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ON SP-ITERATION SCHEMES FOR MULTI-VALUED MAPPINGS IN CAT(0) SPACES

KRITSANA SOKHUMA

Department of Mathematics, Faculty of Science and Technology
Phranakhon Rajabhat University, Bangkok 10220, Thailand
E-mail: k_sokhuma@yahoo.co.th

Abstract. In this paper, we extend known results on convergence of SP-iterations to fixed points of nonexpansive single-valued mappings to a multi-valued version. In order to do so, we prove strong convergence theorems for SP-iteration schemes involving quasi-nonexpansive multi-valued mappings in the framework of CAT(0) spaces.
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1. Introduction

The study of metric spaces without linear structure has played a vital roll in various branches of pure and applied sciences. In particular, existence and approximation results in CAT(0) spaces for classes of single-valued and multi-valued mappings have been studied extensively by many researchers (see [5], [7], [12], [14], [13], [1]).

Iteration schemes for numerical reckoning fixed points of diverse classes of nonlinear operators are available in the literature. The class of nonexpansive mappings via iteration methods has extensively been studied in this regard (see Tan and Xu [19]; Thakur et al. [22]). The class of pseudocontractive mappings in their relation with iteration procedures has been studied by several researchers under suitable conditions (see Yao et al. [24]; Thakur et al. [20]; Dewangan et al. [4]) and applications to variational inequalities are also considered (see [21], [23]). For nonexpansive multi-valued mappings, Sastry and Babu [11] defined a Mann and Ishikawa iteration process in Hilbert spaces. Panyanak [8] and Song and Wang [16] (see also [17]) extended the result of Sastry and Babu [11] to uniformly convex Banach spaces. Recently, Shahzad and Zegeye [15] extended and improved results of (see [11], [8], [16], [17]).

In 2008, Dhompongsa and Panyanak [6] established Δ-convergence theorems for the Mann and Ishikawa iterations for nonexpansive single-valued mappings in CAT(0) spaces. Inspired by Song and Wang [16], Laowang and Panyanak [7] extended the result of Dhompongsa and Panyanak [6] for multi-valued nonexpansive mappings in a CAT(0) space.

In 2011, W. Phuengrattana and S. Suantai [10] introduced the SP-iterative process. The SP-iteration is defined by $u_1 \in E$ and

$$\begin{cases} w_n = (1-\gamma_n)u_n + \gamma_n f(u_n), \\ v_n = (1-\beta_n)w_n + \beta_n f(w_n), \\ u_{n+1} = (1-\alpha_n)v_n + \alpha_n f(v_n), \end{cases} \tag{1.1}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

In 2015, R.P. Pathak et al. [9] introduce a Noor-type iteration process for nonexpansive multi-valued mappings and prove strong convergence theorems for the proposed iterative process in CAT(0) spaces. Let K be a nonempty convex subset of a complete CAT(0) space X. The sequence of Noor-type iterates is defined by $q_1 \in K$,

$$\begin{cases} u_n = (1-\gamma_n)\, q_n \oplus \gamma_n r_n, \\ v_n = (1-\beta_n)\, q_n \oplus \beta_n r_n', \\ q_{n+1} = (1-\alpha_n)\, q_n \oplus \alpha_n r_n'', \end{cases} \tag{1.2}$$

where $r_n \in Tq_n, r_n' \in Tu_n, r_n'' \in Tv_n$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $[a,b] \subset [0,1]$.

Motivated by the above facts in this paper, we extend known results on convergence of SP-iterations to fixed points of nonexpansive single-valued mappings to a multi-valued version. In addition, we prove strong convergence theorems for SP-iteration schemes involving quasi-nonexpansive multi-valued mappings in the framework of CAT(0) spaces.

2. Preliminaries

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset R$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points x_1,x_2,x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i,\overline{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x,y \in \triangle$ and all comparison points $\overline{x},\overline{y} \in \overline{\triangle}$,

$$d(x,y) \leq d_{\mathbb{E}^2}(\overline{x},\overline{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [3]. In fact (cf. [2], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

Lemma 2.1. ([6]) *Let (X, d) be a CAT(0) space.*

(i) For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1 - t)d(x, y).$$

(ii) For $x, y, z \in X$ and $t \in [0, 1]$, we have

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z).$$

(iii) For $x, y, z \in X$ and $t \in [0, 1]$, we have

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2.$$

We will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying Lemma 2.1(i). Now we define preliminaries for the construction of multi-valued nonexpansive mapping.

Let K be the subset of CAT(0) space X. Then:

• The distance from $x \in X$ to K is defined by

$$\operatorname{dist}(x, K) = \inf\{d(x, y) : y \in K\}.$$

• The diameter of K is defined by

$$\operatorname{diam}(K) = \sup\{d(u, v) : u, v \in K\}.$$

Let $CB(K)$ denote the family of nonempty closed bounded subsets of K. The Hausdorff metric H on $CB(K)$ is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} \operatorname{dist}(x, B),\ \sup_{y\in B} \operatorname{dist}(y, A)\right\}$$

for $A, B \in CB(K)$, where $\operatorname{dist}(x, B) = \inf\{d(x, z), z \in B\}$.

Let $T : X \to 2^X$ be a multi-valued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points will be denoted by $\operatorname{Fix}(T)$.

Definition 2.2. A multi-valued mapping $T : K \to CB(K)$ is called:

(i) nonexpansive, if $H(T(x), T(y)) \leq d(x, y)$ for all $x, y \in K$;

(ii) quasi-nonexpansive, if $\operatorname{Fix}(T) \neq \emptyset$, and $H(Tx, Tp) \leq d(x, p)$ for all $x \in K$ and $p \in \operatorname{Fix}(T)$.

Example 2.3. Let $K = [0, \infty)$with the usual metric and $T : K \to CB(K)$ be defined by

$$Tx = \begin{cases} \{0\}, & \text{if } x \leq 1, \\ \left[x - \frac{3}{4}, x - \frac{1}{3}\right], & \text{if } x > 1. \end{cases}$$

Indeed, it is clear that $\mathrm{Fix}(T) = \{0\}$ and for any x we have

$$H(T(x), T(0)) \leq |x - 1|\,.$$

Hence, T is quasi-nonexpansive. However, if $x = 2, y = 1$ we get

$$H(Tx, Ty) > |x - 1| = 1.$$

Clearly this will imply T is not nonexpansive.

3. Main results

Now we introduce the notion of the proposed multi-valued version of the SP-iteration process for a nonexpansive mapping T. Let K be a nonempty convex subset of a complete CAT(0) space X. The sequence of SP-iterates is defined by $x_1 \in K$,

$$\begin{cases} w_n = (1 - \gamma_n)\, x_n \oplus \gamma_n z_n, \\ y_n = (1 - \beta_n)\, w_n \oplus \beta_n z'_n, \\ x_{n+1} = (1 - \alpha_n)\, y_n \oplus \alpha_n z''_n, \end{cases} \tag{3.1}$$

where $z_n \in Tx_n, z'_n \in Tw_n, z''_n \in Ty_n$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $[a, b] \subset [0, 1]$.

Lemma 3.1. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : K \to CB(K)$ be a quasi-nonexpansive multi-valued mapping with Fix(T) $\neq \emptyset$ and for which $Tp = \{p\}$ for each $p \in$ Fix(T). Let $\{x_n\}$ be the SP-iterates defined by (3.1) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[a, b] \subset (0, 1)$. Then:*
(i) $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in$ Fix(T),
(ii) $\lim_{n\to\infty} dist(Tx_n, x_n) = 0$.

Proof. Let $p \in \mathrm{Fix}(T)$. Then, using (3.1) and Lemma 2.1(ii), we have

$$\begin{aligned} d(w_n, p) &= d((1-\gamma_n)x_n \oplus \gamma_n z_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(z_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n \mathrm{dist}(z_n, Tp) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n H(Tx_n, Tp) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) = d(x_n, p). \end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)w_n \oplus \beta_n z'_n, p) \\ &\leq (1-\beta_n)d(w_n, p) + \beta_n d(z'_n, p) \\ &\leq (1-\beta_n)d(w_n, p) + \beta_n \mathrm{dist}(z'_n, Tp) \\ &\leq (1-\beta_n)d(w_n, p) + \beta_n H(Tw_n, Tp) \\ &\leq (1-\beta_n)d(w_n, p) + \beta_n d(w_n, p) = d(w_n, p). \end{aligned} \tag{3.3}$$

Again,

$$\begin{aligned} d(x_{n+1},p) &= d((1-\alpha_n)y_n \oplus \alpha_n z''_n, p) \\ &\le (1-\alpha_n)d(y_n,p) + \alpha_n d(z''_n,p) \\ &\le (1-\alpha_n)d(y_n,p) + \alpha_n \mathrm{dist}(z''_n, Tp) \\ &\le (1-\alpha_n)d(y_n,p) + \alpha_n H(Ty_n, Tp) \\ &\le (1-\alpha_n)d(y_n,p) + \alpha_n d(y_n,p) = d(y_n,p). \end{aligned} \tag{3.4}$$

Using (3.2), (3.3) and (3.4), we have

$$d(x_{n+1},p) \le d(y_n,p) \le d(w_n,p) \le d(x_n,p). \tag{3.5}$$

Hence, the sequence $\{d(x_n,p)\}$ is decreasing and bounded below. It now follows that $\lim_{n\to\infty} d(x_n,p)$ exists for any $p \in \mathrm{Fix}(T)$. From Lemma 2.1(iii), we have

$$\begin{aligned} d^2(x_{n+1},p) &= d^2((1-\alpha_n)y_n \oplus \alpha_n z''_n, p) \\ &\le (1-\alpha_n)d^2(y_n,p) + \alpha_n d^2(z''_n,p) - \alpha_n(1-\alpha_n)d^2(y_n,z''_n) \\ &\le (1-\alpha_n)d^2(y_n,p) + \alpha_n \mathrm{dist}^2(z''_n,Tp) - \alpha_n(1-\alpha_n)d^2(y_n,z''_n) \\ &\le (1-\alpha_n)d^2(y_n,p) + \alpha_n H^2(Ty_n,p) - \alpha_n(1-\alpha_n)d^2(y_n,z''_n) \\ &\le (1-\alpha_n)d^2(y_n,p) + \alpha_n d^2(y_n,p) \\ &= d^2(y_n,p). \end{aligned}$$

Similarly,

$$\begin{aligned} d^2(y_n,p) &= d^2((1-\beta_n)w_n \oplus \beta_n z'_n, p) \\ &\le (1-\beta_n)d^2(w_n,p) + \beta_n d^2(z'_n,p) - \beta_n(1-\beta_n)d^2(w_n,z'_n) \\ &\le (1-\beta_n)d^2(w_n,p) + \beta_n \mathrm{dist}^2(z'_n,Tp) - \beta_n(1-\beta_n)d^2(w_n,z'_n) \\ &\le (1-\beta_n)d^2(w_n,p) + \beta_n H^2(Tw_n,Tp) - \beta_n(1-\beta_n)d^2(w_n,z'_n) \\ &\le (1-\beta_n)d^2(w_n,p) + \beta_n d^2(w_n,p) \\ &= d^2(w_n,p). \end{aligned}$$

Again, from Lemma 2.1(iii), we have

$$\begin{aligned} d^2(w_n,p) &= d^2((1-\gamma_n)x_n \oplus \gamma_n z_n, p) \\ &\le (1-\gamma_n)d^2(x_n,p) + \gamma_n d^2(z_n,p) - \gamma_n(1-\gamma_n)d^2(x_n,z_n) \\ &\le (1-\gamma_n)d^2(x_n,p) + \gamma_n \mathrm{dist}^2(z_n,Tp) - \gamma_n(1-\gamma_n)d^2(x_n,z_n) \\ &\le (1-\gamma_n)d^2(x_n,p) + \gamma_n H^2(Tx_n,Tp) - \gamma_n(1-\gamma_n)d^2(x_n,z_n) \\ &\le (1-\gamma_n)d^2(x_n,p) + \gamma_n d^2(x_n,p) - \gamma_n(1-\gamma_n)d^2(x_n,z_n) \\ &= d^2(x_n,p) - \gamma_n(1-\gamma_n)d^2(x_n,z_n). \end{aligned}$$

Therefore,

$$\begin{aligned} d^2(x_{n+1},p) &\le d^2(y_n,p) \\ &\le d^2(w_n,p) \\ &\le (1-\gamma_n)d^2(x_n,p) + \gamma_n d^2(x_n,p) - \gamma_n(1-\gamma_n)d^2(x_n,z_n) \\ &= d^2(x_n,p) - \gamma_n(1-\gamma_n)d^2(x_n,z_n). \end{aligned}$$

Also

$$\gamma_n(1-\gamma_n)d^2(x_n, z_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p).$$

This implies that,

$$a(1-b)d^2(x_n, z_n) \leq \gamma_n(1-\gamma_n)d^2(x_n, z_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p)$$

and so $\sum_{n=1}^{\infty} a(1-b)d^2(x_n, z_n) < \infty$. Hence, $\lim_{n\to\infty} d^2(x_n, z_n) = 0$.
Thus $\lim_{n\to\infty} d(x_n, z_n) = 0$. We have $\text{dist}(x_n, Tx_n) \leq d(x_n, z_n) \to 0$ as $n \to \infty$.
Therefore,

$$\lim_{n\to\infty} \text{dist}(Tx_n, x_n) = 0. \qquad \square$$

Now we prove a strong convergence theorem for SP-iteration process for multi-valued mappings.

Theorem 3.2. *Let K be nonempty closed convex subset of a complete CAT(0) space X. Let $T : K \to CB(K)$ be a quasi-nonexpansive multi-valued mappings such that Fix(T) $\neq \emptyset$ and for which $Tp = \{p\}$ for each $p \in Fix(T)$. Let $\{x_n\}$ be the SP-iterates defined by (3.1) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[a, b] \subset (0, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\lim_{n\to\infty} \inf dist(x_n, Fix(T)) = 0$.*

Proof. Necessity is obvious. To prove the sufficiency, suppose that

$$\lim_{n\to\infty} \inf \text{dist}(x_n, \text{Fix}(T)) = 0.$$

As in the proof of Lemma 3.1,we have $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in \text{Fix}(T)$. This implies that

$$\text{dist}(x_{n+1}, \text{Fix}(T)) \leq d(x_n, p)$$

so that $\lim_{n\to\infty} \text{dist}(x_n, \text{Fix}(T))$ exists. Thus $\lim_{n\to\infty} \text{dist}(x_n, \text{Fix}(T)) = 0$.
Therefore, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, p_k) < \frac{1}{2^k}$$

for some $\{p_k\} \subset \text{Fix}(T)$ and for all k. By Lemma 3.1 we have

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

Hence

$$d(p_{k+1}, p_k) \leq d(x_{n_{k+1}}, p_{k+1}) \leq d(x_{n_{k+1}}, p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

Consequently, $\{p_k\}$ is a Cauchy sequence in K and hence converges to some $q \in K$.
Since

$$\text{dist}(p_k, Tq) \leq H(Tp_k, Tq) \leq d(p_k, q)$$

and $p_k \to q$ as $k \to \infty$. This implies that $\text{dist}(q, Tq) = 0$ and so $q \in \text{Fix}(T)$ and thus $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} d(x_n, q)$ exists, it follows that $\{x_n\}$ converges strongly to q. $\square$

Theorem 3.3. *Let K be nonempty closed convex subset of a complete CAT(0) space X. Let $T : K \to CB(K)$ be a quasi-nonexpansive multi-valued mapping such that Fix$(T) \neq \emptyset$ and for which $Tp = \{p\}$ for each $p \in$ Fix(T). Let $\{x_n\}$ be the SP-iterates defined by (3.1) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[a, b] \subset (0, 1)$. Assume that T is hemicompact and continuous, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.1, we have $\lim_{n\to\infty} \operatorname{dist}(Tx_n, x_n) = 0$. Since T is hemicompact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in K$ such that $\lim_{k\to\infty} x_{n_k} = q$. From continuity of T, we find that $d(x_{n_k}, Tx_{n_k}) \to d(q, Tq)$. As a result, we have $d(q, Tq) = 0$ and so $q \in \operatorname{Fix}(T)$. By Lemma 3.1, we find that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \operatorname{Fix}(T)$, hence $\{x_n\}$ converges strongly to q. □

Theorem 3.4. *Let K be nonempty closed convex subset of a complete CAT(0) space X. Let $T : K \to CB(K)$ be a quasi-nonexpansive multi-valued mapping such that $Fix(T) \neq \emptyset$ and for which $Tp = \{p\}$ for each $p \in Fix(T)$. Let $\{x_n\}$ be the SP-iterates defined by (3.1) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[a, b] \subset (0, 1)$. Assume that there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$ such that*

$$dist(x, Tx) \geq f(dist(x, Fix(T)))$$

for all $x \in K$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 3.1, we have $\lim_{n\to\infty} \operatorname{dist}(Tx_n, x_n) = 0$. Hence, from the assumption we obtain $\lim_{n\to\infty} d(x_n, \operatorname{Fix}(T)) = 0$. The rest of the conclusion now follows from Theorem 3.2. □

The following corollaries are direct consequences of Theorems 3.2, 3.3 and 3.4.

Corollary 3.5. *Let K be nonempty closed convex subset of a complete CAT(0) space X. Let $T : K \to CB(K)$ be a nonexpansive multi-valued mapping such that $Fix(T) \neq \emptyset$ and for which $Tp = \{p\}$ for each $p \in Fix(T)$. Let $\{x_n\}$ be the SP-iterates defined by (3.1) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[a, b] \subset (0, 1)$. Then $\{x_n\}$converges strongly to a fixed point of T if and only if $\lim_{n\to\infty} dist(x_n, Fix(T)) = 0$.*

Corollary 3.6. *Let K be nonempty closed convex subset of a complete CAT(0) space X. Let $T : K \to CB(K)$ be a nonexpansive multi-valued mapping such that $Fix(T) \neq \emptyset$ and for which $Tp = \{p\}$ for each $p \in Fix(T)$. Let $\{x_n\}$ be the SP-iterates defined by (3.1) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[a, b] \subset (0, 1)$. Assume that T is hemicompact and continuous, then $\{x_n\}$ converges strongly to a fixed point of T.*

Corollary 3.7. *Let K be nonempty closed convex subset of a complete CAT(0) space X. Let $T : K \to CB(K)$ be a nonexpansive multi-valued mapping such that $Fix(T) \neq \emptyset$ and for which $Tp = \{p\}$ for each $p \in Fix(T)$. Let $\{x_n\}$ be the SP-iterates defined by (3.1) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[a, b] \subset (0, 1)$. Assume that there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$ such that*

$$dist(x, Tx) \geq f(dist(x, Fix(T)))$$

for all $x \in K$. Then $\{x_n\}$ converges strongly to a fixed point of T.

For a single-valued mapping, we obtain the following corollary.

Corollary 3.8. *Let K be nonempty closed convex subset of a complete CAT(0) space X. Let $T : K \to K$ be a quasi-nonexpansive mapping such that $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be the SP-iterates defined by*

$$\begin{aligned} w_n &= (1-\gamma_n)x_n \oplus \gamma_n T x_n, \\ y_n &= (1-\beta_n)w_n \oplus \beta_n T w_n, \\ x_{n+1} &= (1-\alpha_n)y_n \oplus \alpha_n T y_n, \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in $[a,b] \subset (0,1)$. Assume that there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x,Tx) \geq f(d(x, Fix(T)))$$

for all $x \in K$. Then $\{x_n\}$ converges strongly to a fixed point of T.

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