as $n \to \infty$. From (1) of Definition 3.1 and

$$H(Ty_{n-1}^{i-1}, Ty_{n-1}^i) \le \beta(H(Ty_{n-1}^{i-1}, Ty_{n-1}^i), d(y_{n-1}^{i-1}, y_{n-1}^i)) \cdot d(y_{n-1}^{i-1}, y_{n-1}^i),$$

it follows that $\{H(Ty_{n-1}^{i-1}, Ty_{n-1}^{i})\}_{n\in\mathbb{N}}$ is a bounded sequence. By (2) of Definition 3.1, we obtain

$$\limsup_{n\to\infty} \beta(H(Ty_{n-1}^{i-1}, Ty_{n-1}^{j}), d(y_n^{i-1}, y_n^{i})) < \frac{1}{s}. \tag{3.7}$$

We denote $\pi_n^i = H(Ty_{n-1}^{i-1}, y_{n-1}^{i})$ for each $i = 1, 2, \cdots, N$. Then there exist positive integer n_0^i and $\gamma^i \in [0, 1)$ such that $\beta(\pi_n^i, d_n^i) < \gamma^i$ for all $n \geq n_0^i$, where

$$\limsup_{n\to\infty} \beta(\pi_n^i, d_n^i) < \gamma^i < \frac{1}{s}.$$

Thus, for each $i = 1, 2, \cdots, N$, we have

$$\beta(\pi_n^i, d_n^i) < \gamma < \frac{1}{s}$$

for all $n \geq n_0$, where $\gamma = \max_{1\leq i\leq N} \gamma^i$ and $n_0 = \max_{1\leq i\leq N} n_0^i$. For each $i = 1, 2, \cdots, N$ and $n \geq n_0 + 1$, we obtain

$$\begin{aligned}
d_n^i &\leq \beta(\pi_{n-1}^i d_{n-1}^i) \cdot d_{n-1}^i + \beta^{k_n^i}(\pi_{n-1}^i, d_{n-1}^i) \\
&\leq \cdots \\
&\leq \prod_{j=0}^{n-1} \beta(\pi_j^i, d_j^i) \cdot d_0^i + \sum_{m=1}^{n-1} \prod_{j=m+1}^{n} \beta(\pi_{j-1}^i, d_{j-1}^i) \beta^{k_m^i}(\pi_{j-1}^i, d_{j-1}^i) + \beta^{m_n^i}(\pi_{n-1}^i, d_{n-1}^i) \\
&\leq \gamma^{n-n_0} \prod_{j=0}^{n_0} \beta(\pi_j^i, d_j^i) \cdot d_0^i \\
&\quad + \sum_{m=1}^{n-1} \prod_{j=\max\{n_0+1, m+1\}}^{n} \beta(\pi_{j-1}^i, d_{j-1}^i) \beta^{k_m^i}(\pi_{j-1}^i, d_{j-1}^i) + \gamma^n.
\end{aligned}$$

We denote by B the second term of the last inequality and so

$$\begin{aligned}
B &= \sum_{m=1}^{n-1} \prod_{j=\max\{n_0+1, m+1\}}^{n} \beta(\pi_{j-1}^i, d_{j-1}^i) \beta^{k_m^i}(\pi_{m-1}^i, d_{m-1}^i) \\
&\leq \gamma^{n-n_0} \sum_{m=1}^{n_0} \beta^{k_m^i}(\pi_{m-1}^i) + \sum_{m=n_0+1}^{n-1} \gamma^{n-m} \beta^{k_m^i}(\pi_{m-1}^i, d_{m-1}^i) \\
&\leq Q_1^i \gamma^n + \sum_{m=n_0+1}^{n-1} \gamma^{n-m+k_m^i} \\
&\leq Q_1^i \gamma^n + \gamma^n \frac{\gamma^{k_{n_0}^i - n_0 - 1}}{1-\gamma} \leq Q_2^i \gamma^n,
\end{aligned}$$

where Q_1^i and Q_2^i are nonnegative real numbers. Thus we have

$$d_n^i \leq \gamma^{n-n_0} \prod_{j=0}^{n_0} \beta(\pi_j^i, d_0^i) \cdot d_0^i + Q_2^i \gamma^n + \gamma^n \leq Q^i \gamma^n, \tag{3.8}$$

where Q^i is a nonnegative real number. Hence it follows that, for any $n \geq n_0 + 1$,

$$d(x_n, x_{n+1}) = d(y_n^0, y_n^N) \leq \sum_{i=1}^{N} d_n^i \leq \sum_{i=1}^{N} Q^i \gamma^n.$$

This implies from $s\gamma < 1$ that, for any $n \geq n_0 + 1$ and $m \in \mathbb{N}$,

$$\begin{aligned} d(x_n, x_{n+m}) &\leq s \cdot d(x_n, x_{n+1}) + s^2 \cdot d(x_{n+1}, x_{n+2}) + \cdots \\ &\quad + s^{n+m-1} \cdot d(x_{n+m-2}, x_{n+m-1}) + s^{n+m} \cdot d(x_{n+m-1}, x_{n+m}) \\ &\leq \sum_{i=1}^{N} Q^i \Big[s\gamma^n + s^2\gamma^{n+1} + \cdots + s^{n+m}\gamma^{n+m-1} \Big] \\ &= \sum_{i=1}^{N} Q^i \frac{s\gamma^n(1-(s\gamma)^m)}{1-s\gamma} \\ &\leq \sum_{i=1}^{N} Q^i \frac{s\gamma^n}{1-s\gamma}. \end{aligned}$$

Since $s\gamma^n \to 0$ and $n \to \infty$, $\{x_n\}$ is a b-Cauchy sequence in X. By the b-completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$. We now denote

$$E = \{n \in \mathbb{N} : x_n = x^*\} \text{ and } card(E) = \text{ the cardinal number of } E.$$

If $card(E) = \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \subset E$ i.e., $x_{n_k} = x^*$ for all $k \in \mathbb{N}$. Since $\{x_{n_k+1}\}$ is a sequence in Tx^* and $x_{n_k+1} \to x^*$, it follows that $x^* \in Tx^*$.

If $card(E) < \infty$, then there exists $l \in \mathbb{N}$ such that $x_n \neq x^*$ for all $n \in \mathbb{N}$ with $n \geq l$, which implies that $H(Tx_n, Tx^*) > 0$ and $d(x_n, x^*) > 0$ for any $n \geq l$. Since $x_n \to x^*$ and $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$, by (2), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x^*) \in E(G)$ for any $k \in \mathbb{N}$. Hence we have

$$\begin{aligned} d(x^*, Tx^*) &\leq s\Big[d(x^*, x_{n_k+1}) + d(x_{n_k+1}, Tx^*)\Big] \\ &\leq sd(x^*, x_{n_k+1}) + sH(Tx_{n_k}, Tx^*) \\ &\leq sd(x^*, x_{n_k+1}) + s\beta(H(Tx_{n_k}, Tx^*), d(x_{n_k}, x^*)) \cdot d(x_{n_k}, x^*). \end{aligned}$$

Taking the limit supremum as $k \to \infty$, we have $d(x^*, Tx^*) = 0$. Since Tx^* is b-closed, it follows that $x^* \in Tx^*$. This complete the proof. □

Since every metric space is b-metric space with $s = 1$, following result is directly obtained by Theorem 3.5.

Corollary 3.6. *Let (X, d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Suppose that $T : X \to CB(X)$ is a generalized Mizoguchi-Takahashi G-contraction. Assume that there exist $N \in \mathbb{N}$ and $x_0 \in X$ such that*

(1) $[x_0]_G^N \cap Tx_0 \neq \emptyset$;

(2) *for any sequence* $\{x_n\}$ *in* X, *if* $x_n \to x$ *and* $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$ *for all* $n \in \mathbb{N}$, *then there is a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $(x_{n_k}, x) \in E(G)$ *for all* $k \in \mathbb{N}$.
Then there exists $x \in X$ *such that* $x \in Tx$.

Remark 3.7. If we put $\beta(u,v) = \alpha(v)$ in Corollary 3.6, then we have Theorem 1.5 (Sultana et al. [24]).

Remark 3.8. If we put $\beta(u,v) = \alpha(v)$, $E(G) = X \times X$ and $N = 1$, then we obtain Mizoguchi's Theorem (Mizoguchi et al. [20]) as a corollary.

By putting $\beta(u,v) = \frac{\varphi(v)}{v}$ in Theorem 3.5, we obtain the following.

Corollary 3.9. *Let* (X,d) *be a complete b-metric space with coefficient* $s \geq 1$ *and* $G = (V(G), E(G))$ *be a directed graph such that* $V(G) = X$. *Suppose that* $T : X \to CB(X)$ *is a set-valued mapping such that*

(1) *for all* $x, y \in X$, $x \neq y$ *and* $(x,y) \in E(G)$

$$H(Tx, Ty) \leq \varphi(d(x,y)),$$

where $\varphi : [0,\infty) \to [0,1)$ *such that* $\varphi(v) < v$ *and* $\limsup_{v \to u^+} \frac{\varphi(v)}{v} < 1$ *for all* $v \in [0,\infty)$, *and if* $u \in Tx$ *and* $v \in Ty$ *are such that* $d(u,v) \leq d(x,y)$, *then* $(u,v) \in E(G)$.

Assume that there exist $N \in \mathbb{N}$ *and* $x_0 \in X$ *such that*

(2) $[x_0]_G^N \cap Tx_0 \neq \emptyset$;

(3) *for any sequence* $\{x_n\}$ *in* X, *if* $x_n \to x$ *and* $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$ *for all* $n \in \mathbb{N}$, *then there is a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $(x_{n_k}, x) \in E(G)$ *for all* $k \in \mathbb{N}$.
Then there exists $x \in X$ *such that* $x \in Tx$.

Definition 3.10 ([18])**.** A function $\phi : [0,\infty) \to [0,\infty)$ is said to be a *weakly lower semi-continuous function* (shortly, a w.l.s.c. function) if, for any bounded sequence $\{u_n\} \subset (0,\infty)$, we have

$$\liminf_{n \to \infty} \phi(u_n) > 0.$$

We denote by $W_{lsc}(R)$ the collection of all w.l.s.c. function.

Corollary 3.11. *Let* (X,d) *be a complete b-metric space with coefficient* $s \geq 1$ *and* $G = (V(G), E(G))$ *be a directed graph such that* $V(G) = X$. *Suppose that* $T : X \to CB(X)$ *is a set-valued mapping such that*

(1) *for all* $x, y \in X$, $x \neq y$ *and* $(x,y) \in E(G)$

$$H(Tx, Ty) \leq d(x,y) - \phi(d(x,y)),$$

where $\phi \in W_{lsc}(R)$ *and* $\phi(0) = 0$, $\phi(v) < sv$ *for all* $v \in (0,\infty)$;

(2) *if* $u \in Tx$ *and* $v \in Ty$ *are such that* $d(u,v) \leq d(x,y)$, *then* $(u,v) \in E(G)$.

Assume that there exist $N \in \mathbb{N}$ *and* $x_0 \in X$ *such that*

(3) $[x_0]_G^N \cap Tx_0 \neq \emptyset$;

(4) *for any sequence* $\{x_n\}$ *in* X, *if* $x_n \to x$ *and* $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$ *for all* $n \in \mathbb{N}$, *then there is a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $(x_{n_k}, x) \in E(G)$ *for all* $k \in \mathbb{N}$.
Then there exists $x \in X$ *such that* $x \in Tx$.

Proof. Define $\beta(u,v) = \dfrac{1}{s} - \dfrac{\phi(u)}{u}$ for all $u, v > 0$ and let $\{u_n\} \subseteq [0,\infty)$ be a bounded sequence. Then we have $\liminf_{n\to\infty} \phi(u_n) > 0$ and so $\liminf_{n\to\infty} \dfrac{\phi(u_n)}{u_n} > 0$. This implies that

$$\limsup_{n\to\infty} \Big(\frac{1}{s} - \frac{\phi(u_n)}{u_n}\Big) = \frac{1}{s} - \liminf_{n\to\infty} \frac{\phi(u_n)}{u_n} < \frac{1}{s},$$

which means that $\beta \in \Lambda_s$. Also, we obtain

$$H(Tx,Ty) \leq \beta(H(Tx,Ty), d(x,y)) \cdot d(x,y).$$

Therefore, all conditions of Theorem 3.5 are satisfied and so T has a fixed point. This completes the proof. □

By using Theorem 3.5, we get the following results for single-valued mappings.

Corollary 3.12. *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Suppose that $f : X \to X$ satisfies the following condition: there exists $\beta \in \Lambda_s$ such that, for any $x, y \in X$, $x \neq y$ with $(x,y) \in E(G)$,*

$$(f(x), f(y)) \in E(G), \quad d(f(x), f(y)) \leq \beta(d(f(x), f(y)), d(x,y)) d(x,y).$$

Assume that there exist $N \in \mathbb{N}$ and $x_0 \in X$ such that
(1) $Tx_0 \in [x_0]_G^N$;
(2) if $f^n(x_0) \to x$ and $f^n(x_0) \in [x_{n-1}]_G^N$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{f^{n_k}(x_0)\}$ of $\{f^n(x_0)\}$ such that $(f^{n_k}(x_0), x) \in E(G)$ for all $k \in \mathbb{N}$.
Then $\{f^n(x_0)\}$ converges to a fixed point of f.

Next, we give an example to illustrate Theorem 3.5.

Example 3.13. Let $X = [0,\infty)$ and a b-metric $d : X \times X \to [0,\infty)$ be defined by $d(x,y) = |x-y|^2$ for all $x, y \in X$. It is easy to see that (X,d) be a complete b-metric space with coefficient $s = 2$. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and

$$E(G) = \Big\{\Big(\frac{1}{2^n}, \frac{1}{2^{n+1}}\Big) : n \in \mathbb{N}\Big\} \cup \Big\{\Big(0, \frac{1}{2^n}\Big) : n \in \mathbb{N}\Big\} \cup \{(0,0)\}.$$

Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} \{0, \frac{x}{2}\} & \text{if } 0 \leq x < 1; \\ \{x, x+1\} & \text{if } x > 0. \end{cases}$$

Choosing $x_0 = \dfrac{1}{4}$ and $x_1 = \dfrac{1}{8} \in Tx_0$, we have $(x_0, x_1) = \Big(\dfrac{1}{4}, \dfrac{1}{8}\Big)$. We choose $\beta(u,v) = \dfrac{1}{2}$, then $\beta \in \Lambda_s$. To prove that T is a s-generalized MT G-contraction, let $x, y \in X$ be such that $(x,y) \in E(G)$.

If $(x,y) = (0,0)$, it is obvious that T satisfies (1) and (2) of Definition 3.1.

If $(x,y) = \left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right)$, then we obtain

$$H(Tx,Ty) = \frac{|\frac{1}{2^n} - \frac{1}{2^{n+1}}|^2}{4} \leq \frac{1}{2}d(x,y) = \beta(H(Tx,Ty), d(x,y))d(x,y)$$

and, for any $y \in Tx$ and $v \in Ty$, we have $(u,v) \in E(G)$.

If $(x,y) = \left(\frac{1}{2^n}, 0\right)$, then we obtain

$$H(Tx,Ty) = \frac{|\frac{1}{2^n} - 0|^2}{4} \leq \frac{1}{2}d(x,y) = \beta(H(Tx,Ty), d(x,y))d(x,y)$$

and, for any $y \in Tx$ and $v \in Ty$, we have $(u,v) \in E(G)$. Hence T is a s-generalized Mizoguchi-Takahashi G-contraction. It is easy to see that the condition (2) of Theorem 3.5 are satisfied with $N = 1$. Therefore, all conditions of Theorem 3.5 are satisfied and we see that 0 is a fixed point of T.

Remark 3.14. It is noted that Nadler's Theorem [21] cannot be applied in Example 3.13 with usual metric space, because for any $k \in [0,1)]$,

$$H(T0, T2) = 3 \nleq kd(0,2).$$

4. Applications to ϵ-chainable b-metric spaces

In 1961, Eldelstein [15] introduced a uniformly locally contractive mapping on an ϵ-chainable metric space and prove some fixed point theorems for this type of mappings.

In this section, we show the existence of fixed points for generalized Mizoguchi-Takahashi contractions on an ϵ-chainable b-metric space. The following definitions are needed.

Definition 4.1. A b-metric space (X,d) with coefficient $s \geq 1$ is said to be *ϵ-chainable* if, for any $a, b \in X$, there exist $N \in \mathbb{N}$ and a sequence $\{y^i\}_{i=0}^N$ such that $y^0 = a$ and $y^N = b$ and $d(y^{i-1}, y^i) < \epsilon$ for each i with $1 \leq i \leq N$.

Now, we establish the following theorem for set-valued mappings on ϵ-chainable b-metric spaces as an application of Theorem 3.5.

Theorem 4.2. *Let (X,d) be a complete ϵ-chainable b-metric space with coefficient $s \geq 1$ and $T : X \to CB(X)$ be a set-valued mapping such that there exists $\beta \in \Lambda_s$ satisfying*

$$H(Tx,Ty) \leq \beta(H(Tx,Ty), d(x,y)) \cdot d(x,y)$$

for all $x, y \in X$ with $x \neq y$ and $d(x,y) < \epsilon$. Then T has a fixed point.

Proof. We define $G = (V(G), E(G))$ by $V(G) = X$ and

$$E(G) = \{(x,y) \in X \times X : d(x,y) < \epsilon\}.$$

Then $E(G) \supseteq \Delta$ and G has no parallel edges. It is easily to see that T is a s-generalized Mizoguchi-Takahasji G-contraction. Let $x_0 \in X$ and $x_1 \in Tx_0$. Since (X,d) is an ϵ-chainable, there exists a path $(y^i)_{i=0}^N$ from x_0 to x_1, i.e., $y^0 = x_0$, $y^N = x_1$ and

$(y^{i-1}, y^i) \in E(G)$ for each $i = 1, 2, \cdots, N$. This implies that there exists $N \in \mathbb{N}$ such that $[x_0]_G^N \cap Tx_1 \neq \emptyset$.

Next, let $\{x_n\}$ be a sequence in X such that $x_n \to z \in X$. Then there exists a positive integer M such that

$$d(x_n, z) < \epsilon$$

for all $n \geq M$. Thus we can obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, z) \in E(G)$ for all $k \in \mathbb{N}$. So all the conditions of Theorem 3.5 are satisfied. Therefore, T has a fixed point. This completes the proof. □

The following result is obtained immediately by Theorem 4.2.

Corollary 4.3. *Let (X, d) be a complete ϵ-chainable metric space and $T : X \to CB(X)$ be a set-valued mapping such that there exists $\beta \in \Lambda$ satisfying*

$$H(Tx, Ty) \leq \beta(H(Tx, Ty), d(x, y)) \cdot d(x, y)$$

for all $x, y \in X$ with $x \neq y$ and $d(x, y) < \epsilon$. Then T has a fixed point.

The following results is obtained by setting the function β in Corollary 4.3 to be $\beta(u, v) = \alpha(u)$.

Corollary 4.4 ([24])**.** *Let (X, d) be a complete ϵ-chainable metric space and $T : X \to CB(X)$ be a set-valued mapping such that there exists $\alpha : [0, \infty) \to [0, 1)$ such that $\limsup_{r \to t^+} \alpha(r) < 1$ for all $t \geq 0$ satisfying*

$$H(Tx, Ty) \leq \alpha(d(x, y)) \cdot d(x, y)$$

for all $x, y \in X$ with $x \neq y$ and $d(x, y) < \epsilon$. Then T has a fixed point.

5. Applications to integral equations

In this section, we give an application of fixed point method to study the existence of solutions for some integral equations.

Let $X = C([0, I], \mathbb{R})$ be the set of real continuous functions defined on closed interval $[0, I]$ where $I > 0$, and define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \|(x - y)^2\|_\infty = \sup_{t \in [0, I]} (x(t) - y(t))^2$$

for all $x, y \in X$. Then (X, d) is a complete b-metric space with coefficient $s = 2$.

Consider the integral equation

$$x(t) = p(t) + \int_0^t S(t, u) f(u, x(u)) du, \tag{9.1}$$

where $f : [0, I] \times \mathbb{R} \to \mathbb{R}$ and $p : [0, I] \to \mathbb{R}$ are two continuous functions and $S : [0, I] \times [0, I] \to [0, \infty)$ is a function such that $S(t, \cdot) \in L^1([0, I])$ for all $t \in [0, I]$.

Let $T : X \to X$ be the operator defined by

$$T(x)(t) = p(t) + \int_0^t S(t, u) f(u, x(u)) du. \tag{9.2}$$

Now, we prove the existence of a solution of the integral equation (9.1) by using Corollary 3.12.

Theorem 5.1. *Let $X = C([0, I], \mathbb{R})$ and $G = (V(G), E(G))$ be a directed graph with $V(G) = X$. Suppose that the following conditions hold:*

(1) *for all $u \in [0, I]$ and $x, y \in X$ with $(x, y) \in E(G)$, we have*

$$|f(u, x(u)) - f(u, y(u))| \leq \eta(x, y)|x(u) - y(u)| \tag{9.3}$$

where

$$\Big\| \int_0^I S(t, u)\eta(x, y)du \Big\|_\infty \leq \sqrt{\varphi(\|(x - y)^2\|_\infty)}$$

and $\varphi : [0, \infty) \to [0, 1)$ such that $\limsup_{r \to t^+} \varphi(r) < \frac{1}{s}$ for all $t \in [0, \infty)$;

(2) *for any $x, y \in X$ with $(x, y) \in E(G)$, $(Tx, Ty) \in E(G)$;*

(3) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(4) *if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then the integral equation (9.1) has a solution in X.

Proof. We see that a fixed point of (9.2) is a solution of (9.1). Let $x, y \in X$ be such that $(x, y) \in E(G)$. From the condition (1), it follows that, for all $t \in [0, I]$,

$$\begin{aligned}
|T(x)(t) - T(y)(t)|^2 &= \Big[\Big| \int_0^I S(t, u)[f(u, x(u)) - f(u, y(u))]du \Big|\Big]^2 \\
&\leq \Big[\int_0^I S(t, u)\Big|f(u, x(u)) - f(u, y(u))\Big|du \Big]^2 \\
&\leq \Big[\int_0^I S(t, u)\eta(x, y)\sqrt{|x(u) - y(u)|^2}du \Big]^2 \\
&\leq \Big[\int_0^I S(t, u)\eta(x, y)\sqrt{\|(x(u) - y(u))^2\|_\infty}du \Big]^2 \\
&= \|(x(u) - y(u))^2\|_\infty \Big[\int_0^I S(t, u)\eta(x, y)du \Big]^2.
\end{aligned}$$

Then we have

$$\Big\|(T(x) - T(y))^2\Big\|_\infty \leq \|(x - y)^2\|_\infty \Big\| \int_0^I S(t, u)\eta(x, y)du \Big\|_\infty^2$$

and so, for all $x, y \in X$ with $(x, y) \in E(G)$, we obtain

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y),$$

where $\varphi \in \Lambda_s$. Therefore, all the conditions of Corollary 3.12 are satisfied and hence the operator T has a fixed point, that is, a solution of the integral equation (9.1) exists in X. This completes the proof. □

Acknowledgements. The authors would like to thank the Thailand Research Fund under the project RTA5780007 and Chiang Mai University, Chiang Mai, Thailand for the financial support. The first author would like to thanks Chiang Mai Rajabhat University, Chiang Mai, Thailand for the financial support. The third author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and future Planning (2014R1A2A2A01002100).

References

[1] M.U. Ali, T. Kamran, M. Postolache, *Fixed point theorems for multivalued G-contractions in Hausdorff b-gauge spaces*, J. Nonlinear Sci. Appl., **8**(2015), no. 5, 847-855.

[2] I.A. Bakhtin, *The contraction mapping principle in quasi-metric spaces*, (Russian), Functional Anal., **30**(1989), 26-37.

[3] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fundam. Math., **3**(1922), 133-181.

[4] I. Beg, A.R. Butt, *The contraction principle for set valued mappings on a metric space with graph*, Comput. Math. Appl., **60**(2010), 1214-1219.

[5] M. Berinde, V. Berinde, *On a general class of multi-valued weakly Picard mappings*, J. Math Anal. Appl., **326**(2007), 772-782.

[6] V. Berinde, M. Pacurar, *The role of Pompeiu-Hausdorff metric in fixed point theory*, Creat. Math. Inform., **22**, 143-150.

[7] V. Berinde, *Generalized contractions in quasi metrics paces*, Seminar on Fixed Point Theory, 1993, 3-9.

[8] M. Boriceanu, *Fixed point theory for multivalued generalized contraction on a set with two b-metrics*, Stud. Univ. Babeş-Bolyai Math., **54**(2009), no. 3, 3-14.

[9] M. Boriceanu, M. Bota, A. Petruşel, *Multivalued fractals in b-metric spaces*, Cent. Eur. J. Math., **8**(2010), 367-377.

[10] S. Czerwick, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti del Semin. Mat. e Fisico dell' Univ. di Modena, **46**(1998), 263-276.

[11] S. Czerwick, K. Dlutek, S.L. Singh, *Round-off stability of iteration procedures for set-valued operators in b-metric spaces*, J. Natural Physical Sci., **11**(2007), 87-94.

[12] S. Czerwick, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1**(1993), 5-11.

[13] T. Dinevari, M. Frigon, *Fixed point results for multivalued contractions on a metric space with a graph*, J. Math. Anal. Appl., **405**(2013), 507-517.

[14] W.S. Du, *On coincidence point and fixed point theorems for nonlinear multivalued maps*, Topology and its Applications, **159**(2012), 49-56.

[15] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc., **12**(1961), 7-10.

[16] A. Erduran, M. Abbas, *Fixed point theory for suzuki type (θ, L)-weak multivalued operators*, Fixed Point Theory, **16**(2015), no. 2, 303-312.

[17] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136**(2008), 1359-1373.

[18] M. Javahernia, A. Razani, F. Khojasteh, *Common fixed point of generalized Mizoguchi-Takahashi's type contractions*, Fixed Point Theory Appl., **2014**, 2014:195.

[19] C. Klanarong, S. Suantai, *Coincidence point theorems for some multi-valued mappings in complete metric spaces endowed with a graph*, Fixed Point Theory Appl., **2015**, 2015:129, 16 pp.

[20] N. Mizoguchi, W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl., **141**(1989), 177-188.

[21] S. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math., **20**(1969), 475-488.

[22] H.K. Pathak, R.P. Agarwal, Y.J. Cho, *Coincidence and fixed points for multi-valued mappings and its application to nonconvex integral inclusions*, J. Comput. Appl. Math., **283**(2015), 201-217.

[23] S. Reich, *Fixed points of contractive functions*, Boll. Un. Mat. Ital., **5**(1972), 26-42.

[24] A. Sultana, V. Vetrivel, *Fixed points of Mizoguchi-Takahashi contraction on a metric space with a graph and applications*, J. Math. Anal. Appl., **417**(2014), 336-344.

[25] J. Tiammee, S. Suantai, *Coincidence point theorems for graph-preserving multi-valued mappings*, Fixed Point Theory Appl. **2014**, 2014:70, 11 pp.
[26] J. Tiammee, S. Suantai, *Coincidence point theorems for multi-valued mappings of Reich-type on metric spaces endowed with a graph*, J. Nonlinear Convex Anal., **16**(2015), no. 2, 365-373.

Received: March 24, 2016; Accepted: October 13, 2016.